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GENERATED SETS OF THE COMPLETE SEMIGROUP BINARI RELATIONS DEFINED BY SEMILATTICES OF THE CLASS $\Sigma_1(X,2)$

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Abstract. In this article, we study generated sets of the complete semi group $B_X(D)$ defined by an $X-$semi lattice D of the class $\Sigma_1(X,2)$.

Key words: **Semi group, semi lattice, binary relation.**

I. INTRODUCTION

1.1. Let X be an arbitrary nonempty set, D is an $X-$semilattice of unions which closed with respect to the set-theoretic union of elements from D, f be an arbitrary mapping of the set X in the set D. To each mapping f we put into correspondence a binary relation α_f on the set X that satisfies the condition

$$\alpha_f = \bigcup_{x\in X}(\{x\}\times f(x)).$$

The set of all such α_f ($f: X\to D$) is denoted by $B_X(D)$. It is easy to prove that $B_X(D)$ is a semigroup with respect to the operation of multiplication of binary relations, which is called a complete semigroup of binary relations defined by an $X-$semilattice of unions D.

We denote by $\varnothing$ an empty binary relation or an empty subset of the set X. The condition $(x,y)\in\alpha$ will be written in the form $x\alpha y$. Further, let $x,y\in X$, $Y\subseteq X$, $\alpha\in B_X(D)$, $\breve{D}=\bigcup_{Y\in D} Y$ and $T\in D$. We denote by the symbols $y\alpha$, $Y\alpha$, $V(D,\alpha)$, X^*, $V(X^*,\alpha)$ and D_T the following sets:

$$y\alpha=\{x\in X \mid y\alpha x\},\ Y\alpha=\bigcup_{y\in Y} y\alpha,\ V(D,\alpha)=\{Y\alpha \mid Y\in D\},$$
$$X^*=\{Y \mid \varnothing\neq Y\subseteq X\},\ V(X^*,\alpha)=\{Y\alpha \mid \varnothing\neq Y\subseteq X\},$$
$$D_T=\{Z\in D \mid T\subseteq Z\}.$$

It is well know the following statements:

Theorem 1.2. *Let $D=\{\breve{D},Z_1,Z_2,\dots,Z_{m-1}\}$ be some finite $X-$semilattice of unions and $C(D)=\{P_0,P_1,P_2,\dots,P_{m-1}\}$ be the family of sets of pairwise nonintersecting subsets of the set X (the set $\varnothing$ can be repeat several time). If φ is a mapping of the semilattice D on the family of sets $C(D)$ which satisfies the condition*

$$\varphi=\begin{pmatrix}\breve{D} & Z_1 & Z_2 & \dots & Z_{m-1}\\ P_0 & P_1 & P_2 & \dots & P_{m-1}\end{pmatrix}$$

and $\hat{D}_Z=D\setminus D_Z$, then the following equalities are valid:

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$$\breve{D} = P_0 \cup P_1 \cup P_2 \cup ... \cup P_{m-1}$$
$$Z_i = P_0 \cup \bigcup_{T \in \hat{D}_{Z_i}} \varphi(T) \qquad ... (1.1)$$

In the sequel these equalities will be called formal.

It is proved that if the elements of the semilattice D are represented in the form (1.1), then among the parameters P_i $(0 < i \le m-1)$ there exist such parameters that cannot be empty sets for D. Such sets P_i are called basis sources, whereas sets P_j $(0 \le j \le m-1)$ which can be empty sets too are called completeness sources.

It is proved that under the mapping φ the number of covering elements of the pre-image of a basis source is always equal to one, while under the mapping φ the number of covering elements of the pre-image of a completeness source either does not exist or is always greater than one (see [1], chapter 11).

Let $P_0, P_1, P_2, ..., P_{m-1}$ be parameters in the formal equalities, $\beta \in B_X(D)$ and

$$\overline{\beta} = \bigcup_{i=0}^{m-1} \left(P_i \times \bigcup_{t \in P_i} t\beta \right) \cup \bigcup_{t' \in X \setminus \breve{D}} \left(\{t'\} \times t'\beta \right). \qquad ...(1.2)$$

The representation of the binary relation β of the form $\overline{\beta}$ will be called subquasinormal.

If $\overline{\beta}$ be the subquasinormal representation of the binary relation β, then for the binary relation $\overline{\beta}$ the following statements are true:

a) $\overline{\beta} \in B_X(D)$;

b) $\beta \subseteq \overline{\beta}$;

c) the subquasinormal representation of the binary relation β is quasinormal;

d) if $\overline{\beta}_1 = \begin{pmatrix} P_0 & P_1 & ... & P_{m-1} \\ P_0\overline{\beta} & P_1\overline{\beta} & ... & P_{m-1}\overline{\beta} \end{pmatrix}$, then $\overline{\beta}_1$ is a mapping of the family of sets $C(D)$ in the set $D \cup \{\varnothing\}$.

e) if $\overline{\beta}_2 : X \setminus \breve{D} \to D$ is a mapping satisfying the condition $\overline{\beta}_2(t') = t'\beta$ for all $t' \in X \setminus \breve{D}$, then

$$\overline{\beta} = \bigcup_{i=0}^{m-1} \left(P_i \times \bigcup_{t \in P_i} t\beta \right) \cup \bigcup_{t' \in X \setminus \breve{D}} \left(\{t'\} \times \overline{\beta}_2(t') \right). \qquad ...(1.3)$$

Remark, that if P_j $(0 \le j \le m-1)$ is such completeness sources, that $P_j = \varnothing$, then the equality $P_j\overline{\beta} = \varnothing$ always is hold. There also exists such a basic sources P_i $(0 \le i \le m-1)$ for which $\bigcup_{t \in P_i} t\beta = \varnothing$, i.e. $P_i\overline{\beta} = \varnothing$.

