

# The Generalized tanh-coth Method

## Applied to Biological Model

### Referent to Nano-solitons of Ionic Wave

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#### Abstract

Based on two previous reports related with soliton-solutions for two equations: the first a biological model and the other a physic model, we consider the generalized tanh-coth method for obtain exact solutions for the first in a more general form that those obtained in previous reports. Additionally, we solve a second nonlinear equation whose relevance consists on the fact that solutions of it can be used for obtain solutions of the first and to other equation related with nano-ionic currents associated with transmission lines. Finally, some conclusion are given.

**Mathematics Subject Classification:** 35C05

**Keywords:** Generalized tanh-coth method; nonlinear evolution equation; exact traveling waves solution; nonlinear transmission line; ionic wave; micro-tubule; soliton

## 1 Introduction

Many computational methods for obtain exact traveling wave solutions to nonlinear partial differential equations (NLPDE's) have been used widely in the literature referent to this theme. The search of new methods or generalized methods is a very relevant task due to the availability of computer symbolic systems and the importance of this line of investigation. The generalized tanh-coth method [1] [2] have been used to solve some NLPDE's showing it effectiveness and simple description. Recently, the authors in [3] have derived two nonlinear equations in two different fields, whose solutions are considered as nano-solitons. We will use the generalized tanh-coth method for solve these two models.

The main idea of this work, is to obtain exact solutions for the following two equations

$$U_{tt}(x, t) - ghU_{xx} + \frac{1}{2h}(U_t)_x^2 = \frac{1}{3}h^2U_{ttxx}, \quad (1)$$

and

$$\frac{d^2w}{d\xi^2} = aw^2 + bw + c, \quad (2)$$

with  $a, b, c$  are nonzero constants,  $h$  is a depth of certain inviscid, incompressible and non-rotating flow of fluid and  $g$  is the gravitational acceleration  $U = U(x, t)$ ,  $w = w(\xi)$  are unknown functions. The equation (1) have relevance in Biology. More exactly, in the reference [3], the authors have realized an analogy between giant ocean solitons and the cellular ionic nano-solitons. They have obtained (1) as the final model which can be used to modeling a nonlinear ionic wave along micro-tubules in living cells. The model given by equation (2) have very near relation with (1) as we will see below and can be derived as the final step in the analysis of a model of MT as nonlinear transmission lines. The solution of the two models have been made using elliptic functions by the authors in [3]. Additionally, the two models described by (1) and (2) were analyzed by the authors in the paper [4] using the extended Jacobian elliptic function expansion method. In follows, we will use the generalized tanh-coth method [1],[2] for solve the two models. The paper is organized as follows: In Sec. 2, we solve (1) using the generalized tanh-coth method. In Sec. 3, we use the same method for obtain exact solutions to (2). Finally, we give a conclusions where we compare the obtained solutions with those derived for the respective authors of the mentioned papers ([3], [4]).

## 2 Exact solution to (1)

We use the wave transformation

$$\begin{cases} U(x, t) = u(\xi), \\ \xi = x - \lambda t + \xi_0, \end{cases} \quad (3)$$

where  $\lambda$  is the speed of the wave and  $\xi_0$  an arbitrary constant. By substituting (3) into (1) and after one integration with respect to  $\xi$  we obtain the following ordinary differential equation

$$-\frac{\lambda^2 h^2}{3} u'''(\xi) + \frac{\lambda^2}{2h} (u'(\xi))^2 + (\lambda^2 - hg)u'(\xi) + k = 0, \quad (4)$$

with  $k$  the integration constant.

Using the idea of the improved generalized tanh-coth method [1],[2], we seek the solution to (4) using the expansion

$$u(\xi) = \sum_{i=0}^M a_i \phi(\xi)^i + \sum_{i=M+1}^{2M} a_i \phi(\xi)^{M-i}, \quad (5)$$

where  $M$  is a positive integer that will be determined by using the balance method and  $\phi = \phi(\xi)$  satisfies the Riccati equation

$$\phi'(\xi) = \alpha + \beta\phi(\xi) + \gamma\phi(\xi)^2. \quad (6)$$

The  $a_i$ ,  $i = 1, 2, \dots, 2M$ ,  $\alpha$ ,  $\beta$ ,  $\gamma$  are constants to be determined later and the solution of (6) in the case  $\beta^2 - 4\alpha\gamma \neq 0$  is given by (see [5]):

$$\phi(\xi) = \frac{-\sqrt{\beta^2 - 4\alpha\gamma} \tanh\left[\frac{1}{2}\sqrt{\beta^2 - 4\alpha\gamma}\xi + \xi_0\right] - \beta}{2\gamma}. \quad (7)$$

