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Sensitivity Analyses in Interval Decision Modelling

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Abstract— **Research efforts of the DECIDE Research Group have resulted in a decision tool capable of handling imprecise information in complex decision situations. Some of the research has been directed towards developing decision analytical representation models and algorithms. The decision tool takes intervals as well as comparative relations as input, and incorporates a level of sensitivity analyses embedded into the representation instead of applying separate analyses on top of the evaluation procedure. Further, besides the embedded sensitivity analysis, an explicit sensitivity analysis is useful in order to point out the most critical probabilities, values, or weights to the decision under consideration. This paper deals with the theory and implementation of sensitivity analyses in a decision tool supporting interval probabilities, values, criteria weights, as well as comparative relations.**

***Index Terms*— decision analysis, decision tree, imprecise information, tornado diagram.**

I. INTRODUCTION

[1]In the market of decision analytic tools today, see e.g. [1] for a survey, most of the software packages are not capable of handling imprecise information. Instead, the input needs to be specified in precise numbers, which is often considered less realistic for decision-makers. The models and software based on them mainly consist of some straightforward set of rules applied to precise numerical estimates of probabilities and values. Then, sensitivity analyses are often not easy to carry out in more than a few dimensions at a time. The decision analytic tool discussed in this paper, DecideIT, coincides functionally with regular decision tree software, such as Precision Tree [2] or TreeAge Pro [3], but also with multi-criteria software such as Expert Choice [4] since it handles probabilities, values, and weights all in the same framework. However, DecideIT also handles imprecise information, not requiring a pointwise precision in the input parameters, but instead being capable of handling intervals as well as comparative statements between parameters, such as "greater than", "more important than", "equal to" or "between 20 and 40 more valuable than" [5].

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There have been quite extensive research efforts in the area of imprecise probabilities, see, e.g., [6] for a collection, but few of the theories have been converted into practical applications. A number of models with representations allowing imprecise statements have been suggested. Some of them use standard probability theory while others contain some specialised formalism. Most of them focus more on representation and probabilistic inference, and less on evaluation, see [7], [8], [9], [10], and [11] for examples. DecideIT has its roots in research on imprecise probabilities but has extended the theory into interval values (utilities) and interval multi-criteria weights. The tool also contains an approach embedding the basic sensitivity analysis into the representation. However, beside the embedded sensitivity analysis a second form of sensitivity analysis, tornado diagrams, are important in order to point out the most critical probabilities, values, or weights. This is also valuable when there is a need to allocate further investigation resources in order to discriminate between the alternatives, yielding insights into which of the parameters to put more effort into investigating.

II. THE DECIDEIT SOFTWARE

In DecideIT, see Fig. 1, the decision-maker can take advantage of both decision trees and multi-criteria models allowing the use of imprecise input in the form of interval and comparative statements.

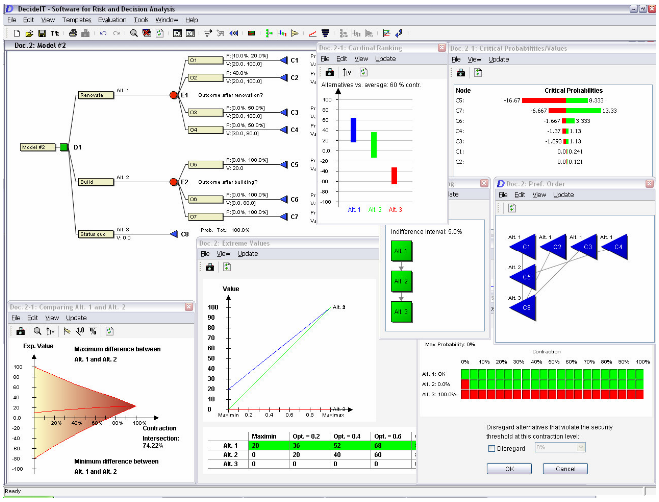

Fig. 1. DecideIT screenshot

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In a multi-criteria model, each end node represents one criterion, which can be modelled in two ways; either directly in the multi-criteria model for criteria not containing probabilistic uncertainty, or in its own decision tree, which then are joined together in a criteria hierarchy. The multi-criteria decision problem can then be analyzed from a one-criterion perspective, or from an all-criteria perspective. The software consists of a graphical user interface, taking advantage of a set of algorithms capable of handling imprecise probabilities, values, and criteria weights, providing a good display of the decision problem and the evaluation results.

