Some sufficient conditions for spirallike functions with argument properties

Muhammad Arif[1], Mohsan Raza[2], Saeed Islam[1], Javed Iqbal[1], Faiz Faizullah[3]

[1]Department of Mathematics, Abdul Wali Khan University Mardan, KPK, Pakistan
[2]Department of Mathematics, GC University Faisalabad, Punjab, Pakistan
[3]College of Electrical and Mechanical Engineering (EME), National University of Sciences and Technology (NUST), Islamabad, Pakistan.
marifmaths@hotmail.com (M. Arif), mohsan976@yahoo.com (M. Raza), proud_pak@hotmail.com (S. Islam)
javedmath@yahoo.com (J. Iqbal), faiz_math@yahoo.com (F. Faizullah)

Abstract. The aim of this paper is to establish certain sufficient conditions for some subclasses of analytic functions using argument properties. Some applications of our work to the generalized Alexander integral operator is also given.

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1. Introduction

Let $A(n)$ denote the class of functions $f(z)$ of the form

$$f(z) = z + \sum_{k=n+1}^{\infty} a_k z^k, \qquad (1.1)$$

which are analytic and multivalent in the open unit disk $\mathfrak{A} = \{z : |z| < 1\}$. By $S_\lambda^*(n,\alpha)$ and $C_\lambda(n,\alpha)$, λ is real with $|\lambda| < \frac{\pi}{2}$, $0 \le \alpha < 1$, $n \in \mathrm{N}$, we mean the subclasses of $A(n)$ consisting of all functions $f(z)$ of the form (1.1) which are defined, respectively, by

$$\operatorname{Re} e^{i\lambda} \frac{zf'(z)}{f(z)} > \alpha \cos\lambda, \ (z \in \mathfrak{A}), \qquad (1.2)$$

$$\operatorname{Re} e^{i\lambda} \left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha \cos\lambda, \ (z \in \mathfrak{A}). \qquad (1.3)$$

We note that for $\alpha = 0$ and $n = 1$, the above classes reduce to the class of spirallike functions introduced by Spacek [1] and the class of Robertson functions studied by Robertson [2] respectively. For more details on the subject of spirallike and Robertson functions, see [3-5].

[1]Corresponding author
E-mail: marifmaths@yahoo.com(M. Arif)

Sufficient conditions were studied by various authors for different subclasses of analytic functions, for some of the related work see [6-12]. The object of the present paper is to obtain sufficient conditions for the classe $S_\lambda^*(n,\alpha)$ and $C_\lambda(n,\alpha)$. We also consider some special cases of our results which lead to various interesting corollaries and relevances of some of these with other known results are also mentioned. We will assume throughout our discussion, unless otherwise stated, that λ is real with $|\lambda| < \frac{\pi}{2}$, $0 \le \alpha < 1$ and $n \in \mathrm{N}$.

We need the following lemma due to Mocanu [13].

Lemma 1.1. If $p(z) \in A(n)$ satisfies the condition

$$|\arg p'(z)| < \frac{\pi}{2}\delta_n \quad (z \in \mathfrak{A}),$$

where δ_n is the unique root of the equation

$$2\tan^{-1}\left[n(1-\delta_n)\right] + \pi(1-2\delta_n) = 0, \qquad (1.4)$$

then $p(z) \in S^*(n,0)$.

2. Main Results

Theorem 2.1. If $f(z) \in A(n)$ satisfies

$$\left| e^{i\lambda} \arg\left(\frac{f(z)}{z}\right) + \left((1-\alpha)\cos\lambda\right) \arg\left\{ e^{i\lambda} \frac{zf'(z)}{f(z)} - \alpha\cos\lambda - i\sin\lambda \right\} \right| < \frac{\pi}{2}\, \delta_n\ (1-\alpha)\cos\lambda\ \ (z \in \mathfrak{A}), \qquad (2.1)$$

where δ_n is the unique root of (1.4), then

$$f(z) \in S_\lambda^*(n, \alpha).$$

Proof. Let us set

$$p(z) = z\left(\frac{f(z)}{z}\right)^{\frac{e^{i\lambda}}{(1-\alpha)\cos\lambda}}$$

$$= z + \frac{e^{i\lambda}\ a_{n+1}}{(1-\alpha)\cos\lambda} z^{n+1} + \ldots \qquad (2.2)$$

for $f(z) \in A(n)$. Then clearly (2.2) shows that $p(z) \in A(n)$.

