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Bulk Arrival Multi-Stage Retrial Queue with Vacation

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Abstract

Bulk arrival retrial queueing model with M stages of service and Bernoulli vacation is considered. Customers arrive in batches according to Poisson process. Server provides M stages of heterogeneous service in succession to each customer. After receiving service from one stage, the customer may either move to next stage or depart the system. If the customer leaves the system, the server takes Bernoulli vacation. The vacation time, service time and retrial time are assumed to be generally distributed. Using generating function technique, the steady state distributions of the server state and the number of customers in the orbit are obtained along with important performance measures. Stochastic decomposition law is verified. The effects of various parameters on the system performance are analysed numerically.

Keywords: Bulk arrival retrial queueing model

1. Introduction

Non availability of the server for primary customers at occasional intervals of time is called vacation. Allowing server to take vacations makes the queueing model more realistic and flexible. Applications arise naturally in call centres with multi task employees, customized manufacturing firms, telecommunication and computer networks etc.

Batch arrival retrial queues with vacation were analysed by many researchers. Aissani (2000) considered bulk arrival queue with exhaustive vacation. Krishna Kumar et al. (2002) studied $M^X/G/1$ retrial queue with Bernoulli schedule. Choudhury (2007) and Senthil Kumar and Arumuganathan (2008) investigated a single server batch arrival retrial queueing system with two phases of heterogeneous service and Bernoulli vacation. Aissani (2009) analysed $M^X/G/1$ energetic retrial queue with vacation. Chang and Ke (2009) discussed a batch arrival retrial queue under modified vacation policy. Ke et al. (2010) developed bulk arrival retrial queueing models with N policy and at most J vacations. Choudhury and Deka (2013) analysed various decomposition properties for batch arrival retrial queue with two phases of service and Bernoulli schedule. Maurya (2013) presented a paper on maximum entropy analysis of $M^X/(G_1,G_2)/1$ retrial queue with second phase optional service and Bernoulli vacation schedule.

There are real life situations where the arriving customer passes successively through several distinct phase service channels in order to complete his service. Shahkar and Badamchi Zadeh (2006) analysed multi phase queueing model with vacation. Salehirad and Badamchi Zadeh (2009) and Abdollahi and Salehirad (2012) studied the multi phase M/G/1 queueing system with random feedbacks. Ebenesar Anna Bagyam and Udaya Chandrika (2013a, 2013b) analysed multi stage retrial queueing model. Bulk arrival multi - stage retrial queueing system with vacation is analysed in this paper.

2. Model Description

Consider a single server retrial queueing system with multi-stages of heterogeneous service. The primary units arrive in batch according to Poisson process with rate λ. It is assumed that at every arrival epoch, a batch of k units arrives with probability C_k, k = 1, 2, ... Let $C(z) = \sum_{k=1}^{\infty} C_k z^k$ be the generating function of the batch size distribution with first two moments m_1 and m_2. If the server is idle at the arrival epoch, then one of the customers starts its service and others join the retrial group. Otherwise all the customers join the retrial group.

The server provides M stages of heterogeneous service in succession. The first stage service is compulsory for all the customers. After completion of first stage, the customer may move to second stage with probability q_1 or depart the system with its complementary probability $\bar{q}_1$. After second stage service completion, the customer may opt third stage service with probability q_2 or leave the system with probability $\bar{q}_2$ and so on upto M^{th} stage. At completion of M^{th} stage service, the customer departs the system. If the customer leaves the system, after receiving i^{th} $(1 \leq i \leq M)$ stage service, then the server may take a vacation with probability ν_i or remain in the system with probability $\bar{\nu}_i$. The schematic representation of this model is shown in Fig. 1.