III. Representation

Models and tools for decision analysis are categorized depending on the nature of the information they handle. If the information, such as probabilities and values (utilities), is in the form of fixed numbers, the models are *zero-order* models. This includes statements such as "the probability of A occurring is 23%" and "the value of consequence B is USD 2.61 million." If the information is in the form of upper and lower bounds (or intervals), the models are *first-order* models. This includes statements such as "the probability of A occurring is between 20% and 30%" and "the value of consequence B is between USD 2.5 and 3 million." Finally, if the information is in the form of distributions of belief over intervals, the models are *second-order* models. The DecideIT tool utilizes modelling techniques from first- and second-order theories.

In first-order decision modelling, the intervals (lower and upper bounds) could be represented by weak inequalities which form constraints on the possible solutions to the set of inequalities (called bases below). A statement such as "the probability of A occurring is between 15% and 25%" is translated into a pair of inequalities $p(A) \geq 15\%$ and $p(A) \leq 25\%$. Each consequence is represented by a variable. A *qualitative* (comparative) statement compares the probabilities (or values) of two consequences occurring with one another, such as "the event C is more likely to occur than D". Those statements also translate into inequalities like $p(C) \geq p(D)$ [12]. The collection of all inequalities (of the same kind) is called a constraint set. There is one each for probabilities, values (utilities), and criteria weights. A constraint set is *consistent* **iff** the system of weak inequalities has a solution. Thus, a consistent constraint set is a set where the constraints are not contradictory.

In the sequel, a generic consistent constraint set X is used. It consists of constraints (inequalities) on the variables x_i, $i \in I$, where I is the index set. For the presentation, it is easiest to think of a one-level decision tree with n nodes and indices $i = 1,\ldots,n$. This maps to consequences c_i. But the implementation in DecideIT handles trees of several levels of depth. A consistent constraint set has definite outer boundaries, i.e. delimiters beyond which there are no consistent points. The *orthogonal hull* describes which parts of the space are incompatible with the constraint set. First, we introduce a shorthand notation for the max- and min-operators.

Definition: Given a consistent constraint set X in variables $\{x_i\}_{i \in I}$ and a function f,
$^{X}\max(f(x)) =_{def} \sup(a \mid \{f(x) > a\} \cup X \text{ is consistent})$.
Similarly,
$^{X}\min(f(x)) =_{def} \inf(a \mid \{f(x) < a\} \cup X \text{ is consistent})$.

The orthogonal hull can now be defined in terms of minima and maxima.

Definition: Given a consistent constraint set X in $\{x_i\}_{i \in I}$, the set of pairs $\{\langle {}^{X}\min(x_i), {}^{X}\max(x_i)\rangle\}_{i \in I}$ is the *orthogonal hull* of the set and is denoted $\langle {}^{X}\min(x_i), {}^{X}\max(x_i)\rangle$.

For convexity reasons, the entire interval between those extremal points is feasible. If the base is consistent, the orthogonal hull can always be determined.

Example 1: Consider the following constraint set P in variables $\{p_i\}_{i \in \{1,2,3,4\}}$:
$p_1 \in [0.20, 0.40]$ $\quad p_1 - p_2 \in [0.10, 0.30]$
$p_2 \in [0.20, 0.45]$ $\quad p_1 - p_3 \in [0.10, 0.40]$
$p_3 \in [0.15, 0.30]$ $\quad p_3 - p_4 \in [-0.10, 0.10]$
$p_4 \in [0.25, 0.40]$ $\quad p_1 + p_2 + p_3 + p_4 = 1$.
A solution vector to the system of inequalities that P represents is (0.30, 0.20, 0.20, 0.30) and thus the constraint set P is consistent.
Next, $^{P}\max(p_1) = 0.40$ and $^{P}\min(p_1) = 0.30$. Repeating for the other three p_i's yields the following orthogonal hull:
$\{\langle 0.30,0.40\rangle, \langle 0.20,0.25\rangle, \langle 0.15,0.225\rangle, \langle 0.25,0.30\rangle\}$.
Compared to the range constraints in the base there are some differences. For example, the upper bound of p_1 has been cut from 0.70 to 0.40, see Fig. 2.