Example 1.1. Let $X = \{1,2,3,4\}$, $D = \{\varnothing, \{1,2\}\}$, then $P_0 = \varnothing$, $P_1 = \{1,2\}$, If $\beta = \{(1,1),(1,2),(2,1),(2,2),(4,1),(4,2)\}$, then $\beta \in B_X(D)$ and subquasinormal representation of a binary relation β has a form

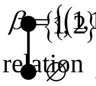

Fig. 1.1

$$\overline{\beta}_1 = \begin{pmatrix} P_0 & P_1 \\ \varnothing & \{1,2\} \end{pmatrix}, \ \overline{\beta}_2 = \begin{pmatrix} 3 & 4 \\ \varnothing & \{1,2\} \end{pmatrix},$$
$$\overline{\beta} = (\varnothing \times \varnothing) \cup (\{1,2\} \times \{1,2\}) \cup (\{3\} \times \varnothing) \cup (\{4\} \times \{1,2\}) =$$
$$= (P_0 \times \varnothing) \cup (P_1 \times \{1,2\}) \cup (\{3\} \times \varnothing) \cup (\{4\} \times \{1,2\}),$$

where P_1 are basic sources and P_0 is completeness sources.

Theorem 1.2. *Let $\alpha, \beta \in B_X(D)$, then $\alpha \circ \beta = \alpha \circ \overline{\beta}$ (see [4], Proposition 2).*

2.1 Let $\Sigma_1(X,2)$ be a class of all $X-$semilattices of unions, whose every element is isomorphic to an $X-$semilattice of unions $D = \{Z_1, \breve{D}\}$ which satisfies the condition $Z_1 \subset \breve{D}$ (see, Fig, 2.1).

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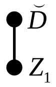

Fig. 2.1

Let $C(D)=\{P_0,P_1\}$, where $P_0,P_1\subset X$, $P_0\cap P_1=\varnothing$ and $\varphi=\begin{pmatrix}\breve{D} & Z_1\\ P_0 & P_1\end{pmatrix}$ is a mapping of the semi lattice D onto the set $C(D)$. Then for the formal equalities of the semilattice D we have a form:

$$\begin{aligned}\breve{D}&=P_0\cup P_1,\\ Z_1&=P_0,\end{aligned}\qquad\ldots(2.1)$$

Here the element P_1 be basis sources, the element P_0 are sources of completeness of the semilattice D. Therefore $|X|\geq 1$.

Definition 2.1. *We say that an element α of the semigroup $B_X(D)$ is external if $\alpha\neq\delta\circ\beta$ for all $\delta,\beta\in B_X(D)\setminus\{\alpha\}$ (see [1], Definition 1.15.1).*

It is well know, that if B is all external elements of the semigroup $B_X(D)$ and B' be any generated set for the $B_X(D)$, then $B\subseteq B'$ (see [1], Lemma 1.15.1).

Lemma 2.1. *Let $D=\{Z_1,\breve{D}\}\in\Sigma_1(X,2)$, $B=\{\alpha\in B_X(D)\,|\,V(X^*,\alpha)=D\}$. If $B\neq\varnothing$, then B is a set external elements of the semigroup $B_X(D)$.*

Proof. Let $D=\{Z_1,\breve{D}\}\in\Sigma_1(X,2)$, $\alpha\in B$ and $\alpha=\delta\circ\beta$ for some $\delta,\beta\in B_X(D)\setminus\{\alpha\}$. Then element δ has quasinormal representation of a form $\delta=(Y_1^\delta\times Z_1)\cup(Y_0^\delta\times\breve{D})$, i.e. $Y_1^\delta\cup Y_0^\delta=X$ and $Y_1^\delta\cap Y_0^\delta=\varnothing$. (see, [1], definition 1.11), By Theorem 1.2 follows that $\alpha=\delta\circ\beta=\delta\circ\overline{\beta}$, where $\overline{\beta}$ is subquasinormal representation of a binary relation β. It is easy to see, that

$$\alpha=\delta\circ\overline{\beta}=(Y_1^\delta\times Z_1\overline{\beta})\cup(Y_0^\delta\times\breve{D}\overline{\beta}).\qquad\ldots(2.2)$$

From the equality $\alpha=\delta\circ\overline{\beta}$ follows that $D=V(X^*,\alpha)\subseteq V(D,\overline{\beta})$ (see [1], Theorem 4.1.1). So, $D=V(D,\overline{\beta})$.

By preposition $\alpha\in B$, i.e. there exists quasinormal representations of a binary relation α of the form $\alpha=(Y_1^\alpha\times Z_1)\cup(Y_0^\alpha\times\breve{D})$, where $|Y_i^\alpha|\geq 1$ for all $i=0,1$ since $\alpha\in B$ (if $Y_j^\delta=\varnothing$ for some j $(0\leq j\leq 1)$, then $V(X^*,\alpha)\neq D$), i.e. $|X|\geq 2$.

For the element Z_1 we consider the following cases.

a) $Z_1=\varnothing$. In this case we have $P_0=\cap D=\varnothing$ and

$$\gamma_1=\begin{pmatrix}\varnothing & P_1\\ \varnothing & \varnothing\end{pmatrix},\ \gamma_2=\begin{pmatrix}\varnothing & P_1\\ \varnothing & \breve{D}\end{pmatrix}$$

are all mappings of the set $\{\varnothing,P_1\}$ in the semilattice D satisfying condition $\gamma_i(\varnothing)=\varnothing$ $(i=1,2)$.