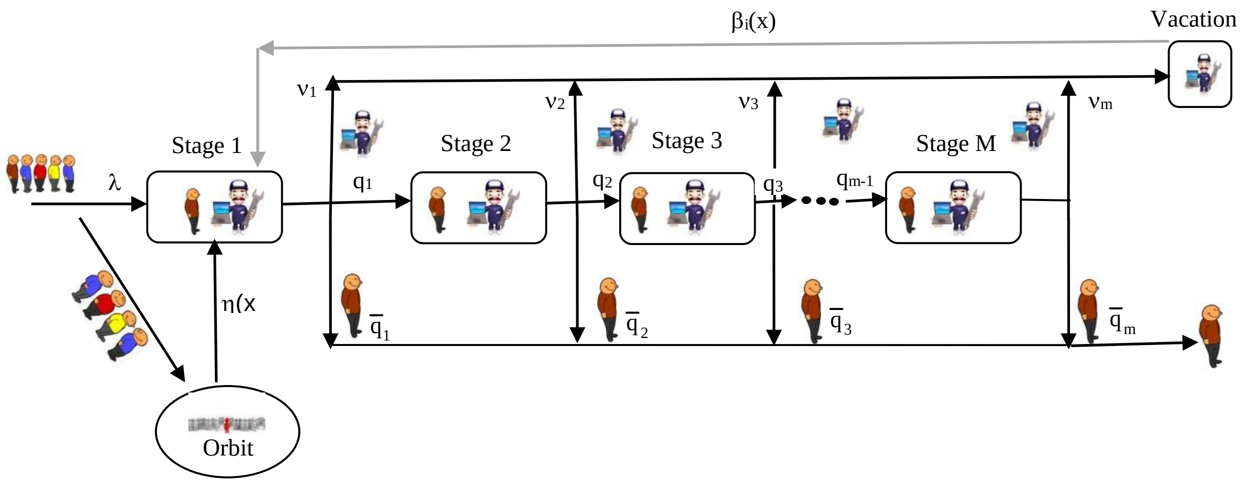

Fig. 1 Bulk Arrival Multi-Stage Retrial Queue with Vacation

The service times, retrial times and vacation times are arbitrarily distributed with distribution function, Laplace-Stieltjes transform, hazard rate function and moments as given in Table 1.

Table 1 Notations

Time	Probability distribution function	Laplace Stieltjes transform	Hazard rate function	n^{th} moment
Retrial	$A(x)$	$A^*(s)$	$\eta(x)$	-
i^{th} stage service $(i = 1, 2, \ldots, M)$	$B_i(x)$	$B_i^*(s)$	$\mu_i(x)$	$\mu_i^{(n)}$
After i^{th} stage service $(i = 1, 2, \ldots, M)$ vacation	$V_i(x)$	$V_i^*(s)$	$\beta_i(x)$	$\beta_i^{(n)}$

Let X (t) be the number of customers in the retrial queue at time t. The server state J (t) is defined as

$$J(t) = \begin{cases} 0 & \text{if the server is idle} \\ i & 1 \le i \le M, \text{if the server is busy with } i^{th} \text{ stage service} \\ M+i & 1 \le i \le M, \text{if the server is on vacation after } i^{th} \text{ stage service} \end{cases}$$

Supplementary variable $\xi(t)$ is defined as

$$\xi(t) = \begin{cases} \text{elapsed retrial time,} & \text{if } J(t) = 0 \\ \text{elapsed service time,} & \text{if } J(t) = i, 1 \le i \le M \\ \text{elapsed vacation time,} & \text{if } J(t) = M+i, 1 \le i \le M \end{cases}$$

The stochastic behaviour of this retrial queueing system can be described as the Markov process $\{N(t); t \ge 0\} = \{J(t), X(t), \xi(t)\}$.

The stability condition for the system under consideration is

$$\lambda\, m_1 \sum_{i-1}^{M} \bar{q}_i\, Q_{i-1}\, (E(S_i) + \nu_i\, \beta_i^{(1)}) + m_1\, (1 - A^*(\lambda)) < 1$$

where $E(S_i) = \sum_{j=1}^{i} \mu_j^{(1)}, Q_i = q_1\, q_2\, q_3 \ldots\, q_i$ with $\bar{q}_M = 1$ and $Q_0 = 1$.