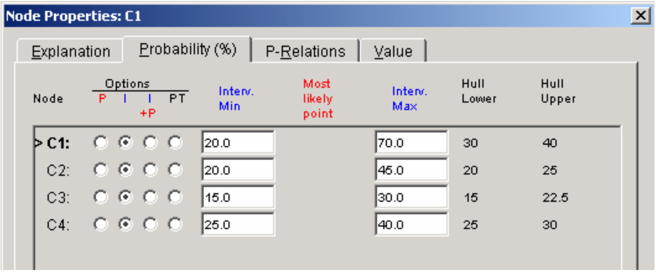

Fig. 2. Probability input and hull calculations

All consistent points (i.e. feasible solution vectors) are found inside the orthogonal hull.

Definition: Given a constraint set X in $\{x_i\}_{i \in I}$ and the orthogonal hull $H = \langle a_i, b_i\rangle$ of X, a *consistent point* is a solution vector $(r_1,\ldots,r_n)$ with $a_i \leq r_i \leq b_i$, $\forall i \in I$.

The *focal point* is a consistent point with a special property. It is the point considered the most likely point (favoured point) by the decision-maker. It is found in one of two ways, or a combination of the two ways.

a) In case the decision-maker has an expressed opinion, it is used as the focal point.

b) Otherwise, the focal point is the mass point relative to an implied second-order distribution over the intervals.

Second-order decision theory devises centre points in each dimension (each variable) which corresponds to components of the mass point of the resulting joint distribution [13]. This is in analogy with mass points of physical bodies, where the mass point is a good representative for the entire body for many modelling purposes. In Fig. 3, an interpretation of the orthogonal hull and the focal point in terms of belief is shown.

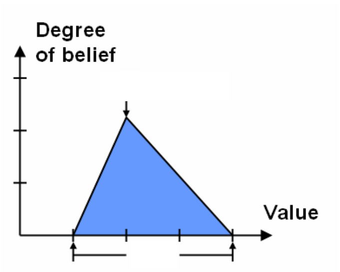

Fig. 3. The orthogonal hull and the focal point

Further, from second-order theory we know that distributions tend towards the mass point, introducing skew into the distributions following multiplications and additions of distribution components. This effect is called the warp effect [14]. To compensate for the effect in DecideIT, skew is represented in the model. A metric should be used that complies with the decision-maker's understanding of the decision problem.

Definition : Given two vectors **a** and **b**, a *distance function* d is a function that satisfies

(i a) $d(\mathbf{a},\mathbf{b}) > 0$ if $\mathbf{a} \neq \mathbf{b}$

(i b) $d(\mathbf{a},\mathbf{a}) = 0$

(ii) $d(\mathbf{a},\mathbf{b}) = d(\mathbf{b},\mathbf{a})$

(iii) $d(\mathbf{a},\mathbf{b}) \leq d(\mathbf{a},\mathbf{c}) + d(\mathbf{c},\mathbf{b})$ for all $\mathbf{c} > 0$.

For the definition to be meaningful in this context, the distance function must be reasonable, even though this does not follow directly from the definition.

Definition: Given two vectors **a** and **b**, a *hull distance function* d is a distance function where $d(\mathbf{a},\mathbf{b}) = (\Sigma a_i - \Sigma b_i)^2$.

In general, the focal point does not need to coincide with the orthogonal hull midpoint. In that case, a set of constraints is said to be skewed, and the concept of skew is introduced to describe this.

