On the extension of the Painlevé property to difference equations

M J Ablowitz†, R Halburd†‡ and B Herbst§

† Department of Applied Mathematics, University of Colorado, Boulder, CO 80309-526, USA
‡ Department of Mathematical Sciences, Loughborough University, Loughborough,
Leicestershire LE11 3TU, UK
§ Department of Applied Mathematics, University of Stellenbosch, Stellenbosch 7602,
South Africa

Abstract. It is well known that the integrability (solvability) of a differential equation is related to
the singularity structure of its solutions in the complex domain—an observation that lies behind the
Painlevé test. A number of ways of extending this philosophy to discrete equations are explored.
First, following the classical work of Julia, Birkhoff and others, a natural interpretation of these
equations in the complex domain as difference or delay equations is described and it is noted
that arbitrary periodic functions play an analogous role for difference equations to that played by
arbitrary constants in the solution of differential equations. These periodic functions can produce
spurious branching in solutions and are factored out of the analysis which concentrates on branching
from other sources. Second, examples and theorems from the theory of difference equations are
presented which show that, modulo these periodic functions, solutions of a large class of difference
equations are meromorphic, regardless of their integrability. It is argued that the integrability of
many difference equations is related to the structure of their solutions at infinity in the complex
plane and that Nevanlinna theory provides many of the concepts necessary to detect integrability in
a large class of equations. A perturbative method is then constructed and used to develop series in z
and the derivative of $\log \Gamma(z)$, where z is the independent variable of the difference equation. This
method provides an analogue of the series developed in the Painlevé test for differential equations.
Finally, the implications of these observations are discussed for two tests which have been studied
in the literature regarding the integrability of discrete equations.

AMS classification scheme numbers: 30D35, 39A10, 39A12

1. Introduction

An ordinary differential equation (ODE) is said to possess the Painlevé property if all of its
solutions are single-valued about all movable singularities. A singularity is said to be movable
if it varies with initial conditions, i.e. it is not a singularity of the equation itself. The observation
that this property is related to integrability has proved most fruitful, cf [4–7, 27]. It is widely
believed that all ODEs that possess the Painlevé property are integrable (i.e. solvable) either
explicitly or via a related linear problem. We also remark, however, that certain equations
solvable via an evolving monodromy problem do not possess the Painlevé property [1–3, 10].

Painlevé and his colleagues identified all second-order ODEs within a large class that
possess the Painlevé property [16, 17, 30, 31]. All of these equations were integrated in terms

of known functions except six which are now known as the Painlevé equations (P_I–P_{VI}). The first two of these are

P_I $$\frac{\mathrm{d}^2 y}{\mathrm{d}z^2} = 6y^2 + z \tag{1}$$

P_{II} $$\frac{\mathrm{d}^2 y}{\mathrm{d}z^2} = 2y^3 + zy + \alpha \tag{2}$$

where α is a constant.

The six Painlevé equations are integrable (solvable) via associated (linear) isomonodromy problems [14, 23]. There are many ways in which one could discretize these equations (Euler's method, for example). Most of these will no longer possess the many special properties that the differential equations (1) and (2) possess; e.g. related isomonodromy problems and solutions that are asymptotically well behaved. However, a number of discretizations of equations (1) and (2) have been studied in the literature and are believed to be integrable (i.e. solvable via an associated linear problem) [15, 24, 26, 29, 32, 33]. For example,

$$y_{n+1} + y_n + y_{n-1} = \frac{\alpha n + \beta + \gamma(-1)^n}{y_n} + \mu$$

$$\frac{\alpha(n+1) + \beta}{y_{n+1} + y_n} + \frac{\alpha n + \beta}{y_n + y_{n-1}} = -y_n^2 + \gamma$$

$$y_{n+1} + y_{n-1} = \frac{\alpha n + \beta}{y_n} + \frac{\gamma}{y_n^2}$$

$$y_{n+1} + y_{n-1} = \frac{\alpha n + \beta}{y_n} + \gamma$$

where α, β and γ are constants, are some known well behaved discretizations of P_I and

$$y_{n+1} + y_{n-1} = \frac{(\alpha n + \beta)y_n + \gamma}{1 - y_n^2}$$

$$\frac{\alpha n + \beta}{y_n y_{n+1} + 1} + \frac{\alpha(n-1) + \beta}{y_{n-1} y_n + 1} = \frac{1}{y_n} - y_n + \alpha n + \gamma$$

are some known special discretizations of P_{II}.

A natural question arises: is it possible to determine *a priori* whether a given difference equation is integrable? In the case of differential equations, at least a partial answer is provided by the Painlevé test cf [4–6, 27]. This property is suggested by the inverse scattering transform, however, it is still not fully understood why the integrability of a real differential equation should be reflected in the singularity structure of its solutions in the complex domain.

The purpose of this paper is to analyse the behaviour of solutions of difference equations in the complex domain through a number of examples and by exploring the implications of several known theorems. This problem was studied by Nörlund, Birkhoff, Batchelder and others earlier this century for linear difference equations. Julia, Nörlund, and more recently, Kimura [25], Shimomura [37] and Yanagihara [40, 41], have derived a number of valuable results for nonlinear difference equations.

We undertake this investigation with the intention of identifying a structure or structures that can be associated with the integrability (or non-integrability) of the difference equation in the spirit of the Painlevé property for ODEs. That is, we wish to find conditions, framed in the language of complex analysis, under which a difference equation would be considered to be of Painlevé type. Some of the conclusions to which we are led are the following.

(a) Arbitrary periodic functions play an analogous role in the solutions of difference equations to that played by constants in the solutions of ODEs and arbitrary functions on characteristic functions on characteristic manifolds in solutions of PDEs. This classical observation plays a central role in our analysis.

(b) Apart from branching due to the above-mentioned arbitrary periodic functions, the solutions of many difference equations can be extended to the complex plane and are meromorphic.

(c) The integrability of a large class of difference equations is associated with the asymptotic structure of its solutions at infinity.

(d) Nevanlinna theory provides a number of important concepts and tools that can be used as detectors of equations that are integrable. In particular, the order of solutions plays a central role.

(e) Perturbative techniques which are difference versions of those used by Painlevé can be constructed and used to find expansions of solutions that are analogues of the Laurent/psi-series for differential equations.