3. Steady State Distribution

By the method of supplementary variable technique, we obtain the following system of steady state equations that governs the dynamics of the system behaviour.

$$\lambda I_0 = \sum_{i=1}^{M} \bar{q}_i\, \bar{\nu}_i \int_0^\infty W_{i,0}(x)\, \mu_i(x)\, dx + \sum_{i=1}^{M} \int_0^\infty V_{i,0}(x)\, \beta_i(x)\, dx \quad (1)$$

$$\frac{d I_n(x)}{dx} = -(\lambda + \eta(x))\, I_n(x), \quad n \ge 1 \quad (2)$$

$$\frac{d W_{i,n}(x)}{dx} = -(\lambda + \mu_i(x))\, W_{i,n}(x) + \lambda\, (1 - \delta_{0n}) \sum_{k=1}^{n} C_k\, W_{i,n-k}(x), \quad n \ge 0;\ i = 1, 2, \ldots, M \quad (3)$$

$$\frac{d V_{i,n}(x)}{dx} = -(\lambda + \beta_i(x))\, V_{i,n}(x) + \lambda\, (1 - \delta_{0n}) \sum_{k=1}^{n} C_k\, V_{i,n-k}(x), \quad n \ge 0;\ i = 1, 2, \ldots, M \quad (4)$$

$$I_n(0) = \sum_{i=1}^{M} \bar{q}_i\, \bar{\nu}_i \int_0^\infty W_{i,n}(x)\, \mu_i(x)\, dx + \sum_{i=1}^{M} \int_0^\infty V_{i,n}(x)\, \beta_i(x)\, dx, \quad n \ge 1 \quad (5)$$

$$W_{1,0}(0) = \lambda C_1 I_0 + \int_0^\infty I_1(x)\,\eta(x)\,dx \tag{6}$$

$$W_{1,n}(0) = \lambda C_{n+1} I_0 + \int_0^\infty I_{n+1}(x)\,\eta(x)\,dx + \lambda \sum_{k=1}^{n} C_k \int_0^\infty I_{n-k+1}(x)\,dx, \quad n \geq 0 \tag{7}$$

$$W_{i,n}(0) = q_{i-1} \int_0^\infty W_{i-1,n}(x)\,\mu_{i-1}(x)\,dx, \qquad n \geq 0\,; \; i = 2, 3, \ldots, M \tag{8}$$

$$V_{i,n}(0) = \bar{q}_i \nu_i \int_0^\infty W_{i,n}(x)\,\mu_i(x)\,dx, \qquad n \geq 0\,; \; i = 1, 2, 3, \ldots, M \tag{9}$$

To solve equations (1) to (9) we define the generating functions

$$I(z, x) = \sum_{n=1}^{\infty} I_n(x)\, z^n$$

$$W_i(z, x) = \sum_{n=0}^{\infty} W_{i,n}(x)\, z^n, \qquad i = 1, 2, \ldots, M \text{ and}$$

$$V_i(z, x) = \sum_{n=0}^{\infty} V_{i,n}(x)\, z^n, \qquad i = 1, 2, \ldots, M$$

Multiplying equations (2) to (9) by z^n with appropriate values of n, summing over n, n = 0, 1, 2, and solving we obtain

$$I(z, x) = I(z, 0)\, e^{-\lambda x} [1 - A(x)] \tag{10}$$

$$W_i(z, x) = W_i(z, 0)\, e^{-\lambda(1 - C(z))x} [1 - B_i(x)], \qquad i = 1, 2, \ldots, M \tag{11}$$

$$V_i(z, x) = V_i(z, 0)\, e^{-\lambda(1 - C(z))x} [1 - V_i(x)], \qquad i = 1, 2, \ldots, M \tag{12}$$

Applying generating function technique and using equation (10) - (12), the equation (5) - (9) yield

$$I(z, 0) = \lambda I_0 \Big[z - C(z) \sum_{i=1}^{M} \bar{q}_i\, Q_{i-1}\, \Lambda_i^*(\lambda(1 - C(z))) \big(\bar{\nu}_i + \nu_i\, V_i^*\,(\lambda(1 - C(z)))\big)\Big] / D(z) \tag{13}$$

$$W_i(z, 0) = \lambda I_0\, A^*(\lambda)\, (1 - C(z))\, Q_{i-1}\, \Lambda_{i-1}^*\,(\lambda(1 - C(z))) / D(z), \qquad i = 1, 2, 3, \ldots, M \tag{14}$$

$$V_i(z, 0) = \bar{q}_i \nu_i Q_{i-1} \lambda I_0\, A^*(\lambda)\, (1 - C(z))\, \Lambda_i^*\,(\lambda(1 - C(z))) / D(z), \qquad i = 1, 2, \ldots, M \tag{15}$$

where

$$D(z) = [A^*(\lambda) + C(z)(1 - A^*(\lambda)] \sum_{i=1}^{M} \bar{q}_i Q_{i-1} \Lambda_i^* (\lambda(1 - C(z)))(\bar{\nu}_i + \nu_i V_i^* (\lambda(1 - C(z)))) - z \quad (16)$$