Definition: Given a constraint set X in $\{x_i\}_{i \in I}$, two real vectors $\mathbf{a} = (a_1,\ldots,a_n)$ and $\mathbf{b} = (b_1,\ldots,b_n)$ of the orthogonal hull $\langle a_i, b_i \rangle$ of X, a hull distance function d and a focal point **r**. The *skew* s of the base X with respect to **r** is $s = \frac{d(\mathbf{r},a) - d(\mathbf{r},b)}{d(\mathbf{a},\mathbf{b})}$.

Thus, a frame is asymmetric (or skewed) if the focal point does not coincide with the orthogonal hull midpoint. It can be seen from the definition that s = 0 if a set of constraints is symmetric, s > 0 if the set is skewed rightwards (toward higher values), and s < 0 if the set is skewed leftwards (toward lower values).

When a set of constraints is skewed, there exists a way of aiding the decision-maker in avoiding this asymmetry by using the symmetric hull instead of the orthogonal. It is constructed by adjusting the interval of each hull dimension from one side so that the focal point and midpoint coincide.

Definition: Given a constraint set X in $\{x_i\}_{i \in I}$, the orthogonal hull $\langle a_i, b_i \rangle$ of X, and a focal point $(r_1,\ldots,r_n)$. Let $d_i = \min(r_i - a_i, b_i - r_i)$, $\forall i \in I$. The *symmetric hull* is $\langle r_i - d_i, r_i + d_i \rangle$.

The generic term *hull* is used in the sequel, meaning the type of hull (orthogonal or symmetric) used in a particular decision situation.

Note: If the symmetric hull coincides with the orthogonal hull, then the skew is zero. This follows from $d(\mathbf{r},\mathbf{a}) - d(\mathbf{r},\mathbf{b}) = 0$ if the midpoint $\frac{a_i + b_i}{2}$ for each index $i \in I$ is equal to the component r_i of the focal point **r**.

Example 2: Consider a decision situation involving two consequence sets C_1 and C_2. C_1 has ten consequences while C_2 has only one. There are no probability constraints and the probability focal point for C_1 is 0.1 for each consequence. While the orthogonal hull covers all consistent probability assignments, i.e. [0, 1] for each probability variable, the symmetric hull is symmetric around the focal point, i.e. [0, 0.2]. Let the value base contain:

$v_{11} \in [1.00, 1.00]$
$v_{1i} \in [0.00, 0.00]$ for i = 2,...,10
$v_{21} \in [0.10, 0.20]$

Using the orthogonal hull, the consequence set C_1 is the preferred one. This is counter to many decision-makers' appreciation of the example. Using the symmetric hull, on the other hand, the consequence set C_2 is the preferred one and the result is stable. This result is perceived to be more indicative by many decision-makers.

There are two levels of possible richness in the representation. Either the constraint sets constitute all information given by the decision-maker, or it is supplemented by cores. Thus, a base (the entity collecting all information) consists of a constraint set possibly together with a core. Constraints and core intervals have different roles in specifying a decision situation. The constraints represent "negative" information, which vectors are not part of the solution sets. The contents of constraints specify which ranges are infeasible by excluding them from the solutions. This is in contrast to core intervals, which represent "positive" information in the sense that the decision-maker enters information about sub-intervals that are felt to be the most central ones and that no further discrimination is desirable or even possible within those ranges. This has the effect of pointing out a sub-interval instead of only a focal point as being the most central (i.e. most believed in).

Definition: Given a constraint set X in $\{x_i\}_{i\in I}$ and the orthogonal hull $\langle a_i,b_i\rangle$ of X, a *core interval* of x_i is an interval $[c_i, d_i]$ such that $a_i \le c_i \le d_i \le b_i$. A *core* $|c_i, d_i|$ of $\{x_i\}_{i\in I}$ is a set of core intervals $\{[c_i, d_i]\}_{i\in I}$, one for each x_i.

As for constraint sets, the core might not be meaningful in the sense that it may contain no possible variable assignments able to satisfy all the inequalities. This is quite similar to the concept of consistency for constraint sets, but for core intervals, the requirement is slightly different. It is required that the focal point is contained within the core.

Definition: Given a consistent constraint set X in $\{x_i\}_{i\in I}$ and a focal point $\mathbf{r} = (r_1,\ldots,r_n)$, the core $|c_i,d_i|$ of $\{x_i\}_{i\in I}$ is *permitted with respect to* ***r*** **iff** $c_i \le r_i \le d_i, \forall i\in I$.