Substituting (5) into (4), and balancing  $u'''(\xi)$  with  $(u'(\xi))^2$  we obtain

$$M + 3 = 2(M + 1),$$

so that

$$M = 1.$$

Therefore, (5) reduces to

$$u(\xi) = a_0 + a_1\phi(\xi) + (t)(\phi(\xi))^{-1}. \quad (8)$$

Now, substituting (8) into (4), taking in account (6) and equating to zero the coefficients of all powers of  $\phi(\xi)$ , we get a set of algebraic equations for  $a_0$ ,  $a_1$ ,  $a_2$ ,  $\alpha$ ,  $\beta$ ,  $\gamma$  and  $k$ . Solving it with aid the *Mathematica* we obtain a lot of solutions from which we consider only the following

$$a_0 = a_0; \quad a_1 = 4h^3\gamma; \quad a_2 = \frac{3(-gh + h\lambda^2)}{4\gamma\lambda^2}; \quad k = 0, \quad (9)$$

which give us the more general expression. Using (9), (8), (3), and reversing the substitutions we have the following general solution to (1):

$$U(x, t) = u(\xi) = a_0 + 4h^3\gamma\phi(\xi) + \frac{3(-gh^2 + h\lambda^2)}{4\gamma\lambda^2}\phi(\xi)^{-1}, \quad (10)$$

where  $\phi(\xi) = -\frac{\sqrt{3}\sqrt{-\frac{gh-\lambda^2}{h^2\lambda^2}} \tanh\left(\frac{1}{4}\sqrt{3}\sqrt{-\frac{gh-\lambda^2}{h^2\lambda^2}}\xi\right)}{4\gamma}$ ,  $\xi = x - \lambda t + x_0$  and  $a_0, \lambda, \gamma$  are arbitrary constants.

### 3 Exact solution for equation (2)

Substituting (5) into (2) and balancing  $w''(\xi)$  with  $u^2(\xi)$  we have  $M+2 = 2M$ . We obtain the relation  $M = 2$ , and then (5) take the form

$$w(\xi) = a_0 + a_1\phi(\xi) + a_2\phi^2(\xi) + a_3\phi(\xi)^{-1} + a_4\phi^{-2}(\xi). \quad (11)$$

Substituting (11) into (2), using (6) and after simplifications we have a equation in the variable  $\phi(\xi)$ . Equating to zero the coefficients of all powers of  $\phi(\xi)$ , we get a set of algebraic equations for  $a_0, a_1, a_2, a_3, a_4, \alpha, \beta, \gamma$  and  $c$ . Solving the system with respect to these unknowns with aid of *Mathematica* we obtain a big set of solutions. However, for sake of simplicity, we consider the following two, which give us the more general expressions:

$$\begin{cases} a_0 = -\frac{2\sqrt{b^2-4ac}-3\beta^2+b}{2a}, & a_1 = \frac{6\beta\gamma}{a}, & a_2 = \frac{6\gamma^2}{a}, & \alpha = \frac{\beta^2\gamma-\sqrt{b^2\gamma^2-4ac\gamma^2}}{4\gamma^2}, \\ a_3 = 0, & a_4 = 0, \end{cases} \quad (12)$$

and

$$\begin{cases} a_0 = \frac{-2b-\sqrt{b^2-4ac}}{4a}, & a_1 = a_3 = 0, & a_2 = \frac{6\gamma^2}{a}, \\ a_4 = \frac{3(b^2-4ac)}{128a\gamma^2}, & \gamma = -\frac{b^2-4ac}{16\beta}, & \beta = 0, \end{cases} \quad (13)$$

Respect to (12), in accordance with (7) and (11) we have the following solution to (2),

$$w(\xi) = -\frac{2\sqrt{b^2-4ac}-3\beta^2+b}{2a} + \frac{6\beta\gamma}{a}\phi(\xi) + \frac{6\gamma^2}{a}\phi^2(\xi), \quad (14)$$

where

$$\phi(\xi) = -\frac{\sqrt[4]{b^2-4ac} \tanh\left(\frac{1}{2}\sqrt[4]{b^2-4ac}\xi\right) + \beta}{2\gamma},$$

$\gamma, \beta$  and  $c$  arbitrary constants.

Now, respect to (13), in accordance with (7) and (11) we have the following solution to (2),

$$w(\xi) = \frac{-2b - \sqrt{b^2 - 4ac}}{4a} + \frac{6\gamma^2}{a}\phi(\xi)^2 + \frac{3(b^2 - 4ac)}{128a\gamma^2}\phi(\xi)^{-2}, \quad (15)$$

where

$$\phi(\xi) = -\frac{\sqrt[4]{b^2 - 4ac} \tanh\left(\frac{1}{4}\sqrt[4]{b^2 - 4ac}\xi\right)}{4\gamma},$$

$\gamma$  and  $c$  arbitrary constants.

## 4 Conclusions

Using the generalized tanh-coth method, we have obtained solutions to (1) and (2). We can see that (4) can be converted to (2) if we use the change of variable  $w(\xi) = u'(\xi)$  and redefining  $a = \frac{3}{2h^3}$  and  $b = \frac{3(\lambda^2 - gh)}{\lambda^2 h^2}$ ,  $k = c$ . Clearly, the solution obtained here for (2) is more general than those obtained in [4], which is more general than those obtained in [3]. On the other hand, from (14) and (15) we can obtain others (different and more general) solutions to (1). The importance of the model given by (1) is that it is associated with one related with nonlinear ionic waves along micro-tubules in living cells (see [3]). The relevance of the model given by (2) consists in its relation with (1) as we mentioned previously. Moreover, as can be seen in [3], (2) has to do with the solutions of a model of a MT as nonlinear transmission line (Eq. (51) in [3]). More exactly, solutions of the model

$$U_x + AU_{xxx} + BU + Cu_t + DUU_t = 0,$$

which correspond to transmission lines for suitable constant  $A, B, C, D$ , can be derived from the solutions obtained here for (2). For more details respect to this last model we refer to [3].

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