Differentiating (2.2), we have

$$p'(z) = \left(\frac{f(z)}{z}\right)^{\frac{e^{i\lambda}}{(1-\alpha)\cos\lambda}} \left\{ \frac{e^{i\lambda}}{(1-\alpha)\cos\lambda}\left[\frac{zf'(z)}{f(z)} - 1\right] + 1 \right\} \qquad (2.3)$$

which gives

$$|\arg p'(z)| = \left| \arg\left(\frac{f(z)}{z}\right)^{\frac{e^{i\lambda}}{(1-\alpha)\cos\lambda}} + \arg\left\{ e^{i\lambda} \frac{zf'(z)}{f(z)} - \alpha\cos\lambda - i\sin\lambda \right\} \right|.$$

Thus using (2.1), we have

$$|\arg p'(z)| \le \frac{\pi}{2}\, \delta_n\ \left(z \in \mathfrak{A}\right),$$

where δ_n is the root of (1.4). Hence, using Lemma 1.1, we have $p(z) \in S^*(n, 0)$.

From (2.3), we can write

$$\frac{zp'(z)}{p(z)} = \frac{1}{(1-\alpha)\cos\lambda}\left[e^{i\lambda} \frac{zf'(z)}{f(z)} - (\alpha\cos\lambda + i\sin\lambda) \right].$$

Since $p(z) \in S^*(n, 0)$, it implies that $\mathrm{Re}\frac{zp'(z)}{p(z)} > 0$. Therefore, we get

$$\frac{1}{(1-\alpha)\cos\lambda}\left[\mathrm{Re}\left(e^{i\lambda} \frac{zf'(z)}{f(z)}\right) - \alpha\cos\lambda\right] = \mathrm{Re}\frac{zp'(z)}{p(z)} > 0$$

or

$$\mathrm{Re}\left(e^{i\lambda} \frac{zf'(z)}{f(z)}\right) > \alpha\cos\lambda.$$

and this implies that $f(z) \in S_\lambda^*(n, \alpha)$.

Making $\lambda = 0$ in Theorem 2.1, we have

Corollary 2.2. If $f(z) \in A(n)$ satisfies

$$\left| \arg\left(\frac{f(z)}{z}\right) + (1-\alpha)\arg\left\{\frac{zf'(z)}{f(z)} - \alpha\right\} \right| < \frac{\pi}{2}\, \delta_n\ (1-\alpha)\ \ (z \in \mathfrak{A}),$$

then $f(z) \in S^*(n, \alpha)$.

Further if we take $n = 1$ in Corollary 2.2, we get the following result proved by Uyanik et al [12].

Corollary 2.3. If $f(z) \in A$ satisfies

$$\left| \arg\left(\frac{f(z)}{z}\right) + (1-\alpha)\arg\left\{\frac{zf'(z)}{f(z)} - \alpha\right\} \right| < \frac{\pi}{2}\, \delta_1\ (1-\alpha),$$

where δ_1 is the unique root of the equation

$$2\tan^{-1}\left[(1-\delta_1)\right] + \pi(1 - 2\delta_1) = 0,$$

then $f(z)$ belongs to the class of starlike functions of order α.

Theorem 2.4. If $f(z) \in A$ satisfies

$$\left| e^{i\lambda} \arg\left(f'(z)\right) + (1-\alpha)\cos\lambda \arg\left\{ e^{i\lambda}\left(\frac{zf''(z)}{f'(z)} + 1\right) - \alpha\cos\lambda - i\sin\lambda \right\} \right| < \frac{\pi}{2}\delta_n\ (1-\alpha)\cos\lambda,\ \ (z \in \mathfrak{A}),$$

where δ_n is the unique root of (1.4), then $f(z) \in C_\lambda(n, \alpha)$.

Proof. Let us set

$$p(z) = \int_0^z \left(f'(t)\right)^{\frac{e^{i\lambda}}{(1-\alpha)\cos\lambda}} dt = z + \frac{e^{i\lambda} a_{1+n}}{(1-\alpha)\cos\lambda} z^{n+1} + \ldots.$$

Also let

$$g(z) = z\left(f'(z)\right)^{\frac{e^{i\lambda}}{(1-\alpha)\cos\lambda}} = z + \frac{(n+1)e^{i\lambda} a_{n+1}}{(1-\alpha)\cos\lambda} z^{n+1} + \ldots.$$

Then clearly $p(z)$ and $g(z) \in A(n)$. Now

$$g(z) = z\left(f'(z)\right)^{\frac{e^{i\lambda}}{(1-\alpha)\cos\lambda}}.$$

Differentiating logarithmically and then simple computation gives us

$$|\arg g'(z)| = \left| \arg\left(f'(z)\right)^{e^{i\lambda}/(1-\alpha)\cos\lambda} + \arg\left\{ e^{i\lambda}\left(\frac{zf''(z)}{f'(z)} + 1\right) - \alpha\cos\lambda - i\sin\lambda \right\} \right| < \frac{\pi}{2}\delta_n.$$