If $X=\breve{D}$, then from the formal equalities (2.1) follows that $\varnothing\cup P_1=\breve{D}$. In this case $P_0=\varnothing$, $P_1=\breve{D}$, $X\setminus\breve{D}=\varnothing$, $\overline{\beta}=(\varnothing\times\varnothing)\cup(P_1\times\breve{D})\cup\varnothing=X\times\breve{D}$ (see equality 1.2). It easy to see $V(D,\overline{\beta})=D$ and

$$\begin{aligned}\alpha=\delta\circ\overline{\beta}&=\left((Y_1^\delta\times Z_1)\cup(Y_0^\delta\times\breve{D})\right)\circ(X\times D)=\\ &=\left((Y_1^\delta\times Z_1)\circ(X\times D)\right)\cup\left((Y_0^\delta\times\breve{D})\circ(X\times D)\right)=\\ &=(Y_1^\delta\times\breve{D})\cup(Y_0^\delta\times\breve{D})=X\times\breve{D}\notin B\end{aligned}$$

since $V(X^*,\alpha)=\{\breve{D}\}\neq D$. So, $X\neq\breve{D}$.

In the sequel we suppose, that $|X\setminus\breve{D}|\geq 1$.

For the binary relations γ_i $(i=1,2)$ we consider the following cases.

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Let $\gamma_1 = \begin{pmatrix} \varnothing & P_1 \\ \varnothing & \varnothing \end{pmatrix}$. By formal equality 2.1 follows that $P_1 = \breve{D} \neq \varnothing$. If $\overline{\gamma}_1$ be a mapping of the set $X \setminus \breve{D}$ on the set $D \setminus \{\varnothing\} = \{\breve{D}\}$ (by preposition $|X \setminus \breve{D}| \geq 1$), then

$$\overline{\beta} = (P_1 \times \varnothing) \cup \bigcup_{t' \in X \setminus \breve{D}} \left(\{t'\} \times \overline{\gamma}_1(t') \right), \qquad \ldots (2.3)$$

$V(D, \overline{\beta}) = D$ (see equality 1.2) and from the formal equality (2.1) and equalities (2.2), (2.3) follows that

$$\begin{aligned} &Z_1\overline{\beta} = \varnothing\overline{\beta} = \varnothing, \\ &\breve{D}\overline{\beta} = \varnothing\overline{\beta} \cup P_1\overline{\beta} = \varnothing \cup \varnothing = \varnothing, \\ &\alpha = \delta \circ \overline{\beta} = \left(Y_1^\delta \times Z_1\overline{\beta}\right) \cup \left(Y_0^\delta \times \breve{D}\overline{\beta}\right) = \\ &= \left(Y_1^\delta \times \varnothing\right) \cup \left(Y_0^\delta \times \varnothing\right) = X \times \varnothing = \varnothing \notin B \end{aligned}$$

since $V(X^*, \alpha) = \{\varnothing\} \neq D$.

If $\gamma_2 = \begin{pmatrix} \varnothing & P_1 \\ \varnothing & \breve{D} \end{pmatrix}$ and $\overline{\gamma}_2$ be a mapping of the set $X \setminus \breve{D}$ on the semilattice $D \setminus \{\breve{D}\}$. So, if

$$\overline{\beta} = (P_1 \times \breve{D}) \cup \bigcup_{t' \in X \setminus \breve{D}} \left(\{t'\} \times \overline{\gamma}_2(t') \right), \qquad \ldots (2.4)$$

then $V(D, \overline{\beta}) = D$. From the formal equality (2.1) and equalities (2.2), (2.4) we have:

$$\begin{aligned} &Z_1\overline{\beta} = \varnothing\overline{\beta} = \varnothing, \\ &\breve{D}\overline{\beta} = \varnothing\overline{\beta} \cup P_1\overline{\beta} = \varnothing \cup \breve{D} = \breve{D}, \\ &\alpha = \delta \circ \overline{\beta} = \left(Y_1^\delta \times Z\overline{\beta}\right) \cup \left(Y_0^\delta \times \breve{D}\overline{\beta}\right) = \\ &= \left(Y_1^\delta \times \varnothing\right) \cup \left(Y_0^\delta \times \breve{D}\right) = \delta. \end{aligned}$$

But, the equality $\alpha = \delta$ contradict the condition, that $\delta \in B_X(D) \setminus \{\alpha\}$. That is in this case $\alpha \notin B$.

So, from the cases $a)$ follows that B is a set external elements of semigroup $B_X(D)$ since the mappings γ_i $(i = 1, 2)$ are all mappings of the set $\{\varnothing, P_1\}$ in the semilattice D satisfying condition $\gamma_i(\varnothing) = \varnothing$.

b) $Z_1 \neq \varnothing$. Then $P_0 = \cap D \neq \varnothing$ and

$$\sigma_1 = \begin{pmatrix} P_0 & P_1 \\ Z_1 & Z_1 \end{pmatrix}, \ \sigma_2 = \begin{pmatrix} P_0 & P_1 \\ Z_1 & \breve{D} \end{pmatrix}, \ \sigma_3 = \begin{pmatrix} P_0 & P_1 \\ \breve{D} & Z_1 \end{pmatrix} \text{ and } \sigma_4 = \begin{pmatrix} P_0 & P_1 \\ \breve{D} & \breve{D} \end{pmatrix}$$

are all mappings of the set $\{P_0, P_1\}$ in the semilattice D.