Substituting the expression of I(z, 0), $W_i(z, 0)$ and $V_i(z, 0)$ in the equations (10) - (12) and integrating with respect to x from 0 to ∞. We get

$$I(z) = I_0 [1 - A^*(\lambda)] [z - C(z) \sum_{i=1}^{M} \bar{q}_i Q_{i-1} \Lambda_i^* (\lambda(1 - C(z))) (\bar{\nu}_i + \nu_i V_i^*(\lambda(1 - C(z))))] / D(z) \quad (17)$$

$$W_i(z) = I_0 A^*(\lambda) Q_{i-1} \Lambda_{i-1}^* (\lambda(1 - C(z))) [1 - B_i^* (\lambda(1 - C(z)))] / D(z) \quad i = 1, 2, ..., M \quad (18)$$

$$V_i(z) = I_0 A^*(\lambda) \bar{q}_i \nu_i Q_{i-1} \Lambda_i^* (\lambda(1 - C(z))) [1 - V_i^* (\lambda(1 - C(z)))] / D(z), \quad i = 1, 2, ..., M \quad (19)$$

The normalizing condition yields

$$I_0 = N_1 / A^*(\lambda) \quad (20)$$

where $N_1 = 1 - m_1 (1 - A^*(\lambda)) - \lambda m_1 \sum_{i=1}^{M} \bar{q}_i Q_{i-1} (E(S_i) + \nu_i \beta_i^{(1)})$ (21)

The probability generating function of the number of customers in the orbit is

$$P_q(z) = I_0 + I(z) + \sum_{i=1}^{M} [W_i(z) + V_i(z)]$$

$$= I_0 A^*(\lambda) (1 - z) / D(z) \quad (22)$$

The probability generating function of the number of customers in the system is

$$P_s(z) = I_0 + I(z) + \sum_{i=1}^{M} [z W_i(z) + V_i(z)]$$

$$= I_0 A^*(\lambda) (1 - z) \sum_{i=1}^{M} \bar{q}_i Q_{i-1} \Lambda_i^* (\lambda(1 - C(z))) / D(z) \quad (23)$$

4. Performance Measures

In this section the performance measures for the system under consideration are derived.

- The steady state probability that the system is empty is I_0.
- The steady state probability that the server is idle in the non empty system is given by

$$I = \lim_{z \to 1} I(z)$$

$$= (1 - A^*(\lambda))[\lambda m_1 \sum_{i=1}^{M} \bar{q}_i Q_{i-1} (E(S_i) + \nu_i \beta_i^{(1)}) + m_1 - 1] / A^*(\lambda) \quad (24)$$

- The mean number of customers in the system when the server is idle is

$$L_I = \lim_{z \to 1} I'(z)$$

$$= \frac{(1 - A^*(\lambda)) N_4}{N_1 A^*(\lambda)} + \frac{N_3 N_2 (1 - A^*(\lambda))}{N_1 A^*(\lambda)} \quad (25)$$

where

$$N_2 = m_2 (1 - A^*(\lambda)) / 2 + \lambda m_1^2 (1 - A^*(\lambda)) \sum_{i=1}^{M} \bar{q}_i Q_{i-1} [E(S_i) + \nu_i \beta_i^{(1)}]$$

$$+ \lambda^2 m_1^2 \sum_{i=1}^{M} \bar{q}_i Q_{i-1} [E(S_i^2) + 2 \nu_i \beta_i^{(1)} + \nu_i \beta_i^{(2)}] / 2$$

$$+ \lambda m_2 \sum_{i=1}^{M} \bar{q}_i Q_{i-1} [E(S_i) + \nu_i \beta_i^{(1)}] / 2$$

$$N_3 = \lambda m_1 \sum_{i=1}^{M} \bar{q}_i Q_{i-1} (E(S_i) + \nu_i \beta_i^{(1)}) + m_1 - 1$$