Example 3: Reconsider the constraint set P from Example 1. Recall that the constraint set is

$p_1 \in [0.20, 0.40]$ $p_3 \in [0.15, 0.30]$

$p_2 \in [0.20, 0.45]$ $p_4 \in [0.25, 0.40]$

and that $\mathbf{r} = (r_1,\ldots,r_4) = (0.32, 0.25, 0.17, 0.26)$ is the focal point. Let the core suggested by the decision-maker be

$p_1 \in [0.30, 0.35]$ $p_3 \in [0.15, 0.20]$

$p_2 \in [0.22, 0.25]$ $p_4 \in [0.25, 0.28]$.

Now r_1 is contained in the core interval of p_1, and the same is true for the other three p_i's. Thus the core is permitted. The interpretation of, for example, the information about p_1 is that, according to the decision-maker, the value of p_1 is not below 0.20 and not above 0.40. In addition, the most plausible values for p_1 are between 0.30 and 0.35.

Formally, there is always a core, even if it is left unspecified in case of core statements not being employed by the decision-maker. Then, the core is simply the single focal point. Thus, a base is a collection of constraints and the core that belongs to the variables in the set. The idea with a base is to represent a class of functions over a finite, discrete set of consequences.

Definition: Given a set $\{x_i\}_{i\in I}$ of variables and a focal point **r**, a *base* X in $\{x_i\}_{i\in I}$ consists of a constraint set X_C in $\{x_i\}_{i\in I}$ and a core X_K of $\{x_i\}_{i\in I}$. The base X is *consistent* if X_C is consistent and X_K is permitted with respect to **r**.

A probability base contains a collection of probability statements in the form of constraints and a core.

Definition: Given a set $\{C_{ik}\}_{k\in K}$ of disjoint and exhaustive consequences, a base P in $\{p_{ik}\}_{k\in K}$, $K = \{1,\ldots,m_i\}$, and a discrete, finite probability mass function $\prod:C\rightarrow[0,1]$ over $\{C_{ik}\}$. Let p_{ik} denote the function value $\prod(C_{ik})$. $\prod$ obeys the standard probability axioms, and thus $p_{ik} \in [0,1]$ and $\sum_k p_{ik} = 1$ are default constraints in the constraint set P_C.
P is called a *probability base*.

Thus, a probability base represents a set of discrete probability distributions. The core P_K can be thought of as an attempt to estimate a class of mass functions by estimating the individual discrete function values.

Requirements similar to those for probability variables can be found for value variables. There are apparent similarities between probability and value statements but there are differences as well. The normalization ($\sum_k p_{ik} = 1$) requires the probability variables of a set of exhaustive and mutually exclusive consequences to sum to one. No such dimension reducing constraint exists for the value variables.

Definition: Given a set $\{C_{ik}\}_{k\in K}$ of disjoint and exhaustive consequences, a base V in $\{v_{ik}\}_{k\in K}$, $K = \{1,\ldots,m_i\}$, and a discrete, finite value function $\Omega:C \rightarrow [0, 1]$. Let v_{ik} denote the function value $\Omega(C_{ik})$. Because of the range of Ω, $v_{ik} \in [0, 1]$ are default constraints in the constraint set V_C.
V is called a *value base*.

Using the concepts of consequence, constraint, core, and base, DecideIT models the decision-maker's situation in a *decision frame*.

Definition: Given a decision situation with m alternatives $(A_1,\ldots,A_m)$, each with m_i consequences, and statements about the probabilities and values of those consequences. A *decision frame* is a structure $\langle C, P, V\rangle = \langle\{\{C_{ik}\}_{m_i}\}_m, P, V\rangle$ containing the following representation of the situation:

- For each alternative A_i the corresponding consequence set $\{C_{ik}\}_{k\in K_i}$ for $K_i = \{1,\ldots,m_i\}$.
- A probability base P containing all probability statements in the form of constraints and a core.
- A value base V containing all value statements in the form of constraints and a core.