If $X = \breve{D}$ and $\sigma_i(P_0) = \sigma_i(P_1)$ ($i = 1, 4$), then from the formal equalities (2.1) follows that $\sigma_i(P_1) = \breve{D}$ and in this case $\overline{\beta} = X \times \sigma_i(P_1)$ (see equality 1.2). So, $\overline{\beta} = X \times Z_1$ or $\overline{\beta} = X \times \breve{D}$. Both case $V(D, \overline{\beta}) = D$ and

$$\begin{aligned} &\alpha = \delta \circ \overline{\beta} = \left(\left(Y_1^\delta \times Z_1\right) \cup \left(Y_0^\delta \times \breve{D}\right)\right) \circ \left(X \times Z_1\right) = \\ &= \left(\left(Y_1^\delta \times Z_1\right) \circ \left(X \times Z_1\right)\right) \cup \left(\left(Y_0^\delta \times \breve{D}\right) \circ \left(X \times Z_1\right)\right) = \\ &= \left(Y_1^\delta \times Z_1\right) \cup \left(Y_0^\delta \times Z_1\right) = X \times Z_1 \notin B, \end{aligned}$$

$$\begin{aligned} &\alpha = \delta \circ \overline{\beta} = \left(\left(Y_1^\delta \times Z_1\right) \cup \left(Y_0^\delta \times \breve{D}\right)\right) \circ \left(X \times \breve{D}\right) = \\ &= \left(\left(Y_1^\delta \times Z_1\right) \circ \left(X \times \breve{D}\right)\right) \cup \left(\left(Y_0^\delta \times \breve{D}\right) \circ \left(X \times \breve{D}\right)\right) = \\ &= \left(Y_1^\delta \times \breve{D}\right) \cup \left(Y_0^\delta \times \breve{D}\right) = X \times \breve{D} \notin B \end{aligned}$$

since $V(X^*, \alpha) \neq D$. So, $X \neq \breve{D}$.

In the sequel we suppose, that $|X \setminus \breve{D}| \geq 1$.

For the binary relations σ_i $(i = 1, 2, 3, 4)$ we consider the following cases.

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Let $\sigma_1 = \begin{pmatrix} P_0 & P_1 \\ Z_1 & Z_1 \end{pmatrix}$. In this case we have $P_0 \neq \varnothing$ and $P_1 \neq \varnothing$. If $\bar{\sigma}_1$ be a mapping of the set $X \setminus \breve{D}$ on the set $D \setminus \{Z_1\} = \{\breve{D}\}$ (by preposition $\left|X \setminus \breve{D}\right| \geq 1$), then

$$\bar{\beta} = \left(\left(P_0 \cup P_1 \right) \times Z_1 \right) \cup \bigcup_{t' \in X \setminus \breve{D}} \left(\{t'\} \times \bar{\sigma}_1(t') \right), \qquad \ldots (2.5)$$

$V\left(D, \bar{\beta}\right) = D$ and from the formal equality (2.1) and equalities (2.2), (2.5) follows that

$$\begin{aligned} & Z_1\bar{\beta} = P_0\bar{\beta} = Z_1, \\ & \breve{D}\bar{\beta} = P_0\bar{\beta} \cup P_1\bar{\beta} = Z_1 \cup Z_1 = Z_1, \\ & \alpha = \delta \circ \bar{\beta} = \left(Y_1^{\delta} \times Z_1\bar{\beta}\right) \cup \left(Y_0^{\delta} \times \breve{D}\bar{\beta}\right) = \\ & = \left(Y_1^{\delta} \times Z_1\right) \cup \left(Y_0^{\delta} \times Z_1\right) = X \times Z_1 \notin B \end{aligned}$$

since $V\left(X^*, \alpha\right) = \{Z_1\} \neq D$.

If $\sigma_2 = \begin{pmatrix} P_0 & P_1 \\ Z_1 & \breve{D} \end{pmatrix}$ and $\bar{\sigma}_2$ is mapping of the set $X \setminus \breve{D}$ in the semilattice D. So, if

$$\bar{\beta} = \left(P_0 \times Z_1 \right) \cup \left(P_1 \times \breve{D} \right) \cup \bigcup_{t' \in X \setminus \breve{D}} \left(\{t'\} \times \bar{\sigma}_2(t') \right), \qquad \ldots (2.6)$$

then $V\left(D, \bar{\beta}\right) = D$. From the formal equality (2.1) and equalities (2.2), (2.6) we have:

$$\begin{aligned} & Z_1\bar{\beta} = P_0\bar{\beta} = Z_1, \\ & \breve{D}\bar{\beta} = P_0\bar{\beta} \cup P_1\bar{\beta} = Z_1 \cup \breve{D} = \breve{D}, \\ & \alpha = \delta \circ \bar{\beta} = \left(Y_1^{\delta} \times Z_1\bar{\beta}\right) \cup \left(Y_0^{\delta} \times \breve{D}\bar{\beta}\right) = \\ & = \left(Y_1^{\delta} \times Z_1\right) \cup \left(Y_0^{\delta} \times \breve{D}\right) = \delta \notin B \setminus \{\alpha\}. \end{aligned}$$

But, the equality $\alpha = \delta$ contradict the condition, that $\delta \in B_X(D) \setminus \{\alpha\}$. That is in this case $\alpha \notin B$.

If $\sigma_3 = \begin{pmatrix} P_0 & P_1 \\ \breve{D} & Z_1 \end{pmatrix}$ and $\bar{\sigma}_3$ is a mapping of the set $X \setminus \breve{D}$ in the semilattice D. So, if

$$\bar{\beta} = \left(P_0 \times \breve{D} \right) \cup \left(P_1 \times Z_1 \right) \cup \bigcup_{t' \in X \setminus \breve{D}} \left(\{t'\} \times \bar{\sigma}_3(t') \right), \qquad \ldots (2.7)$$

then $V\left(D, \bar{\beta}\right) = D$. From the formal equality (2.1) and equalities (2.2), (2.7) we have:

$$\begin{aligned} & Z_1\bar{\beta} = P_0\bar{\beta} = \breve{D}, \\ & \breve{D}\bar{\beta} = P_0\bar{\beta} \cup P_1\bar{\beta} = \breve{D} \cup Z_1 = \breve{D}, \\ & \alpha = \delta \circ \bar{\beta} = \left(Y_1^{\delta} \times Z_1\bar{\beta}\right) \cup \left(Y_0^{\delta} \times \breve{D}\bar{\beta}\right) = \\ & = \left(Y_1^{\delta} \times \breve{D}\right) \cup \left(Y_0^{\delta} \times \breve{D}\right) = X \times \breve{D} \notin B \end{aligned}$$

since $V\left(X^*, \alpha\right) = \{\breve{D}\} \neq D$.