The tests we describe are not necessary conditions for integrability, just as the Painlevé property is not necessary for the integrability of a differential equation.

Recently, two tests have been proposed as discrete analogues of the Painlevé test. The *singularity confinement* method of Grammaticos and co-workers [18, 36] involves examining solutions (sequences of iterates) of a discrete equation which pass arbitrarily close to a singular point of the equation. The associated test is applicable to a certain class of equations (reversible, strictly rational) and associates integrability with the preservation of information beyond the singularity. This test has been used successfully to find a number of integrable discrete equations. Later we discuss how our analytical observations can be related to this singularity confinement method. Also, the *discrete Painlevé property* has been introduced by Conte and Musette together with an associated test [11, 12]. We will explore some of the strengths and weaknesses of this property and how it relates to our asymptotic analysis.

This paper is structured as follows. In section 2 we discuss why difference (or delay) equations allow a natural extension of their solutions to the complex domain as opposed to discrete (or lattice) equations which are defined on the integers only. Thus we adopt the approach of Birkhoff *et al*. In this setting arbitrary functions of period h/m (where h is the step size and m is some positive integer) play a role analogous to that played by constants in the theory of differential equations. These periodic functions can introduce complicated branching into solutions of the most benign difference equations and so they must be 'factored out' or otherwise dealt with in the analysis as we are only concerned with singularities other than those generated by the periodic functions. In some sense, branching due to the arbitrary periodic functions in solutions of difference equations is analogous to branching about characteristic manifolds in solutions of partial differential equations.

After factoring out possible branching due to the periodic functions we note that the solutions of a large class of difference equations (including many chaotic examples) are meromorphic in the finite plane. This leads us to the conclusion that for a great many difference equations the natural place where the integrability (or otherwise) could be encoded is in the structure of the singularity *at infinity*. This suggests that Nevanlinna theory (value distribution theory for meromorphic functions) is an important ingredient in the study of integrable discrete or difference systems.

In section 3 we describe some fundamental concepts from Nevanlinna theory and we note that all meromorphic solutions of many classes of integrable differential and difference

equations, including the important examples of the Riccati and Painlevé equations, are of finite order. Section 4 describes, via examples, a method to obtain a series representation of a solution of a difference equation that is the natural analogue of the expansions obtained in Painlevé's analysis of differential equations. In section 5 we discuss some of the existing tests that appear in the literature in light of our results.

2. Difference equations in the complex domain

We begin by examining some simple discrete equations and the problems that arise when we try to extend their solutions to the complex domain. Consider the discrete (or lattice) equation

$$u_{n+1} - u_n = \frac{1}{(n-c)^N}. \tag{3}$$

If we specify u_0 we can determine u_n for all $n \in \mathbb{N}$, namely

$$u_n = u_0 + \sum_{i=0}^{n-1} \frac{1}{(i-c)^N}. \tag{4}$$

Note that this solution involves an unbounded number of terms and, from a calculational point of view, involves as much work to evaluate as iterating equation (3). Note that if c is a positive integer then u_n is infinite for all $n \geqslant c$. This is also related to the singularity confinement property which we discuss in section 5.1.

We wish to explore the singularity structure of solutions to equations such as (3) in the complex domain. There appears to be no compelling way to extend the solution (4) off the integers uniquely because the independent variable, n, appears as an endpoint in the sum. Since equation (3) is only defined on the integers, there is no constraint on extensions $f(z)$ of u_n to the complex domain ($f(n) = u_n, n \in \mathbb{N}$).

We find it convenient to reinterpret discrete (or lattice) equations such as (3) as difference (or delay) equations such as

$$y(z+1) - y(z) = \frac{1}{(z-c)^N} \tag{5}$$

where z is complex. The homogeneous part of equation (5) is

$$\Delta y(z) := y(z+1) - y(z) = 0.$$

The general solution of this equation is any function of period 1 (or $1/m$ for some positive integer m). In fact, it is well known that arbitrary periodic functions (which we refer to later as $\pi(z)$) play the same role in solutions of difference equations as that played by arbitrary constants in the solutions of differential equations. Since these functions can introduce any type of singularity into solutions of trivial difference equations we will factor them out of our analysis.

Recall that the gamma function is meromorphic and satisfies

$$\Gamma(z+1) = z\Gamma(z). \tag{6}$$

On taking the logarithm of equation (6) we obtain

$$\log \Gamma(z+1) - \log \Gamma(z) = \log z. \tag{7}$$

Replacing z with $z - c$ in equation (7) and differentiating N times with respect to z, we see that the general solution of equation (5) is

$$y(z) = \frac{(-1)^{N-1}}{(N-1)!} \left(\frac{d}{dz} \right)^N \log \Gamma(z - c) + \pi(z) \tag{8}$$

where π is a periodic function (as described above). It is important to note that other than the singularities due to π, the only singularities of $y(z)$ are poles. This contrasts markedly with the solution of the analogous differential equation

$$\frac{dy}{dz} = \frac{1}{(z - c)^N} \tag{9}$$

which, in the case $N = 1$, has a logarithmic singularity at $z = c$.

Since essential branching (i.e. branching that does not arise from the arbitrary periodic function) is absent in the solutions of the first-order linear difference equation (9), in stark contrast to the differential case, we are naturally led to ask whether essential branching is absent from nonlinear difference equations. Note that equations (5) and (9) have fixed singularities at $z = c$. Throughout the rest of this paper we consider only equations whose solutions have movable singularities. Although there is to date no complete theory of difference equations, the following results are of importance. These theorems were largely motivated by the work of Kimura [25] on the iteration of analytic functions.

Theorem 1 (Shimomura [37]). *For any polynomial $P(y)$, the difference equation*

$$y(z + 1) = P(y(z)) \tag{10}$$

has a non-trivial entire solution.

Theorem 2 (See Yanagihara [40]). *For any rational function $R(y)$, the difference equation*

$$y(z + 1) = R(y(z)) \tag{11}$$

has a non-trivial meromorphic solution.

Yanagihara has extended theorem 2 to a limited class of higher-order difference equations [41]. Note that since equations (10) and (11) are autonomous then the general solution of equation (10) is a particular entire function of $z - \pi(z)$ and the general solution of equation (11) is a particular meromorphic function of $z - \pi(z)$, where $\pi(z)$ is an arbitrary function of period one.

