A Study on Unsteady Flow of an Oldroyd B-Fluid due to Torsional vibrations of a Circular Disc using Bessel functions

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***Abstract*— Primary flow of an oldroyd-B fluid elastic viscous fluid due to the torsional vibrations of a circular disc is considered. Integral Solution for the velocity field and the expression of the torque have been found**

Keywords*— *oldroyd B- fluid, torsional vibrations, torque, circular disc.

I. INTRODUCTION

Let (v_r, v_ϕ, v_z) (1) be the components of velocity in (r, ϕ, z) directions in a system of cylindrical polar co-ordinates. The disc is defined by

z=0, $r \leq a$ (2)

for primary flow when we neglect quadratic terms of inertia and the components of secondary flow[4,5,6] there remains only the azimuthal components of velocity $v_\phi = (v_r, v_\phi, v_z)$, $v_r = v_z = 0$ (3)

$\frac{\partial}{\partial \phi}(\,) = 0$ and $v_\phi = v = (v_r, v_\phi, v_z)$ (4)

where the suffix ϕ has been dropped for convenience.

II. MATHEMATICAL FORMULATION

The only non-zero component of rate of strain tensor[1,3] is $\rho_{r\phi}$and the equations of motion reduces to

$$\left(1+\lambda_1\frac{\partial}{\partial t}\right)\left(\frac{v^2}{r}\right)=\frac{1}{\rho}\left(1+\lambda_2\frac{\partial}{\partial t}\right)\left(\frac{\partial p}{\partial r}\right) \qquad (5)$$

$$\left(1+\lambda_1\frac{\partial}{\partial t}\right)\left(\frac{\partial v}{\partial t}\right)=\vartheta\left(1+\lambda_2\frac{\partial}{\partial t}\right)\left(\nabla^2-\frac{1}{r^2}\right)v \qquad (6)$$

where

$$\nabla^2=\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{\partial^2}{\partial z^2}, \qquad \vartheta=\frac{\eta_0}{\rho}$$

In the case of steady motion $\frac{\partial}{\partial t}(\,) = 0$

Equation (5) then gives

$$\frac{\rho v^2}{r}=\frac{\partial p}{\partial r}$$

this shows that the radial variation of pressure simply supplies the force necessarily to keep the fluid elements moving in a circular path[10].

In the case of unsteady motion integrating equation (5) with the condition that initially

when t=0 v=0 and $-\frac{\partial p}{\partial r}=0$, we obtain

$$\frac{\rho v^2}{r}-\frac{\partial p}{\partial r}=0 \qquad (7)$$

This equation gives the pressure field when one we find the velocity field from equation (6). In equation (6) if we put $\lambda_1=\lambda_2=0$ we obtain the classical Newtonian[6] equation of motion

$$\frac{\partial v}{\partial t}=\vartheta\left(\nabla^2-\frac{1}{r^2}\right)v \qquad (8)$$

where

$\nabla^2=\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{\partial^2}{\partial z^2}$,

$v=\frac{\eta_0}{\rho}$ is the kinematic viscosity of the fluid.

Now, we observe that equation (6) is a relation expressing the balance for the instaneous [5] rate of increase of angular momentum of a cylindrical shell of fluid undet the action of the couple exerted by friction at its inner and outer faces as influenced by the time-relaxation and retardation parameters $\lambda_1 and\, \lambda_2$ respectively. This can be seen by noting that the tangential stress $p_{r\phi}$ as an element of the surface of a cylinder of radius *r* is given by [7]

$$\left(1+\lambda_1\frac{\partial}{\partial t}\right)p_{r\phi}=\eta_0\left(1+\lambda_2\frac{\partial}{\partial t}\right)\left(\frac{\partial v}{\partial r}-\frac{v}{r}\right) \qquad (9)$$

Thus the couple T extended on the fluid inside a cylindrical surface of radius *r* by the fluid outside in it per unit length is given by

$$\left(1+\lambda_1\frac{\partial}{\partial t}\right)T=2\pi\eta_0 r^2\left(1+\lambda_2\frac{\partial}{\partial t}\right)\left(\frac{\partial v}{\partial r}-\frac{v}{r}\right) \qquad (10)$$

The balance of the rate of change of angular momentum of the fluid in a cylindrical shell per unit length and unit thickness and the couple acting on it is then given by

$$\left(1+\lambda_1\frac{\partial}{\partial t}\right)\frac{\partial}{\partial t}(2\pi\rho r^2 v)=\frac{\partial}{\partial r}\left[2\pi\eta_0 r^2\left(1+\lambda_2\frac{\partial}{\partial t}\right)\left(\frac{\partial v}{\partial r}-\frac{v}{r}\right)\right] \quad (11)$$

It may be easily seen that equation (6) follows from equation (11).

Again equation (6) may be used in to investigate the flow changes produced by starting or stopping the rotation of the disc. Here we consider the motion of incompressible oldroyd model [7] of electro viscous fluid generated from rest by torsional oscillatory performed by a circular disc defied by $z=0\,, r\leq a$.

We have to solve equation (6) under appropriate boundary conditions

The Boundary Conditions are

$$v=r\Omega\; on\; z=0, r\leq a$$

and the condition for shear stress is

$$\left[\frac{\partial v}{\partial z}\right]_{z=0}=0\;\; for\, r>a\,, v\to 0\; as\; r\to\infty$$

where Ω is the angular velocity of the disc.