Let $\sigma_4 = \begin{pmatrix} P_0 & P_1 \\ \breve{D} & \breve{D} \end{pmatrix}$. If $\bar{\sigma}_4$ be a mapping of the set $X \setminus \breve{D}$ on the set $D \setminus \{\breve{D}\} = \{Z_1\}$, then

$$\bar{\beta} = \left(\left(P_0 \cup P_1 \right) \times \breve{D} \right) \cup \bigcup_{t' \in X \setminus \breve{D}} \left(\{t'\} \times \bar{\sigma}_4(t') \right), \qquad \ldots (2.8)$$

$V\left(D, \bar{\beta}\right) = D$ and from the formal equality (2.1) and equalities (2.2), (2.8) follows that

$$\begin{aligned} & Z_1\bar{\beta} = P_0\bar{\beta} = \breve{D}, \\ & \breve{D}\bar{\beta} = P_0\bar{\beta} \cup P_1\bar{\beta} = \breve{D} \cup \breve{D} = \breve{D}, \\ & \alpha = \delta \circ \bar{\beta} = \left(Y_1^{\delta} \times Z_1\bar{\beta}\right) \cup \left(Y_0^{\delta} \times \breve{D}\bar{\beta}\right) = \\ & = \left(Y_1^{\delta} \times \breve{D}\right) \cup \left(Y_0^{\delta} \times \breve{D}\right) = X \times \breve{D} \notin B \end{aligned}$$

since $V\left(X^*, \alpha\right) = \{\breve{D}\} \neq D$.

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So, from the cases $b)$ follows that B is a set external elements of semigroup $B_X(D)$ since the mappings $\sigma_1 - \sigma_4$ are all mappings of the set $\{P_0, P_1\}$ in the semilattice D.

Lemma 2.1 is proved.

Corollary 2.1. *Let* $D = \{\varnothing, \breve{D}\} \in \Sigma_1(X,2)$ *and* $B = \{\alpha \in B_X(D) \mid V(X^*, \alpha) = D\}$. *Then the following statements are true:*

1) *If* $|X \setminus \breve{D}| \geq 1$ *and* $Z_1 = \varnothing$, *then* $\alpha = X \times D$ *do not generating by elements of the set* B;

2) *If* $X = \breve{D}$ *and* $Z_1 \neq \varnothing$, *then* $\alpha = X \times Z_1$ *do not generating by elements of the set* B;

3) *If* $X = \breve{D}$ *and* $Z_1 = \varnothing$, *then* $\alpha = \varnothing$ *and* $\alpha = X \times \breve{D}$ *do not generating by elements of the set* B.

Proof. Let $|X \setminus \breve{D}| \geq 1$, $Z_1 = \varnothing$ and $\delta, \beta \in B$. Then quasinormal representation of a binary relation δ has a form $\delta = (Y_1^\delta \times \varnothing) \cup (Y_0^\delta \times \breve{D})$, where $Y_1^\delta, Y_0^\delta \notin \{\varnothing\}$ and

$$\delta \circ \beta = (Y_1^\delta \times \varnothing\beta) \cup (Y_0^\delta \times \breve{D}\beta) = (Y_1^\delta \times \varnothing) \cup (Y_0^\delta \times \breve{D}\beta) \neq X \times \breve{D}$$

since $Y_1^\delta \neq \varnothing$.

Therefore, if $|X \setminus \breve{D}| \geq 1$ and $Z_1 = \varnothing$, then $\alpha = X \times D$ do not generating by elements of the set B.

The statement $1)$ of the Corollary 2.1 is proved.

Let $X = \breve{D}$ and $Z_1 \neq \varnothing$. If δ and β are such elements of the set B, that $\delta \circ \beta = X \times Z_1$, then quasinormal representation of a binary relation δ has a form $\delta = (Y_1^\delta \times Z_1) \cup (Y_0^\delta \times \breve{D}) = (Y_1^\delta \times Z_1) \cup (Y_0^\delta \times X)$, where $Y_1^\delta, Y_0^\delta \notin \{\varnothing\}$ and

$$\delta \circ \beta = (Y_1^\delta \times Z_1\beta) \cup (Y_0^\delta \times \breve{D}\beta) = (Y_1^\delta \times Z_1\beta) \cup (Y_0^\delta \times X\beta) = X \times Z_1,$$

i.e. $Z_1\beta = X\beta = Z_1$. Of the equality $X\beta = Z_1$ follows that $t\beta = Z_1$ for all $t \in X$ since Z_1 be smallest element of the semilattice D. So, the equality $\beta = X \times Z_1$ is true. Last equality contradict the condition $\beta \in B$ since $V(X^*, \beta) = \{Z_1\} \neq D$. Therefore, binary relation $\alpha = X \times Z_1$ do not generating by elements of the set B.

The statement $2)$ of the Corollary 2.1 is proved.