Theorem 1 says that even the quintessential chaotic difference equation, the logistic equation [8, 28],

$$y(z + 1) = \mu y(z)(1 - y(z)) \tag{12}$$

where μ is a parameter, which includes the (non-integrable) discretization of the Riccati equation (when $\mu = 1$),

$$y(z + 1) - y(z) + y^2(z) = 0 \tag{13}$$

has solutions of the form

$$y(z) = w_\mu(z - \pi(z)) \tag{14}$$

where w_μ is a non-trivial entire function (which depends on the choice of parameter μ). Viewed as a map, equation (12) is chaotic on its (non-empty) Julia set (see, for example, [13]). If all

the solutions of equation (12) have this form then the only singularities of y arise from the periodic function π.

Thus we see that the same (meromorphic) singularity structure in the finite complex plane is found throughout a wide class of difference equations which includes both integrable and non-integrable examples. The solutions of these equations show no essential branching (i.e. branching due to sources other than the arbitrary periodic functions). It appears that for this class the only way in which solutions can differ in their singularity structure is through their behaviours at infinity.

Given that the solutions of first-order rational difference equations admit no essential branching, yet solutions of most first-order rational differential equations do show branching about movable singularities, it is natural to enquire into the process whereby branching arises in the process of taking a continuous limit in which a difference equation becomes a differential equation. To this end we consider the difference equation

$$y(z+1) - y(z) = hF(y(z)) \tag{15}$$

where h is a small parameter and F is a rational function of $y(z)$. In order to take the (standard) continuum limit we perform the change of variables

$$y(z) = u(x) \qquad x = hz$$

which gives

$$\frac{u(x+h) - u(x)}{h} = F(u(x)). \tag{16}$$

We take the limit $h \to 0$ such that x is fixed to obtain

$$\frac{du}{dx} = F(u(x)). \tag{17}$$

Recall that solutions of equation (16) are single-valued (modulo the periodic function) while, for generic choices of F (for example, $F(u) = u^3$), the general solution of equation (17) is branched. Note that since $x = hz$ is held fixed as $h \to 0$, then $z \to \infty$. Hence the behaviour of solutions of equation (17) for finite values of x is related to the behaviour of solutions of equation (15) at $z = \infty$.

3. Nevanlinna theory

The relationship between the value distribution of meromorphic functions and their behaviour at infinity is the principal subject of Nevanlinna theory. In this section we describe some of the key quantities of Nevanlinna theory; namely the characteristic, order and type of a meromorphic function. We consider differential and difference equations whose meromorphic solutions are of finite order and type.

The Nevanlinna characteristic

$$T(r, f) = N(r, f) + m(r, f)$$

of a meromorphic function $f(z)$ is the sum of the *counting function $N(r, f)$* which is a measure of the number of poles within $|z| < r$ and the *proximity function $m(r, f)$* which has contributions from the segments of $|z| = r$ where $|f(z)|$ is large. Explicitly, the counting function is defined by

$$N(r, f) = \int_0^r \frac{n(t, f) - n(0, f)}{t} \, dt + n(0, f) \log r$$

where $n(r, f)$ is the number of poles (counting multiplicities) of f in $|z| \leqslant r$. The proximity function is defined by

$$m(r, f) = \frac{1}{2\pi} \int_0^{2\pi} \log^+ \left| f(r e^{i\theta}) \right| d\theta$$

where

$$\log^+ x = \max(0, \log x).$$

The *order* of a meromorphic function f is defined by

$$\sigma(f) = \limsup_{r \to \infty} \frac{\log T(r, f)}{\log r}. \tag{18}$$

If f is entire, this agrees with the definition

$$\sigma(f) = \limsup_{r \to \infty} \frac{\log \log M(r, f)}{\log r} \tag{19}$$

where

$$M(r, f) = \max_{|z| = r} |f(z)|.$$

Since the order is a natural measure of the growth of a function at infinity, we consider differential and difference equations whose meromorphic solutions are of finite order. In this case the type of f is defined to be

$$\tau(f) = \limsup_{r \to \infty} r^{-\sigma} T(r, f).$$

It is a classical result due to Malmquist (see, for example, Hille [21]) that any first-order differential equation of the form

$$\frac{dy}{dz} = P(z, y)$$

where P is polynomial in $y(z)$ and analytic in z, that admits non-rational meromorphic solutions is a special case of the Riccati equation:

$$\frac{dy}{dz} = a_0(z) + a_1(z) y + a_2(z) y^2.$$

All meromorphic solutions of the Riccati equation (where the coefficients a_j are rational functions of z) are of finite order [39]. The Weierstrass and Jacobi elliptic functions are of order two and satisfy first-order second-degree equations of the form

$$\left(\frac{dy}{dz} \right)^2 = P(y)$$

where P is a polynomial of degree at most four (see [21]). Moving on to second-order differential equations, we note that, of the six Painlevé equations P_I–P_{VI}, only P_I, P_{II} and P_{IV} have meromorphic general solutions. The order of the non-rational solutions of these equations are also finite (being $\frac{5}{2}$, 3 and 4, respectively) [9, 21]. We note that, as outlined above, Nevanlinna theory is not directly applicable to equations such as the P_{VI} because its general solution is branched about fixed singularities.

Next let us consider difference equations. For first-order difference equations, Yanagihara [40] has shown that if the equation

$$y(z+1) = \frac{a_0(z) + a_1(z)y + \cdots + a_p(z)y^p}{b_0(z) + b_1(z)y + \cdots + b_q(z)y^q} \tag{20}$$

where a_i and b_i are polynomials, admits a non-rational meromorphic solution of finite order, then $\max(p, q) = 1$ (i.e. equation (20) is the integrable difference Riccati equation). Note that this result implies that all non-rational meromorphic solutions of the logistic equation (12) are of infinite order for $\mu \neq 0$. For example, if $\mu = 4$, the general solution of equation (12) is

$$y(z) = \tfrac{1}{2} \left\{ 1 - \cosh \left(2^z \pi(z) \right) \right\}$$

where π is an arbitrary periodic function. So for $\mu = 4$, the logistic equation (12) is integrable in the sense that it is solvable but its solution grows very rapidly and is badly behaved numerically.

In the appendix we prove the following.