Therefore, by using Lemma 1.1, we have

$$g(z) = zp'(z) \in S^*(n,0)$$

which implies that $p(z) \in C(n,0)$. Since

$$\frac{zp''(z)}{p'(z)} = \frac{e^{i\lambda}}{(1-\alpha)\cos\lambda}\left\{\frac{zf''(z)}{f'(z)}\right\},$$

therefore

$$\operatorname{Re}\left(1+\frac{zp''(z)}{p'(z)}\right) = \frac{1}{(1-\alpha)\cos\lambda}\left\{\operatorname{Re} e^{i\lambda}\left(1+\frac{zf''(z)}{f'(z)}\right)-\alpha\cos\lambda\right\}.$$

Since $p(z) \in C(n,0)$, So

$$\frac{1}{(1-\alpha)\cos\lambda}\left\{\operatorname{Re} e^{i\lambda}\left(1+\frac{zf''(z)}{f'(z)}\right)-\alpha\cos\lambda\right\} > 0,$$

or

$$\operatorname{Re} e^{i\lambda}\left(1+\frac{zf''(z)}{f'(z)}\right) > \alpha\cos\lambda.$$

It follows that $f(z) \in C_\lambda(n,\alpha)$.

Taking $\lambda = 0$ in Theorem 2.4, we have

Corollary 2.5. If $f(z) \in A(n)$ satisfies

$$\left|\arg\left(f'(z)\right) + (1-\alpha)\arg\left\{\frac{zf''(z)}{f'(z)}+1-\alpha\right\}\right| < \frac{\pi}{2}\delta_n(1-\alpha),$$

then $f(z) \in C(n,\alpha)$.

Further If we take $n = 1$ in Corollary 2.5, we get the following result proved in [12].

Corollary 2.6. If $f(z) \in A$ satisfies

$$\left|\arg f'(z) + (1-\alpha)\arg\left\{\frac{zf''(z)}{f'(z)}+1-\alpha\right\}\right| < \frac{\pi}{2}\delta_1(1-\alpha),$$

for $0 \le \alpha < 1$, then $f(z) \in C(\alpha)$, the class of convex functions of order α.

Remark 2.1. If we put $\alpha = 0$ in Corollary 2.6, we get the result proved in [13].

3. Generalized Integral Operator

For $f(z) \in A(n)$, we consider

$$G(z) = \int_0^z \left(\frac{f(t)}{t}\right)^\gamma dt$$

$$= z + \frac{\gamma\, a_{1+n}}{n+1} z^{n+1} + \dots \qquad (3.1)$$

Clearly $G(z) \in A(n)$ and when $\gamma = 1$ then (3.1) reduces to the well-known Alexander integral operator [14].

Theorem 3.1. If $\gamma \ge 1$ and $f(z) \in A(n)$ with

$$\left|\arg\left(\frac{f(z)}{z}\right)^\gamma + \arg\left\{\gamma\left(\frac{zf'(z)}{f(z)}-1\right)+1\right\}\right| < \frac{\pi}{2}\delta_n, \qquad (3.2)$$

then $f(z) \in S^*(n,0)$.

Proof. From (3.1), we get

$$G'(z) = \left(\frac{f(z)}{z}\right)^\gamma. \qquad (3.3)$$

Differentiating (3.3), logarithmically, we get

$$\frac{zG''(z)}{G'(z)} = \gamma\left(\frac{zf'(z)}{f(z)}-1\right). \qquad (3.4)$$

Then by simple computation, we have,

$$\left|\arg\left(zG''(z)+G'(z)\right)\right| = \left|\arg\left(\frac{f(z)}{z}\right)^\gamma + \arg\left\{\gamma\left(\frac{zf'(z)}{f(z)}-1\right)+1\right\}\right| < \frac{\pi}{2}\delta_n,$$

where we have used (3.2). Therefore

$$\left|\arg\left(zG''(z)+G'(z)\right)\right| < \frac{\pi}{2}\delta_n,$$

By using Theorem 2.4 with $\alpha = 0$ and $\lambda = 0$, we have $G(z) \in C_0(n,0)$.

From (3.4), we can write

$$\operatorname{Re}\left(1+\frac{zG''(z)}{G'(z)}\right) = \gamma \operatorname{Re}\frac{zf'(z)}{f(z)} - \gamma + 1,$$

since $G(z) \in C_0(n,0)$. Therefore we have

$$\operatorname{Re}\frac{zf'(z)}{f(z)} > \left(1-\frac{1}{\gamma}\right),$$

which shows $f(z) \in S^*(n,0)$, where ☹¤ 1.

Conclusion:

In this paper we established certain sufficient conditions for some subclasses of analytic functions using argument properties. We also gave some applications of our work to the generalized Alexander integral operator..

In future work, the formal approach will be used to develop logical connection between analytic function and formal specification. Formal specification are the mathematical approaches used for many applications [15-26].

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