Let $X = \breve{D}$, $Z_1 = \varnothing$. If $\alpha = \delta \circ \beta$ for some $\delta, \beta \in B$, then quasinormal representations of binary relations δ and β has a form

$$\delta = (Y_1^\delta \times \varnothing) \cup (Y_0^\delta \times \breve{D}) = (Y_1^\delta \times \varnothing) \cup (Y_0^\delta \times X),$$
$$\beta = (Y_1^\beta \times \varnothing) \cup (Y_0^\beta \times \breve{D}) = (Y_1^\beta \times \varnothing) \cup (Y_0^\beta \times X),$$

where $Y_1^\delta, Y_0^\delta, Y_1^\beta, Y_0^\beta \notin \{\varnothing\}$ since $V(X^*, \delta) = V(X^*, \beta) = D$ ($\delta, \beta \in B$). So, we have:

$$\begin{aligned}
\alpha = \delta \circ \beta &= \left((Y_1^\delta \times \varnothing) \cup (Y_0^\delta \times X)\right) \circ \left((Y_1^\beta \times \varnothing) \cup (Y_0^\beta \times X)\right) = \\
&= \left((Y_1^\delta \times \varnothing) \circ (Y_1^\beta \times \varnothing)\right) \cup \left((Y_1^\delta \times \varnothing) \circ (Y_0^\beta \times X)\right) \cup \\
&\cup \left((Y_0^\delta \times X) \circ (Y_1^\beta \times \varnothing)\right) \cup \left((Y_0^\delta \times X) \circ (Y_0^\beta \times X)\right) = \varnothing \cup \varnothing \cup \\
&\cup (Y_1^\delta \times \varnothing) \cup (Y_0^\delta \times X) = \delta
\end{aligned}$$

since $X \cap Y_1^\beta = Y_1^\beta \neq \varnothing$ and $X \cap Y_0^\beta = Y_0^\beta \neq \varnothing$. Bat from the equalities $\alpha = \varnothing$, $\alpha = \delta$ or $\alpha = X \times \breve{D}$, $\alpha = \delta$ respectively follows that $\alpha \notin B$ and $\alpha \in B$ since $\delta \in B$ by preposition, i.e. $\alpha = \varnothing$ and $\alpha = X \times \breve{D}$ do not generating by elements of the set B.

The statement $3)$ of the Corollary 2.1 is proved.

The Corollary 2.1 is proved.

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Theorem 2.1. *Let $D=\{Z_1,\breve{D}\}\in\Sigma_1(X,2)$. If $B=\{\alpha\in B_X(D)\,|\,V(X^*,\alpha)=D\}$, then the following statements are true:*

a) *if $|X\setminus\breve{D}|\geq 1$ and $Z_1\neq\varnothing$, then B is irreducible generating set for the semigroup $B_X(D)$;*

b) *if $|X\setminus\breve{D}|\geq 1$ and $Z_1=\varnothing$, then $B\cup\{X\times\breve{D}\}$ is irreducible generating set for the semigroup $B_X(D)$.*

c) *if $X=\breve{D}$ and $Z_1\neq\varnothing$, then $B\cup\{X\times Z_1\}$ is irreducible generating set for the semigroup $B_X(D)$;*

d) *if $X=\breve{D}$ and $Z_1=\varnothing$, then $B\cup\{\varnothing,X\times\breve{D}\}$ is irreducible generating set for the semigroup $B_X(D)$.*

Proof. Let $|X\setminus\breve{D}|\geq 1$, $Z_1\neq\varnothing$ and α is any element of the semigroup $B_X(D)\setminus B$. Then binary relation α has a quasinormal representation of the form $\alpha=(Y_1^\alpha\times Z_1)\cup(Y_0^\alpha\times\breve{D})$, where $Y_1^\alpha=\varnothing$ or $Y_0^\alpha=\varnothing$ ($V(X^*,\alpha)\neq D$ since $\alpha\notin B$).

Let $Y_1^\alpha=\varnothing$. Then $\alpha=X\times\breve{D}$ and for any $\delta=(Y_1^\delta\times Z_1)\cup(Y_0^\delta\times\breve{D})\in B$ and for $\beta=(Z_1\times\breve{D})\cup((X\setminus Z_1)\times Z_1)$ we have $\beta\in B$ since $Z_1\neq\varnothing$ by preposition and $X\setminus\breve{D}\supset X\setminus Z_1\neq\varnothing$. So, the following equalities are hold:

$$\begin{aligned}\delta\circ\beta&=((Y_1^\delta\times Z_1)\cup(Y_0^\delta\times\breve{D}))\circ((Z_1\times\breve{D})\cup((X\setminus Z_1)\times Z_1))=\\&=((Y_1^\delta\times Z_1)\circ(Z_1\times\breve{D}))\cup((Y_1^\delta\times Z_1)\circ((X\setminus Z_1)\times Z_1))\cup\\&\cup((Y_0^\delta\times\breve{D})\circ(Z_1\times\breve{D}))\cup((Y_0^\delta\times\breve{D})\circ((X\setminus Z_1)\times Z_1))=\\&=(Y_1^\delta\times\breve{D})\cup\varnothing\cup(Y_0^\delta\times\breve{D})\cup(Y_0^\delta\times Z_1)=(Y_1^\delta\times\breve{D})\cup(Y_0^\delta\times\breve{D})=X\times\breve{D}=\alpha\end{aligned}$$

since $Z_1\subset\breve{D}$ by preposition.

Let $Y_0^\alpha=\varnothing$. Then $\alpha=X\times Z_1$ and for any $\delta=(Y_1^\delta\times Z_1)\cup(Y_0^\delta\times\breve{D})\in B$ and for $\beta=(\breve{D}\times Z_1)\cup((X\setminus\breve{D})\times\breve{D})$ we have $\beta\in B$ since $X\setminus\breve{D}\neq\varnothing$ $(|X\setminus\breve{D}|\geq 1)$. So, the following equalities are true:

$$\begin{aligned}\delta\circ\beta&=((Y_1^\delta\times Z_1)\cup(Y_0^\delta\times\breve{D}))\circ((\breve{D}\times Z_1)\cup((X\setminus\breve{D})\times\breve{D}))=\\&=((Y_1^\delta\times Z_1)\circ(\breve{D}\times Z_1))\cup((Y_1^\delta\times Z_1)\circ((X\setminus\breve{D})\times\breve{D}))\cup\\&\cup((Y_0^\delta\times\breve{D})\circ(\breve{D}\times Z_1))\cup((Y_0^\delta\times\breve{D})\circ((X\setminus\breve{D})\times\breve{D}))=\\&=(Y_1^\delta\times Z_1)\cup\varnothing\cup(Y_0^\delta\times Z_1)\cup\varnothing=(Y_1^\delta\times Z_1)\cup(Y_0^\delta\times Z_1)=X\times Z_1=\alpha\end{aligned}$$

since $\breve{D}\supset Z_1$ by preposition.