Theorem 3. *If the second-order difference equation*

$$y(z+1) + y(z-1) = R(z, y(z)) := \frac{a_0(z) + a_1(z)y + \cdots + a_p(z)y^p}{b_0(z) + b_1(z)y + \cdots + b_q(z)y^q} \tag{21}$$

where a_i and b_i are polynomials, admits a non-rational meromorphic solution of finite order, then $\max(p, q) = 2$.

Note that this class of equations includes all the discretizations of P_{I} and P_{II} of the form (21) that are known to have associated linear problems. This class also contains equations solvable in terms of elliptic functions [18, 22, 34]. For example, from the addition formulae for Jacobi elliptic functions it follows that the equation

$$y(z+1) + y(z-1) = \frac{2y(z)}{1 - y^2(z)} \tag{22}$$

has the general solution

$$y(z) = \pi_1(z) \, \mathrm{cn} \left[\Omega(z - \pi_2(z)), \kappa(z) \right] \tag{23}$$

where π_1 and π_2 are arbitrary periodic functions, $\mathrm{cn}(\Omega) = \mathrm{dn}^2(\Omega)$, and the modulus of the elliptic function is $\kappa(z) = i\pi_1(z) \, \mathrm{sn}(\Omega) / \mathrm{dn}(\Omega)$. For suitable π_1 and π_2 (e.g. constant), the general solution (23) is of finite order.

We remark that theorem 3 does not give a complete classification of second-order rational equations of the form (21) with respect to the order of solutions. In particular, we prove the following theorem in the appendix.

Theorem 4. *Any entire non-polynomial solution of the equation*

$$y(z+1) + y(z-1) = a(z) + b(z)y(z) + cy^2(z) \tag{24}$$

where a and b are polynomials and $c \neq 0$ is a constant, is of infinite order.

The proof of theorem 3 readily extends to a proof of the following.

Theorem 5. *If the second-order difference equation*

$$y(z+1)y(z-1) = R(z, y(z)) := \frac{a_0(z) + a_1(z)y + \cdots + a_p(z)y^p}{b_0(z) + b_1(z)y + \cdots + b_q(z)y^q} \tag{25}$$

where the a_i, b_i are polynomials, admits a non-rational meromorphic solution of finite order, then $\max(p, q) = 2$.

The class of equations of the form (25) includes integrable difference versions of the third and fourth Painlevé equations [35]. If $R(z, y(z)) = y^2(z)$ then equation (25) has the general solution $y(z) = \pi_1(z)[\pi_2(z)]^z$, where π_1 and π_2 are periodic. Hence, for π_1 and π_2 constant, y has finite order. Note that theorems 3 and 5 do not single out only those equations that one would consider to be of Painlevé type.

Throughout this section we have only considered equations with polynomial or rational dependence on the independent variable. In general, the rate of growth of a meromorphic solution of differential and difference equations depends on the rate of growth of coefficients in the equation.

4. Series expansions

In this section we consider a particular perturbation of equation (22), namely

$$y(z+1) + y(z-1) = \frac{(2 + \epsilon f(z))y(z)}{1 - y^2(z)} \tag{26}$$

where ϵ is a small parameter and f is an analytic function. The techniques developed here can be generalized to analyse equation (21) with $\max(p, q) = 2$. It is sufficient for our purposes to consider this example. We will develop a series expansion for solutions of equation (26). In general, this expansion will be a Laurent series in $z - z_0$ and the digamma function $\psi(z - z_0) = \mathrm{d} \log \Gamma(z - z_0)/\mathrm{d}z$. In these expansions ψ plays a role analogous to the role played by log in the expansions of solutions of differential equations. We will see that demanding that these expansions have no ψ dependence leads to the known integrable cases of equation (26).

We begin by substituting the expansion

$$y(z) = y_0(z) + \epsilon y_1(z) + \epsilon^2 y_2(z) + \cdots + \epsilon^n y_n(z) + \cdots \tag{27}$$

into equation (26). Equating the terms independent of ϵ shows that y_0 satisfies equation (22). The terms proportional to ϵ gives

$$y_1(z+1) - 2\frac{(1 + y_0^2(z))}{(1 - y_0^2(z))^2}y_1(z) + y_1(z-1) = \frac{y_0(z)f(z)}{1 - y_0^2(z)}. \tag{28}$$

Differentiating equation (22) with respect to z shows that $\mathrm{d}y_0/\mathrm{d}z$ solves the homogeneous part of equation (28). Following the method of variation of parameters, we substitute

$$y_1(z) = u(z)y_0'(z) \tag{29}$$

into equation (28), which yields

$$u(z) - u(z-1) = \frac{v(z)}{y_0'(z)y_0'(z-1)} \tag{30}$$

where v satisfies

$$v(z+1) - v(z) = \frac{y_0(z)y_0'(z)f(z)}{1 - y_0^2(z)}. \tag{31}$$

To illustrate our method in a concrete way, we choose the simplest non-trivial solution of equation (22),

$$y_0(z) = \frac{1}{\xi} \qquad \text{where} \quad \xi = z - z_0 \tag{32}$$

and z_0 is arbitrary. We will later drop this restriction and illustrate a method that uses properties of the general solution (23) of equation (22).

Substituting (32) into equation (31) gives

$$v(z + 1) - v(z) = \frac{f(\xi + z_0)}{(\xi - 1)\xi(\xi + 1)}$$

$$= A(\xi) + \frac{1}{2} \left\{ \frac{f(z_0 + 1)}{\xi - 1} - \frac{2f(z_0)}{\xi} + \frac{f(z_0 - 1)}{\xi + 1} \right\} \qquad (33)$$

where A is analytic. Equation (33) gives

$$v(z) = Q(\xi) + \tfrac{1}{2} \{ f(z_0 + 1)\psi(\xi - 1) - 2f(z_0)\psi(\xi) + f(z_0 - 1)\psi(\xi + 1) \} + \pi_1(z).$$

Hence, using the fact that the digamma function, ψ, satisfies

$$\psi(\xi + 1) - \psi(\xi) = \frac{1}{\xi}$$

we have

$$v(z) = B(\xi) + \frac{f(z_0 - 1)}{2\xi} - \frac{f(z_0 + 1)}{2(\xi - 1)} + c\psi(\xi) + \pi_1(\xi) \qquad (34)$$

where π_1 is an arbitrary periodic function, B is analytic and

$$c(z_0) = \tfrac{1}{2} \{ f(z_0 - 1) - 2f(z_0) + f(z_0 + 1) \}.$$

Note that if $c(z_0) \neq 0$ then from equations (29)–(34), y_1 will contain an explicit (linear) dependence on ψ. In fact, if $c(z_0) \neq 0$ then the expansion (27) for y will contain infinitely many ψ-terms. If $c(z_0) = 0$ for all z_0 then $f(z) = az + b$, where a and b are periodic, which we choose to be constant, and equation (26) becomes

$$y(z + 1) + y(z - 1) = \frac{(2 + \epsilon(a + bz))y(z)}{1 - y^2(z)}.$$

This is a known integrable difference equation which, for $b \neq 0$, has a continuum limit to the second Painlevé equation (2) with $\alpha = 0$.