III. SOLUTION OF THE PROBLEM

Assuming solution in the form

$$v = \bar{v}e^{i\sigma t} \qquad (3.1)$$

where $\bar{v}$ is independent of t, we obtain from equation (6)

$$\left(\nabla^2 - \frac{1}{r^2} - k^2\right)\bar{v} = 0 \qquad (3.2)$$

$$k^2 = \frac{i\sigma}{v}\left(\frac{1 + i\sigma\lambda_1}{1 + i\sigma\lambda_2}\right) \qquad (3.3)$$

Let the velocity field be expressed as

$$\bar{v} = v^{(1)} + k^2 v^{(2)} \qquad (3.4)$$

We take $|k|$ to be small so that we may neglect terms of $O(k^4)$ and higher order

The appropriate boundary conditions are

$$\left.\begin{aligned} v^{(1)} &= r\Omega \\ v^{(2)} &= 0 \end{aligned}\right\} on\ z = 0\,, r \le a \qquad (3.5)$$

where Ω is the angular velocity of the disc. As $r \to \infty$, $v^{(1)}$, $v^{(2)}$ tends to zero. The boundary conditions are not enough to assure a unique solution. So we require another condition, i.e., condition of shear stress[19]

$$\left[\frac{\partial v^{(1)}}{\partial z}\right]_{z=0} = 0 = \left[\frac{\partial v^{(2)}}{\partial z}\right]_{z=0} for\ r > a \qquad (3.6)$$

Substituting from (3.4) in to (3.2) and equating the coefficients to zero, various powers of k^2 we obtain up to $O(k^2)$

$$\left(\nabla^2 - \frac{1}{r^2}\right)v^{(1)} = 0 \qquad (3.7)$$

$$\left(\nabla^2 - \frac{1}{r^2}\right)v^{(2)} = v^{(1)} \qquad (3.8)$$

An elementary solution of equation (3.7) may be taken as

$$v^{(1)} = e^{-\lambda z}J_1(\lambda r) \qquad (3.9)$$

Here $J_1(\lambda r)$ is the Bessel function[2] of first kind and order unity, this solution satisfies the boundary conditions that $v^{(1)}$ and $v^{(2)}$ tend to zero as $r \to \infty$. The exponential factor in the solution will be $e^{-\lambda z}$ for the region $z > 0$ and $e^{\lambda z}$ for the region $z < 0$ in the equation (3.9). Now there remain (3.5) and (3.6) to be satisfied. The satisfaction of these boundary conditions is achieved by representing the solution in the integral form[2]

$$v^{(1)} = \frac{4\Omega}{\pi}\int_0^\infty e^{-\lambda z}J_1(\lambda r)\left(\frac{sin\, a\lambda}{\lambda^2} - \frac{a\, cos\, a\lambda}{\lambda}\right)d\lambda \qquad (3.10)$$

Equation (3.8) has a Particular Integral of the type

$$v^{(2)} = -\frac{1}{2\lambda}e^{-\lambda z}J_1(\lambda r) \qquad (3.11)$$

Now, the integral solution of the differential equation (3.8) satisfying the boundary conditions (3.5) is

$$v^{(2)} = -\frac{2\Omega}{\pi}\int_0^\infty e^{-\lambda z}J_1(\lambda r)\left(\frac{\sin a\lambda}{\lambda^3} - \frac{a\cos a\lambda}{\lambda^2}\right)d\lambda \quad (3.12)$$

Since the boundary conditions (3.6) is yet to be satisfied we add the complementary function

$$-\frac{2\Omega a^3}{3\pi}\int_0^\infty \frac{1}{\lambda}e^{-\lambda z}J_1(\lambda r)\cos a\lambda\, d\lambda \qquad (3.13)$$

to the Particular Integral (3.12) and put the solution as

$$v^{(2)} = -\frac{2\Omega}{\pi}\int_0^\infty ze^{-\lambda z}J_1(\lambda r)\left(\sin a\lambda - a\lambda\cos a\lambda\right)\frac{d\lambda}{\lambda^3} - \frac{2\Omega a^3}{3\pi}\int_0^\infty e^{-\lambda z}J_1(\lambda r)\frac{\cos a\lambda}{\lambda}\, d\lambda \quad (3.14)$$

IV. TORQUE

The torque on the disc due to the velocity field $k^2 v^{(2)}$ is obtained by evaluating the integral

$$2\int_0^\infty \left(p_{r\phi}\right)_{z=0} 2\pi r^2 dr = 2\eta_0 k^2 \int_0^a \left[\frac{\partial v^{(2)}}{\partial z}\right]_{z=0} 2\pi r^2 dr$$

$$= \frac{32}{15}\eta_0 a^5 \Omega k^2 e^{i\sigma t}$$

Consequently the torque τ acting on the vibrating disc due to the velocity field

$$\left[v^{(1)} + k^2 v^{(2)}\right] e^{i\sigma t}$$

obtained from (3.10) and (3.14) is

$$T = -\frac{32}{3}\eta_0 a^3 \Omega e^{i\sigma t} - \frac{32}{15}\eta_0 a^5 \Omega e^{i\sigma t}$$

$$= T_0\left(1 + \frac{k^2 a^2}{5}\right)e^{i\sigma t}$$

where $T_0 = -\frac{32}{3}\eta_0 a^3 \Omega$ denotes the torque on the disc due to the velocity $v^{(1)}$ and represents the steady state of torque in case of classical Newtonian fluid.

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