$$N_4 = m_2 / 2 + \mu m_1^2 \sum_{i=1}^{M} \bar{q}_i Q_{i-1} [E(S_i) + \nu_i \beta_i^{(1)}] + \lambda m_2 \sum_{i=1}^{M} \bar{q}_i Q_{i-1} E(S_i) / 2$$

$$+ \lambda m_1^2 \sum_{i=1}^{M} \bar{q}_i Q_{i-1} E(S_i^2) / 2 + \lambda^2 m_1^2 \sum_{i=1}^{M} \bar{q}_i Q_{i-1} \nu_i E(S_i) \beta_i^{(1)}$$

$$+ \lambda^2 m_1^2 \sum_{i=1}^{M} \bar{q}_i Q_{i-1} \nu_i \beta_i^{(2)} + \lambda m_2 \sum_{i=1}^{M} \bar{q}_i Q_{i-1} \nu_i \beta_i^{(1)}$$

and $E(S_i^2) = \sum_{j=1}^{i} (\mu_j^{(1)})^2 + \sum_{j=1}^{i} [\mu_j^{(2)} - (\mu_j^{(1)})^2]$

- The steady state probability that the server is busy is given by

$$W = \lim_{z \to 1} \sum_{i=1}^{M} W_i(z)$$

$$= \lambda m_1 \sum_{i=1}^{M} Q_{i-1} \mu_i^{(1)} \quad (26)$$

- The mean number of customers in the system when the server is busy is

$$L_w = \lim_{z\to 1} \sum_{i=1}^{M} W_i'(z)$$

$$= \sum_{i=1}^{M} Q_{i-1}\, [\lambda^2\, m_1^2\, E(S_i)\, \mu_i^{(1)} + \lambda^2\, m_1^2\, \mu_i^{(2)}/2 + \lambda\, m_2\, \mu_i^{(2)}/2 + \lambda\, m_1\, \mu_i^{(1)}\, N_2 / N_1] \quad (27)$$

- The steady state probability that the server is on vacation is given by

$$V = \lim_{z\to 1} \sum_{i=1}^{M} V_i(z)$$

$$= \lambda\, m_1 \sum_{i=1}^{M} \bar{q}_i\, Q_{i-1}\, \nu_i\, \beta_i^{(1)} \quad (28)$$

- The mean number of customers in the system when the server is on vacation is

$$L_v = \lim_{z\to 1} \sum_{i=1}^{M} V_i'(z)$$

$$= \sum_{i=1}^{M} \nu_i\, \bar{q}_i\, Q_{i-1}\, [\lambda^2\, m_1^2\, E(S_i)\, \beta_i^{(1)} + \lambda^2\, m_1^2\, \beta_i^{(2)}/2 + \lambda\, m_2 \beta_i^{(1)}/2 + \lambda\, m_1 \beta_i^{(1)}\, N_2 / N_1] \quad (29)$$

- The mean number of customers in the orbit is given by

$$L_q = \lim_{z\to 1} P_q'(z)$$

$$= N_2 / N_1 \quad (30)$$

- The mean number of customers in the system is given by

$$L_s = \lim_{z\to 1} P_s'(z)$$

$$= L_q + \lambda\, m_1 \sum_{i=1}^{M} \bar{q}_i\, Q_{i-1}\, E(S_i) \quad (31)$$

5. System Size Distribution at a Departure Epoch

Let $\{\Phi_j, j \geq 0\}$ be the probability that there are j units in the system at a departure epoch. Then

$$\Phi_j = \mathbb{K}_0 \sum_{i=1}^{M} \bar{q}_i \int_0^{\infty} W_{i,j}(x)\ \mu_i(x)\,dx,\ j \geq 0 \tag{32}$$

where $\mathbb{K}_0$ is the normalizing constant.