Next we discuss how to derive this result by perturbing off the general elliptic function solution (23) where for convenience we take π_1 and π_2 to be constant. We begin by locating the singular points of the right-hand side of equation (31). Since y_0 is a (scaled) Jacobi elliptic function, it takes on every value (including ∞) exactly once in each period parallelogram P. There are at most three points $z_\pm$ and z_0 in P where the right-hand side of equation (31) can be singular. These points are defined by $y_0(z_\pm) = \pm 1$ and $y_0(z_0) = \infty$ (y has a simple pole at z_0). On choosing initial conditions such that P is sufficiently large and balancing the poles on both sides of equation (31) we find that either $z_\pm = z_0 \pm 1$ or $z_\pm = z_0 \mp 1$. Without loss of generality we assume $z_\pm = z_0 \pm 1$ and hence y has the expansions

$$y_0(z) = \begin{cases} \dfrac{c_0}{z - z_0} + \mathrm{O}(1) & z \sim z_0 \\[2mm] \pm 1 + c_\pm(z - [z_0 \pm 1]) + \mathrm{O}\left((z - [z_0 \pm 1])^2\right) & z \sim z_0 \pm 1 \end{cases}$$

for some constants c_0 and $c_\pm$. These expansions allow us to calculate residues of the poles of the right-hand side of equation (31) at z_0 and $z_0 \pm 1$. Hence equation (31) has the form (33) where now A is analytic throughout P. The argument then proceeds as above.

In this section we have described a method for difference equations which is analogous to that used by Painlevé in the context of branching of solutions of differential equations (see, e.g. Ince). The ψ function plays the role of log in Painlevé's analysis. We note that ψ is a meromorphic function of finite order but infinite type. Difference equations whose solutions are of finite order and type (up to the arbitrary periodic functions) are natural candidates to be further investigated regarding their integrability.

5. Other approaches

In this section we discuss two properties and their associated tests that have been introduced recently. These are the *singularity confinement property* and the *discrete Painlevé property*.

5.1. The singularity confinement property

Grammaticos and Ramani *et al* [18, 19, 35] have proposed a test for the integrability of discrete equations that involves examining solutions (lists of iterates) that pass arbitrarily close to singularities of the equation. The basic idea is that for a generic (i.e. non-integrable) rational equation, if initial conditions are chosen so that the next iterate becomes infinite, then all future iterates will be infinite (such iterations of infinity are defined perturbatively—see the examples below). In this case the singularity is said to be not confined. Grammaticos and Ramani *et al* claim that the integrability of discrete equations is associated with the preservation of information contained within the initial conditions after the solution has passed through a singularity.

The singularity confinement property has been used by several authors and has identified many discrete equations which possess many of the properties that continuous integrable equations have, for example, the (so-called) discrete Painlevé equations. Indeed, the test was used to identify the first known integrable discretizations of the third, fourth and fifth Painlevé equations [35]. Hence this test has been successful in isolating interesting and important equations.

We remark, however, that recently Hietarinta and Viallet [20] have shown that the equation

$$u_{n+1} + u_{n-1} = u_n + \frac{a}{u_n^2} \tag{35}$$

possesses the singularity confinement property and yet it is (numerically) chaotic for $a \neq 0$. On the other hand, from theorem 3 we see that for $a \neq 0$, non-rational meromorphic solutions of equation (35) have infinite order. From our point of view, such equations should not be included within the class of Painlevé-type difference equations (the infinite-order growth of their solutions indicates that they are too 'ill-posed' in the complex plane). The singularity confinement test also contains a number of restrictions: for example, the maps must be strictly rational and there must be no 'proliferation of pre-images' (see [19]). In particular, the test cannot be used to analyse the logistic equation (12). On the other hand, the discussion here includes all equations of the form (10) as a special case of equation (20).

Consider the reversible, rational, second-order discrete equation

$$y_{n+1} = y_n + \frac{y_n - y_{n-1}}{1 + (y_n - y_{n-1})}. \tag{36}$$

This equation becomes singular when $1 + y_n - y_{n-1} = 0$. In order to explore this singularity we consider the initial conditions

$$y_0 = k \qquad y_1 = k - 1 + \epsilon$$

where k is an arbitrary parameter. Following the ideas of Grammaticos and co-workers [18, 36], we calculate the next few iterates: $y_2 = -\epsilon^{-1} + k + O(\epsilon)$, $y_3 = -\epsilon^{-1} + (k + 1) + O(\epsilon)$, $y_4 = -\epsilon^{-1} + (k + \frac{3}{2}) + O(\epsilon)$. It appears that the singularity is not confined in the sense of Grammaticos, Ramani *et al*. We will now show that this is indeed true, despite the fact that equation (36) has a simple solution.

Let

$$w_n = (y_n - y_{n-1})^{-1}.$$

Equation (36) becomes

$$\Delta w_n \equiv w_{n+1} - w_n = 1$$

which, in turn, sums to give $w_n = n - n_0$, where n_0 is an arbitrary constant. Hence

$$y_{n+1} - y_n = \frac{1}{n - n_0}$$

which sums to give

$$y_n = y_0 + \sum_{m=0}^{n-1} \frac{1}{m - n_0}. \tag{37}$$

We see that equation (36) is integrable and (from equation (37)) $y_n = -\epsilon^{-1} + O(1)$, for $n \geqslant 2$ (so the singularity is not confined).