Thus, each alternative is represented by its consequence set. Since multi-criteria weights are mathematically similar to probabilities, they are not included in the discussion on representation. In DecideIT, they are stored in a third base in the decision frame. The frame is stored in data structures in the tool.

IV. Embedded Sensitivity Analyses

Since the application takes imprecise input, the evaluation of decision alternatives will also reflect that uncertainty, thus providing the decision-maker with *expected value intervals* that might overlap each other, i.e. no single alternative might dominate all the others. Then, DecideIT examines in which parts of the intervals we can obtain a dominating alternative, using an embedded sensitivity analysis. The probabilities, values, and weights are therefore subject to an embedded sensitivity analysis as well as an explicit analysis, shown as a tornado diagram of the influences of the probabilities, values, and weights on the evaluation result. The embedded sensitivity analysis examines in which parts of the intervals we can obtain a dominating alternative. This is done by introducing contraction levels, contracting the intervals, thus moving the interval boundaries closer towards the focal point. The use of contraction levels is an embedded sensitivity analysis where the decision-maker gains a better understanding of the stability of the result and how important the interval boundary points are.

Definition: Given a decision frame $\langle\{C_i\}_m, P, V\rangle$, the *expected value* $E(C_i)$ of a consequence set $C_i = \{C_{ik}\}_{m_i}$ is the sum $\sum_k p_{ik}\cdot v_{ik}$ over all consequences C_{ik} in the set. Further, δ_{ij} denotes the expression $E(C_i) - E(C_j) = \sum_k p_{ik}\cdot v_{ik} - \sum_k p_{jk}\cdot v_{jk}$ over all consequences in the consequence sets C_i and C_j.

In order to assess the overlap between expected value intervals of the alternatives after the evaluation, sensitivity analyses are performed. They are of two kinds, embedded and explicit. An embedded analysis uses built-in properties of the representation to study the stabilities of the results obtained. The *hull cut* is a generalized sensitivity analysis for this purpose. It is reasonable to consider values near the boundaries of the intervals in a constraint set to be less believable than more central values, due to interval constraints being deliberately imprecise (see Fig. 3). If no core is present, the bases are contracted from the hull boundaries toward the focal point.

Definition: Given a decision frame with a base X with the variables $x_1,\ldots,x_n$, $\pi \in [0, 1]$ is a real number, and $\{\pi_i \in [0, 1] : i = 1,\ldots,n\}$ is a set of real numbers. $[a_i, b_i]$ is the interval corresponding to the variable x_i in the solution set of the system of constraints, and $(k_1,\ldots,k_n)$ is a focal point in X. A *π-contraction* of X is to add the interval statements $\{x_i \in [a_i+\pi\cdot\pi_i\cdot(k_i-a_i), b_i-\pi\cdot\pi_i\cdot(b_i-k_i)] : i = 1,\ldots,n\}$ to the base X.

If a core is present, on the other hand, it represents the most believable estimates. It is then desirable to be able to study the bases with differing cut directions, i.e. studying increments or decrements of the core. Thus, if the core itself is not enough to yield desired evaluation results, it can be further cut towards the focal point with varying degrees of contraction.

Definition: Given a base X in $\{x_i\}_{i\in I}$, a set of real numbers $\{a_i,b_i\}_{i\in I}$, a core $|c_i, d_i|$ of $\{x_i\}_{i\in I}$, and a real number $\pi \in [0, 1]$, a *τ-cut* of X is to replace the core by $[c_i+\tau\cdot(a_i-c_i), d_i+\tau\cdot(b_i-d_i)]$. If the set $\{a_i, b_i\}_{i\in I}$ is the hull $\langle a_i, b_i\rangle$ then it is called a *τ-expansion* of X. If $(r_1,\ldots,r_n)$ is a focal point and $a_i = b_i = r_i$, then it is called a *τ-contraction* of X.

As the bases in the frame are contracted, the initially partially ordered alternatives (consequence sets) gradually shift into a total order (except when the focal points coincide). The contraction is carried out in a large number of dimensions at the same time, most often all, making it an embedded general analysis yielding a good overview of the stability of obtained evaluation results.