The statement $a)$ of the Theorem 2.1 is proved.

Let $|X\setminus\breve{D}|\geq 1$ and $Z_1=\varnothing$. for any $\delta=(Y_1^\delta\times\varnothing)\cup(Y_0^\delta\times\breve{D})\in B$ and for $\beta=(\breve{D}\times\varnothing)\cup((X\setminus\breve{D})\times\breve{D})$ we have $\beta\in B$ since $X\setminus\breve{D}\neq\varnothing$ $(|X\setminus\breve{D}|\geq 1)$. So, the following equalities are true:

$$\begin{aligned}\delta\circ\beta&=((Y_1^\delta\times\varnothing)\cup(Y_0^\delta\times\breve{D}))\circ((\breve{D}\times\varnothing)\cup((X\setminus\breve{D})\times\breve{D}))=\\&=((Y_1^\delta\times\varnothing)\circ(\breve{D}\times\varnothing))\cup((Y_1^\delta\times\varnothing)\circ((X\setminus\breve{D})\times\breve{D}))\cup\\&\cup((Y_0^\delta\times\breve{D})\circ(\breve{D}\times\varnothing))\cup((Y_0^\delta\times\breve{D})\circ((X\setminus\breve{D})\times\breve{D}))=\\&=\varnothing\cup\varnothing\cup(Y_0^\delta\times\varnothing)\cup\varnothing=\varnothing=\alpha.\end{aligned}$$

Now, the statement $b)$ of the Theorem 2.1 immediately follows from the statement $1)$ of the Corollary 2.1.

Let $X=\breve{D}$ and $Z_1\neq\varnothing$. If $\delta=(Y_1^\delta\times Z_1)\cup(Y_0^\delta\times\breve{D})$ be any element of the set B and $\beta=(Z_1\times\breve{D})\cup((X\setminus Z_1)\times Z_1)$. It is easy to see, that $\beta\in B$ and

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$$\delta\circ\beta=\left((Z_1\times Z_1)\cup\left((X\setminus Z_1)\times\breve{D}\right)\right)\circ\left((Z_1\times\breve{D})\cup\left((X\setminus Z_1)\times Z_1\right)\right)=$$
$$=\left((Z_1\times Z_1)\circ(Z_1\times\breve{D})\right)\cup\left((Z_1\times Z_1)\circ\left((X\setminus Z_1)\times Z_1\right)\right)\cup$$
$$\cup\left(\left((X\setminus Z_1)\times\breve{D}\right)\circ(Z_1\times\breve{D})\right)\cup\left(\left((X\setminus Z_1)\times\breve{D}\right)\circ\left((X\setminus Z_1)\times Z_1\right)\right)=$$
$$=(Z_1\times\breve{D})\cup\varnothing\cup\left((X\setminus Z_1)\times\breve{D}\right)\cup\left((X\setminus Z_1)\times Z_1\right)=$$
$$=(Z_1\times\breve{D})\cup\left((X\setminus Z_1)\times\breve{D}\right)=X\times\breve{D}$$

since $Z_1\subset\breve{D}$ by preposition.

Now, the statement $c)$ of the Theorem 2.1 immediately follows from the statement $2)$ of the Corollary 2.1.

The statement $d)$ of the Teorem 2.1 immediately follows from the statement $3)$ of the Corollary 2.1.

The Theorem 2.1 is proved.

Theorem 2.2. *Let X be finite a set, $D=\{Z_1,\breve{D}\}\in\Sigma_1(X,2)$.If $|X|=n$, then for the number of the irreducible generated set B' of the semigroup $B_X(D)$ following statements are true:*

a) *if $|X\setminus\breve{D}|\geq 1$ and $Z_1\neq\varnothing$, then $|B'|=2^n-2$;*

b) *if $|X\setminus\breve{D}|\geq 1$ and $Z_1=\varnothing$ or $X=\breve{D}$ and $Z_1\neq\varnothing$, then $|B'|=2^n-1$;*

d) *if $X=\breve{D}$ and $Z_1=\varnothing$, then $|B'|=2^n$.*

Proof. It is well know, that if B is all external elements of the semigroup $B_X(D)$ and B' be any generated set for the $B_X(D)$, then $B\subseteq B'$. By Lemma 2.1 The set $B=\{\alpha\in B_X(D)\,|\,V(X^*,\alpha)=D\}$ is a set external elements of the semigroup $B_X(D)$. It is easy to see, that $B=B_X(D)\setminus\{X\times Z_1,X\times\breve{D}\}$. Of this follows that $|B|=2^n-2$ (see 1.1).

By statement $a)$ of the Theorem 2.1 follows that $B=B'$, i.e. $|B'|=2^n-2$.

By statement $b)$ and $c)$ of the Theorem 2.1 follows that $B'=B\cup\{X\times\breve{D}\}$ or $B'=B\cup\{X\times Z_1\}$, i.e. $|B'|=(2^n-2)+1=2^n-1$.

By statement $d)$ of the Theorem 2.1 follows that $B'=B\cup\{X\times Z_1,X\times\breve{D}\}$, i.e. $|B'|=(2^n-2)+2=2^n$.

Theorem 2.2 is proved.

Example 2.1. Let $X=\{1,2,3\}$ and $D=\{\{1\},\{1,2\}\}$, i.e. $|X\setminus\breve{D}|=1$ and $Z_1=\{1\}\neq\varnothing$ (see statement $a)$).