On the other hand, if we consider equation (36) as a difference equation, the solution of this equation is ($n \mapsto z$)

$$y(z) = \pi_1(z) + \psi(z - \pi_2(z)) \tag{38}$$

where π_1 and π_2 are arbitrary periodic functions and $\psi(z) = \mathrm{d} \log \Gamma(z)/\mathrm{d}z$ (digamma function). This example and others suggest the following. For difference equations with solutions of finite order, the singularity confinement test can be interpreted as a test for the appearance of ψ in expansions of solutions of difference equations. We also note that in terms of the variable z, $y(z)$ is meromorphic for all z modulo π_1 and π_2.

A (natural) continuum limit of equation (36) is

$$y'' + (y')^2 = 0 \tag{39}$$

which has the general solution

$$y(z) = z_1 + \log(z - z_2) \tag{40}$$

where z_1 and z_2 are constants. Although equation (39) does not possess the Painlevé property, we still consider it to be integrable as the single log can be transformed away. Similarly, we consider equation (36) to be integrable because its general solution contains *only one ψ function* (digamma function). However, many of the equations related to Painlevé equations can be considered perturbations of autonomous equations which have elliptic functions as solutions. In this case, if an expansion near a pole of the elliptic function gives rise to a series involving digamma functions arises, then there would be contributions from the infinity of poles of the elliptic function which we take to be an indication of non-integrability. This is a difference analogue of the psi-series which arise in the singularity analysis of differential equations.

5.2. The discrete Painlevé property

Conte and Musette [12] define their discrete Painlevé property as follows. A difference equation,

$$F(x, h, \{u(x + kh), k - k_0 = 0, \ldots, N\}) = 0$$

is said to possess the discrete Painlevé property if and only if there exists a neighbourhood of $h = 0$ within which the general solution $x \mapsto u(x, h)$ has no movable critical singularities as a function of x. Thus this definition demands that the difference equation has a parameter h which allows us to obtain a meaningful continuum limit as h tends to zero. Hence we are restricted to a non-generic subclass of difference equations. In general, a difference equation will not have a parameter or a continuum limit. In the broad sense, the integrability of a difference equation should not depend on the existence of an arbitrary parameter in the equation. This definition also presents us with other difficulties if we are to associate the discrete Painlevé property with integrability.

Under this definition the trivial difference equation

$$y(z + h) - y(z) = 0$$

fails to possess the discrete Painlevé property since its general solution is an arbitrary periodic function. For example,

$$y(z) = \log\left(\cos\left(\frac{2\pi z}{h}\right)\right)$$

is a solution with logarithmic branching for arbitrarily small $h \neq 0$. This kind of difficulty can be removed by adding a caveat to the definition to the effect that we are only concerned with essential branching (branching due to sources other than the periodic functions). With this in mind consider the difference equation

$$u(x + h) - u(x) + hu^2(x) = 0. \tag{41}$$

Under the transformation $z = x/h$, $y(z) = hu(x)$, equation (41) becomes equation (13). Thus equation (41) admits the family of solutions

$$u(x) = \frac{w(hx + \pi(hx))}{h}$$

where π is an arbitrary periodic function and w is an entire function (cf equation (14) and [37]). Also, in the limit $h \to 0$, equation (41) becomes $u' + u^2 = 0$ which has only rational solutions. So ignoring singularities due to the periodic function π (by, for example, restricting π to be entire), we see that u is meromorphic for all h and hence equation (41) possesses the discrete Painlevé property. Equation (41), however, is known to be non-integrable. From our point of view, equation (41) should be excluded from our analysis because its meromorphic solutions are of infinite order.

This test has, nonetheless proved to be successful and yields some information about the asymptotic structure of the solution at infinity. Recall from section 4 that if we take $x = hz$ to be held fixed as we let h become arbitrarily small, then an expansion in h would be an expansion in $1/z$ in terms of the original, unscaled equation (15). These expansions are limits of the expansions described above in section 4.

6. Summary and conclusion

In this paper we have explored the singularity structure of solutions of certain classes of difference equations in the complex domain.

We note that for a large class of difference equations the singularity structure of the solutions is meromorphic in the complex domain, apart from singularities that may arise due to the arbitrary periodic functions. This leads us to conclude that the integrability of many difference equations is encoded in the nature of the singularity at infinity in the same way that the integrability of many differential equations is encoded in the movable singularities in the finite plane. Much research remains to be done to determine exactly what type of singularities at infinity are acceptable.

Our analysis has led us to consider a special class of equations whose solutions (up to the arbitrary periodic functions) are well behaved in the complex plane. In particular, the equations that we consider to be of Painlevé type have solutions that are:

(a) finite order in the sense of Nevanlinna, and
(b) have no digamma functions (i.e. ψ's) in their series expansions as outlined in section 4. The digamma functions play an analogous role to the logs in psi-series (Painlevé) expansions of differential equations.

We have only considered difference analogues of logarithmic singularities in the series method described above. Meromorphic functions such as $\Gamma\left(z+\frac{1}{2}\right)/\Gamma(z)$, which plays the role of $\sqrt{z}$ for difference equations, will be considered elsewhere.

Acknowledgment

Appendix. Order of solutions of second-order rational difference equations

In this appendix we provide a proof of theorem 3 based on arguments used by Yanagihara [40] for first-order equations. We begin by proving the following lemma.

Lemma 1. *Given $\epsilon > 0$ and a meromorphic function y, the Nevanlinna characteristic function T satisfies*

$$T(r, y(z \pm 1)) \leqslant (1 + \epsilon)T(r + 1, y(z)) + \kappa$$

for all $r \geqslant 1/\epsilon$, for some constant κ.

Proof. We will use the following identity which follows from Cartan's identical relation (see, for example, [42]):

$$\frac{\mathrm{d}T(r, f(z))}{\mathrm{d}\log r} = \frac{1}{2\pi} \int_0^{2\pi} n(r, \mathrm{e}^{\mathrm{i}\theta}, f(z)) \,\mathrm{d}\theta \tag{A1}$$

where $n(r, \mathrm{e}^{\mathrm{i}\theta}, f(z))$ is the number of solutions of

$$f(z) = \mathrm{e}^{\mathrm{i}\theta} \tag{A2}$$

(counting multiplicities) inside $|z| \leqslant r$.