V. Explicit Sensitivity Analyses

If the evaluation and subsequent contractions do not point out a single preferred alternative, the next step, to further discriminate the alternatives, could be to gather more information regarding the decision situation. In order to guide the allocation of analysis resources for further information gathering in an efficient way, the variables having the greatest impact on the expected value should be identified. Even if the evaluation points out a preferred alternative, it is important to gain an understanding of the stability of the result, i.e. its sensitivity to changes in the information supplied.

An established way of displaying a one-way sensitivity analysis of several variables in the same output window is by a *tornado diagram*. By showing the sensitivity ranges as bars and sorting the widest range on top, the resulting picture resembles a tornado; see Fig. 4 for an example of a value tornado. The output can be interpreted such that value intervals having greater impact on the expected value are more critical, and information related to these consequences is important to investigate further.

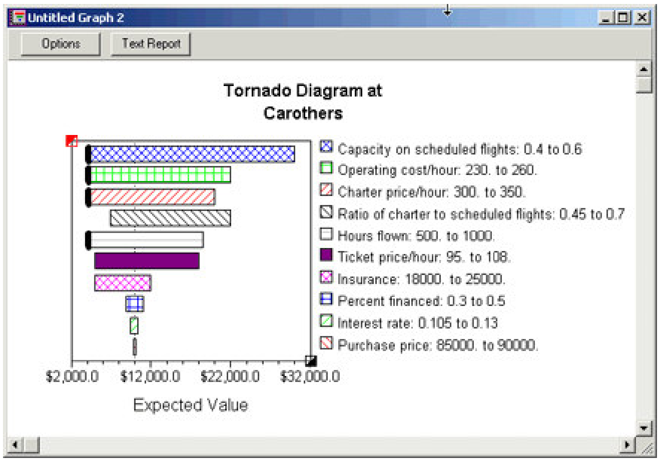

Fig. 4. Tornado diagram (TreeAge Pro software)

An advantage of an interval approach is that the basis for calculating the critical values is already present, i.e. the interval width is given by the decision-maker already from the beginning. The problem, however, is that the parameters may include dependencies in the form of comparative value statements. Since we are aiming to provide the user with critical probabilities and critical weights as well, the normalization constraint provides a further obstacle. Varying each variable in isolation would yield incorrect results due to comparative statements and normalization constraints.

Another problem is that a change in the probability of one node affects all the children of this node, which in turn affects the evaluation result. Similar problems are inherent when dealing with weights or comparative relations. Varying the value of one consequence might affect some other value due to value relations between them.

VI. Interval Tornado Diagrams

In a zero-order decision problem (i.e. no intervals), a tornado consists simply of varying each of the variables, one at a time, either to their specified "best" and "worst" values or by a percentage, for example ±10%. In a first-order (interval) decision tool, there is no obvious counterpart to the zero-order tornado diagram. In an interval evaluation, there is no single output expected value but rather an expected value interval within which the expected value falls when the input variables are kept within their interval input ranges and being consistent with the constraints. The meaning of an input interval constraint (or, to be more precise, the hull) is the upper and lower bounds of the numbers that the variable could assume in the decision problem. But changing each variable within the interval, even letting it assume its lower and upper bounds, will not yield any new information since all possible consistent assignments of variables have already been considered in the expected value range by definition.

To gain an overview of the sensitivity of the results of an interval decision evaluation, another approach is required in order to reflect the sensitivity of the output expected value to changes in the input variables. This is achieved by studying the sensitivity of the best representative of the expected value (the focal point).

Let the reference expected value (EU_0) of an alternative be the expected value of the focal point. EU_0 is considered the best single representative for the expected value interval of the alternative. Since it is calculated using the same expected value formula as all points in the expected value interval, it is a reasonable candidate for an anchor point in a sensitivity analysis. The analysis then studies the effects on the focal point of varying each variable, instantiating it with its lower and upper bounds, respectively. This way, the positive and negative impacts are obtained for each variable.