Then $B_X(D)=\{\alpha_1,\alpha_2,...,\alpha_7,\alpha_8\}$, where

$$\alpha_1=\{(1,1),(2,1),(3,1)\},\ \alpha_2=\{(1,1),(2,1),(3,1),(3,2)\},$$
$$\alpha_3=\{(1,1),(2,1),(2,2),(3,1)\},\ \alpha_4=\{(1,1),(2,1),(2,2),(3,1),(3,2)\},$$
$$\alpha_5=\{(1,1),(1,2),(2,1),(3,1)\},\ \alpha_6=\{(1,1),(1,2),(2,1),(3,1),(3,2)\},$$
$$\alpha_7=\{(1,1),(1,2),(2,1),(2,2),(3,1)\},\ \alpha_8=\{(1,1),(1,2),(2,1),(2,2)(3,1),(3,2)\}.$$

In this case we have $B=\{\alpha_2,\alpha_3,\alpha_4,\alpha_5,\alpha_6,\alpha_7\}$ and

$\circ$	α_2	α_3	α_4	α_5	α_6	α_7
α_2	α_1	α_2	α_2	α_8	α_8	α_6
α_3	α_1	α_3	α_3	α_8	α_8	α_8
α_4	α_1	α_4	α_4	α_8	α_8	α_8
α_5	α_1	α_5	α_5	α_8	α_8	α_8
α_6	α_1	α_6	α_6	α_8	α_8	α_8
α_7	α_1	α_7	α_7	α_8	α_8	α_8

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So, we have that $B=\{\alpha_2,\alpha_3,\alpha_4,\alpha_5,\alpha_6,\alpha_7\}$ is irreducible generated set for the semigroup $B_X(D)$.

Example 2.2. Let $X=\{1,2,3\}$ and $D=\{\varnothing,\{1,2\}\}$, i.e. $|X\setminus\breve{D}|=1$ and $Z_1=\varnothing$ (see statement b)). Then $B_X(D)=\{\alpha_1,\alpha_2,...,\alpha_7,\alpha_8\}$, where

$$\begin{aligned}
&\alpha_1=\varnothing,\ \alpha_2=\{(3,1),(3,2)\},\ \alpha_3=\{(2,1),(2,2)\},\\
&\alpha_4=\{(2,1),(2,2),(3,1),(3,2)\},\ \alpha_5=(1,1),(1,2),\\
&\alpha_6=\{(1,1),(1,2),(3,1),(3,2)\},\ \alpha_7=\{(1,1),(1,2),(2,1),(2,2)\},\\
&\alpha_8=\{(1,1),(1,2),(2,1),(2,2)(3,1),(3,2)\}.
\end{aligned}$$

In this case we have $B=\{\alpha_2,\alpha_3,\alpha_4,\alpha_5,\alpha_6,\alpha_7\}\cup\{\alpha_8\}$ and

$\circ$	α_2	α_3	α_4	α_5	α_6	α_7	α_8
α_2	α_1	α_2	α_2	α_2	α_2	α_2	α_2
α_3	α_1	α_3	α_3	α_3	α_3	α_3	α_3
α_4	α_1	α_4	α_4	α_4	α_4	α_4	α_4
α_5	α_1	α_5	α_5	α_5	α_5	α_5	α_5
α_6	α_1	α_6	α_6	α_6	α_6	α_6	α_6
α_7	α_1	α_7	α_7	α_7	α_7	α_7	α_7
α_8	α_1	α_8	α_8	α_8	α_8	α_8	α_8

So, we have that $B=\{\alpha_2,\alpha_3,\alpha_4,\alpha_5,\alpha_6,\alpha_7\}\cup\{\alpha_8\}$ is irreducible generated set for the semigroup $B_X(D)$.

Example 2.3. Let $X=\{1,2\}$ and $D=\{\{1\},\{1,2\}\}$, i.e. $X=\breve{D}=\{1,2\}$ and $Z_1=\{1\}\neq\varnothing$ (see statement b)). Then $B_X(D)=\{\alpha_1,\alpha_2,\alpha_3,\alpha_4\}$, where

$$\begin{aligned}
&\alpha_1=\{(11),(2,1)\},\alpha_2=\{(11),(2,1),(2,2)\},\\
&\alpha_3=\{(11),(1,2),(2,1)\},\\
&\alpha_4=\{(11),(1,2),(2,1),(2,2)\}.
\end{aligned}$$

In this case we have $B=\{\alpha_2,\alpha_3\}\cup\{\alpha_1\}$ and

$\circ$	α_1	α_2	α_3
α_1	α_1	α_1	α_4
α_2	α_1	α_2	α_4
α_3	α_1	α_3	α_4

So, we have that $B=\{\alpha_2,\alpha_3\}\cup\{\alpha_1\}$ is irreducible generated set for the semigroup $B_X(D)$.

Example 2.4. Let $X=\{1,2\}$ and $D=\{\varnothing,\{1,2\}\}$, i.e. $X=\breve{D}=\{1,2\}$ and $Z_1=\varnothing$ (see statement d)). Then $B_X(D)=\{\alpha_1,\alpha_2,\alpha_3,\alpha_4\}$, where

$$\begin{aligned}
&\alpha_1=\varnothing,\ \alpha_2=\{(2,1),(2,2)\},\\
&\alpha_3=\{(1,1),(1,2)\},\\
&\alpha_4=\{(1,1),(1,2),(2,1),(2,2)\}.
\end{aligned}$$

In this case we have $B=\{\alpha_2,\alpha_3\}\cup\{\alpha_1,\alpha_4\}$ and

$\circ$	α_1	α_2	α_3	α_4
α_1	α_1	α_1	α_1	α_1

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α_2	α_1	α_2	α_2	α_2
α_3	α_1	α_3	α_3	α_3
α_4	α_1	α_4	α_4	α_4

So, we have that $B=\{\alpha_2,\alpha_3\}\cup\{\alpha_1,\alpha_4\}$ is irreducible generated set for the semi group $B_X(D)$.

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