It follows from the definition of n that $n(r + 1, \mathrm{e}^{\mathrm{i}\theta}, y(z))$ is the number of solutions of equation (A2) in $R := \{z : |z| \leqslant r + 1\}$ and $n(r, \mathrm{e}^{\mathrm{i}\theta}, y(z \pm 1))$ is the number of solutions of

$y(z \pm 1) = e^{i\theta}$ inside $|z| \leqslant r$ or, equivalently, the number of solutions of equation (A2) in $R_\pm := \{z : |z \mp 1| \leqslant r\}$. Using the fact that R_+ and R_- are subsets of R, we conclude that $n(r, i\theta, y(z \pm 1)) \leqslant n(r + 1, e^{i\theta}, y(z))$ for all θ and using the identity (A1) we find

$$\frac{dT(r, y(z \pm 1))}{d \log r} \leqslant \frac{dT(r + 1, y(z))}{d \log(r + 1)} \leqslant \left(1 + \frac{1}{r}\right) \frac{dT(r, y(z))}{d \log r}.$$

Integrating this equation for $r \geqslant 1/\epsilon$ establishes the lemma. $\qquad\square$

Proof of theorem 3. Equating the Nevanlinna characteristics of both sides of equation (21), we find

$$T(r, y(z + 1) + y(z - 1)) = T(r, R(z, y(z))) = mT(r, y(z)) + O(\log r) \quad \text{(A3)}$$

where $m = \max(p, q)$ and the second equality in equation (A3) follows from a theorem of Valiron [38]. Using lemma 1, together with the fact that the left-hand side of equation (A3) is bounded above by $T(r, y(z + 1)) + T(r, y(z - 1)) + \log 2$ (see, for example, [21]) gives

$$T(r + 1, y(z)) \geqslant kT(r, y(z)) + g(r) \quad \text{(A4)}$$

where $k = m/2(1 + \epsilon)$ and $|g(r)| < K \log r$ for some K and all r greater than some r_0. Hence for $r > r_0$,

$$T(r + j, y(z)) \geqslant k^j T(r, y(z)) + E_j(r) \quad \text{(A5)}$$

where

$$|E_j(r)| = |k^{j-1} g(r) + k^{j-2} g(r + 1) + \cdots + g(r + j - 1)|$$

$$\leqslant Kk^{j-1} \sum_{i=0}^{j-1} \frac{\log(r + i)}{k^i}$$

$$\leqslant Kk^{j-1} \sum_{i=0}^{\infty} \frac{\log(r + i)}{k^i}.$$

Using $\log(r + i) \leqslant \log r \log i$, for r and i sufficiently large, we note that since $k > 1$ the series converges and hence

$$|E_j(r)| \leqslant Ck^j \log r$$

where C is a positive constant.

A meromorphic function f is rational if and only if $T(r, f) = O(\log r)$. Since, by hypothesis, y is non-rational we can choose r_0 sufficiently large such that for all $r > r_0$, we have $T(r, y(z)) > 2C \log r$ (where we have used the fact that T is an increasing function of $\log r$, cf equation (A1)). Hence equation (A5) implies

$$T(r + j, y(z)) > Ck^j \log r.$$

and so

$$\frac{\log T(r + j, y(z))}{\log(r + j)} > \frac{j \log k + \log(C \log r)}{\log(r + j)}.$$

So for $k > 1$, $\sigma(y) = \infty$ (cf equation (18)). $\qquad\square$

Proof of theorem 4. Without loss of generality we rescale equation (24) so that $c = 3$. Let

$$M(r) := \max_{|z|=r} |y(z)|.$$

By considering the maximum of $|y(z \pm 1)|$ on $|z| = r$ we see that the maximum principle gives

$$M(r+1) \geqslant \max_{|z|=r} |y(z \pm 1)|.$$

Hence

$$2M(r+1) \geqslant \max_{|z|=r} |y(z+1)| + \max_{|z|=r} |y(z-1)|$$

$$\geqslant \max_{|z|=r} |y(z+1) + y(z-1)|$$

$$\geqslant \max_{|z|=r} |a(z) + b(z)y(z) + 3y^2(z)| \tag{A6}$$

where we have used equation (24). Since a and b are polynomials and y is not, there exists an r_0 such that

$$M^2(r) = \max_{|z|=r} |y(z)|^2 \geqslant \max_{|z|=r} |a(z) + b(z)y(z)|$$

for all $r > r_0$. Hence, using equation (A6), we find for sufficiently large r

$$M(r+1) \geqslant M^2(r)$$

and so

$$M(r+i) \geqslant M^{2^i}(r).$$

This gives

$$\frac{\log \log M(r+i)}{\log(r+i)} \geqslant \frac{i \log 2 + \log \log M(r)}{\log(r+i)}$$

which, from equation (19), shows that y has infinite order. $\qquad \square$

The proof of theorem 5 uses the identity $T(r, f_1 f_2) \leqslant T(r, f_1) + T(r, f_2)$ (see, for example, [21]) but is otherwise identical to the proof just given for theorem 3.

References

[1] Ablowitz M J, Chakravarty S and Halburd R 1998 Darboux–Halphen systems and the singularity structure of its solutions *Proc. 4th Int. Conf. on Mathematical and Numerical Aspects of Wave Propagation* ed J A DeSanto (Philadelphia, PA. SIAM) pp 408–12

[2] Ablowitz M J, Chakravarty S and Halburd R 1998 On Painlevé and Darboux–Halphen type equations *The Painlevé Property, One Century Later (CRM Series in Mathematical Physics)* ed R Conte (Berlin: Springer)

[3] Ablowitz M J, Chakravarty S and Herbst B M 1995 Integrability, computability and applications *Acta Appl. Math.* **39** 5–37

[4] Ablowitz M J and Clarkson P A 1991 *Solitons, Nonlinear Evolution Equations and Inverse Scattering (London Mathematical Society Lecture Note Series vol 149)* (Cambridge: Cambridge University Press)

[5] Ablowitz M J, Ramani A and Segur H 1978 Nonlinear evolution equations and ordinary differential equations of Painlevé type *Lett. Nouvo Cimento* **23** 333–8