Each variable influences the focal point when assigned its lower and upper bounds, not least because of dependencies between variables in an interval specified decision problem, such as comparisons (e.g. $v_i > v_k$) and normalisations ($\Sigma_j\, p_j = 1$). The focal point is recomputed and the impact of the disturbance is then measured as the change in expected value for the recomputed focal point compared to the reference.

Procedure: Let Φ be the focal point of an alternative. The value $v_i = V(c_i)$ of each consequence c_i is varied in turn, one at a time.
i) When v_i is assigned its maximum (upper hull bound), the new focal point calculated is Φ_i^+. Let EU_i^+ be the expected value obtained from Φ_i^+.
ii) When v_i is assigned its minimum (lower hull bound), the new focal point calculated is Φ_i^-. Let EU_i^- be the expected value obtained from Φ_i^-.
iii) The positive impact on EU_0 by varying v_i is now $EU_i^+ - EU_0$ and the negative impact on EU_0 is similarly $EU_i^- - EU_0$. Thus, the resulting impact range of each variable has the form of an interval $[EU_i^- - EU_0, EU_i^+ - EU_0]$.

Hence, collecting the intervals for each decision variable for each alternative is the operationalisation of degree of impact in an interval tornado analysis and the resulting impact intervals are displayed in a tornado diagram, sorted in decreasing interval width analogous to the zero-order display. Using the core in lieu of the hull in the same procedure, another diagram (the core tornado) is obtained.

Traditionally, in zero-order models, only value tornados are presented. In DecideIT, tornado diagrams are also created for the critical probabilities or the critical criteria weights, where the positive and negative impacts, by varying probability and weight variables, are calculated in a similar manner. These first-order tornado diagrams, yielding critical values and critical probabilities for decision trees, and critical weights and weighted critical values for multi-criteria models, have been implemented in the software *Decide*IT, see Fig. 5.

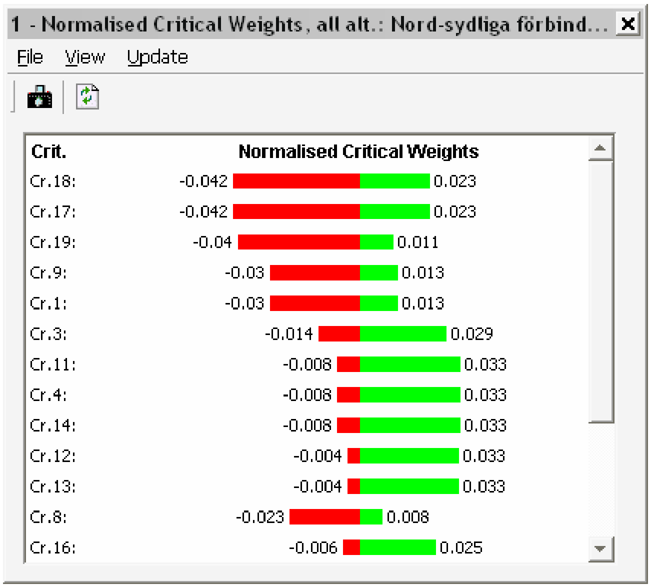

Fig. 5. Tornado diagram (DecideIT software)

A red coloured (dark) bar indicates that the expected value is influenced in a negative way, and a green coloured (light) bar indicates a positive influence on the expected value.

VII. Summary and Concluding Remarks

This paper discusses sensitivity analyses in interval decision models. Two different kinds are treated, embedded analyses and tornado diagrams. The embedded analysis is a generalized sensitivity analysis carried out in many dimensions at the same time. An interval tornado diagram can be used for displaying the individual sensitivities of probabilities and values in decision trees, and of weights and weighted values in multi-criteria models. The difference compared to traditional tornado diagrams is that interval ranges have already been specified by the decision-maker and taken account of in the resulting expected value interval. Furthermore, varying one variable at a time may not be possible due to dependencies between variables derived from, e.g., comparative relations and probability or weight normalisation constraints. We present a solution circumventing this problem, which has also been implemented in the software DecideIT. Using both kinds of sensitivity analyses in conjunction, they provide a thorough picture of the stability of evaluation results obtained.

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