[6] Ablowitz M J, Ramani A and Segur H 1980 A connection between nonlinear evolution equations and ordinary differential equations of P-type. I *J. Math. Phys.* **21** 715–21
Ablowitz M J, Ramani A and Segur H 1980 A connection between nonlinear evolution equations and ordinary differential equations of P-type. II *J. Math. Phys.* **21** 1006–15

[7] Ablowitz M J and Segur H 1977 Exact linearization of a Painlevé transcendent *Phys. Rev. Lett.* **38** 1103–6

[8] Birkhoff G D 1920 Surface transformations and their dynamical applications *Acta Math.* **43** 1–119

[9] Boutroux P 1904 Sur quelques propriétés des fonctions entières *Acta Math.* **28** 97–224

[10] Chakravarty S and Ablowitz M J 1996 Integrability, monodromy, evolving deformations and self-dual Bianchi IX systems *Phys. Rev. Lett.* **76** 857–60

[11] Conte R and Mussette M 1996 A new method to test discrete Painlevé equations *Phys. Lett.* A **223** 439–48

[12] Conte R and Mussette M 1998 Rules of discretization for Painlevé equations *Theory of Nonlinear Special Functions: the Painlevé Transcendents (Montreal, 13–17 May 1996)* ed L Vinet and P Winternitz (Berlin: Springer)

[13] Devaney R L 1986 *An Introduction to Chaotic Dynamical Systems* (Menlo Park, CA: Benjamin-Cummings)

[14] Flaschka H and Newell A C 1980 Monodromy- and spectrum preserving deformations. i *Commun. Math. Phys.* **76** 65–116

[15] Fokas A S, Grammaticos B and Ramani A 1993 From continuous to discrete Painlevé equations *J. Math. Anal. Appl.* **180** 342–60

[16] Fuchs L 1905 Sur quelques équations différentielles linéares du second ordre *C. R. Acad. Sci., Paris* **141** 555–8

[17] Gambier B 1910 Sur les équations différentielles du second ordre et du premier degré dont l'intégrale générale est à points critiques fixes *Acta Math.* **33** 1–55

[18] Grammaticos B, Ramani A and Papageorgiou V 1991 Do integrable mappings have the Painlevé property? *Phys. Rev. Lett.* **67** 1825–8

[19] Grammaticos B, Ramani A and Tamizhmani K M 1994 Nonproliferation of pre-images in integrable mappings *J. Phys. A: Math. Gen.* **27** 559–66

[20] Hietarinta J and Viallet C 1998 Singularity confinement and chaos in discrete systems *Phys. Rev. Lett.* **81** 325–8

[21] Hille E 1959–62 *Analytic Function Theory* vols 1, 2 (Boston, MA: Ginn)

[22] Hirota R 1978 Nonlinear partial difference equations iv. Bäcklund transformation for the discrete-time Toda lequation *J. Phys. Soc. Japan* **45** 321–32

[23] Jimbo M and Miwa T 1981 Monodromy preserving deformation of linear ordinary differential equations with rational coefficients II *Phys. D* **2** 407–48

[24] Joshi N, Burtonclay D and Halburd R G 1992 Nonlinear nonautonomous discrete dynamical systems from a general discrete isomonodromy problem *Lett. Math. Phys.* **26** 123–31

[25] Kimura T 1971 On the iteration of analytic functions *Funkcial. Ekvac.* **14** 197–238

[26] Kitaev A V and Fokas A S 1990 An isomonodromy approach to the theory of two-dimensional quantum gravity *Russ. Math. Surv.* **45** 155–7

[27] Kruskal M D, Joshi N and Halburd R 1997 Analytic and asymptotic methods for nonlinear singularity analysis: a review and extensions of tests for the Painlevé property *Integrability of Nonlinear Systems (Pondicherry, 1996) (Lecture Notes in Physics vol 495)* ed B Grammaticos and K Tamizhmani (Berlin: Springer) pp 171–205

[28] May R M 1976 Simple mathematical models with very complicated dynamics *Nature* **261** 459–67

[29] Nijhoff F W, Satsuma J, Kajiwara K, Grammaticos B and Ramani A 1996 A study of the alternative discrete Painlevé-II equation *Inverse Problems* **12** 697–716

[30] Painlevé P 1900 Mémoire sur les équations différentielles dont l'integrale générale est uniforme *Bull. Soc. Math. France* **28** 201–61

[31] Painlevé P 1902 Sur les équations différentielles du second ordre et d'ordre supérieur dont l'integrale générale est uniforme *Acta Math.* **25** 1–85

[32] Papageorgiou V G, Nijhoff F W, Grammaticos B and Ramani A 1992 Isomonodromic deformation problems for discrete analogues of Painlevé equations *Phys. Lett.* A **164** 57–64

[33] Periwal V and Shevitz D 1990 Unitary matrix models as exactly solvable string theories *Phys. Rev. Lett.* **64** 1326–9

[34] Potts R B 1987 Weierstrass elliptic difference equations *Bull. Aust. Math. Soc.* **35** 43–8

[35] Ramani A, Grammaticos B and Hietarinta J 1991 Discrete versions of the Painlevé equations *Phys. Rev. Lett.* **67** 1829–32

[36] Ramani A, Grammaticos B and Tamizhmani K M 1993 Painlevé analysis and singularity confinement: the ultimate conjecture *J. Phys. A: Math. Gen.* **26** L53–8

[37] Shimomura S 1981 Entire solutions of a polynomial difference equation *J. Fac. Sci. Univ. Tokyo Sect. IA Math.* **28** 253–66

[38] Valiron G 1931 Sur la derivee des fonctions algebroïdes *Bull. Soc. Math. France* **59** 17–39

[39] Wittich H 1954 Einige Eigenschaften der Lösungen von $w = a(z) + b(z)w + c(z)w^2$ *Arch. Math.* **5** 226–32

[40] Yanagihara N 1980 Meromorphic solutions of some difference equations *Funkcial. Ekvac.* **23** 309–26

[41] Yanagihara N 1985 Meromorphic solutions of some difference equations of the nth order *Arch. Ration. Mech. Anal.* **91** 169–92

[42] Zhang G-H 1993 *Theory of Entire and Meromorphic Functions: Deficient and Asymptotic Values and Singular Directions (Translations of Mathematical Monographs vol 122)* (Providence, RI: American Mathematical Society)