Let $\Phi(z) = \sum_{j=0}^{\infty} \Phi_j z^j$ be the probability generating function of $\{\Phi_j, j \geq 0\}$, then the equation (32) yields

$$\begin{aligned}\Phi(z) &= \mathbb{K}_0 \sum_{i=1}^{M} \bar{q}_i \int_0^{\infty} W_i(z, x)\,\mu_i(x)\,dx \\ &= \mathbb{K}_0\, W_i(z, 0)\ B_i^*(\lambda(1-C(z))) \\ &= \mathbb{K}_0\, \lambda\, I_0\ A^*(\lambda)\ (1 - C(z)) \sum_{i=1}^{M} \bar{q}_i\ Q_{i-1}\ \Lambda_i^*\ (\lambda(1 - C(z)))/D(z)\end{aligned} \tag{33}$$

Using the normalizing condition $\Phi(1) = 1$, we get, $\mathbb{K}_0 = 1 / (\lambda\, m_1)$ (34)

Now equation (33) becomes

$$\Phi(z) = I_0\, A^*(\lambda)\,(1 - C(z)) \sum_{i=1}^{M} \bar{q}_i\ Q_{i-1}\ \Lambda_i^*\ (\lambda(1 - C(z)))\,/\,[m_1\, D(z)] \tag{35}$$

The relationship between the probability generating function of the system size distribution at a random epoch and at a departure epoch is given by

$$\Phi(z) = H(z)\, P_s(z) \tag{36}$$

where $H(z) = \dfrac{1-C(z)}{m_1\,(1-z)}$, is the probability generating function of the number of customers placed before an arbitrary tagged customer in a batch in which the tagged customer arrives.

The expected number of units at a departure epoch is

$$\begin{aligned}L_d &= \lim_{z\to 1} \frac{d}{dz}\Phi(z) \\ &= L_B + L_s\end{aligned} \tag{37}$$

where $L_B = \dfrac{m_2}{2\,m_1}$ is the mean residual batch size.

6. Stochastic Decomposition

Theorem

The mean number of customers in the system L_s of the model under study can be expressed as the sum of the mean number of customers in classical bulk arrival multi-stage queue with vacation L_ϕ and the mean number of customers given that the server is idle L_ψ.

Proof

The probability generating function of the number of customers in the system for the classical bulk arrival multi-stage queue with vacation is

$$\phi(z) = [1 - \lambda\, m_1 \sum_{i=1}^{M} \bar{q}_i\, Q_{i-1}(E(S_i) + \nu_i \beta_i^{(1)})] \sum_{i=1}^{M} \bar{q}_i\, Q_{i-1} \Lambda_i^* \,(\lambda(1 - C(z)))\,(1 - z)\, /$$
$$[\sum_{i=1}^{M} \bar{q}_i\, Q_{i-1}\, \Lambda_i^*\,(\lambda(1 - C(z)))\,[\bar{\nu}_i + \nu_i\, V_i^*\,(\lambda(1 - C(z)))] - z] \tag{38}$$

The probability generating function $\psi(z)$ of the number of customers in the system at a random point of time given that the server is idle for the model under consideration is given by

$$\psi(z) = \frac{I_0 + I(z)}{I_0 + I(1)}$$
$$= \frac{I_0\, A^*(\lambda) \sum_{i=1}^{M} \bar{q}_i\, Q_{i-1}\, \Lambda_i^*(\lambda(1-C(z)))[\bar{\nu}_i + \nu_i\, V_i^*(\lambda(1-C(z)))]}{D(z)[1-\lambda\, m_1 \sum_{i=1}^{M} \bar{q}_i\, Q_{i-1}\,(E(S_i) + \nu_i\, \beta_i^{(1)})]} \tag{39}$$

From equations (23), (37) and (38), it is observed that

$$P_s(z) = \phi(z)\, \psi(z) \tag{40}$$

Differentiating the equation (40) with respect to z and taking limit as $z \to 1$ we get

$$L_s = L_\phi + L_\psi$$

7. Special Cases

1. If $C(z) = z$, then the system reduces to multi stage retrial queueing system with vacation.
2. If $C(z) = z$ and $M = 1$ then the results agree with the corresponding results in Sumitha and Udaya Chandrika (2012) without starting failure and orbital search.

3. If M = 2, $A^*(\lambda) \to 1$, $q_1 = 1$ and $\nu_1 = 0$ then the results coincide with the results in Choudhury and Madan (2004).
4. If $C(z) = z$, $A^*(\lambda) \to 1$ and $\nu_i = 0$ then the model coincides with the model of Salehirad and Badamchi Zadeh (2009) without feedback.

8. Numerical Study

This section is devoted to study the effect of the parameters on the system performance.

Assume that the service time, retrial time and vacation time follow exponential distribution with respective parameters μ_i, η and β_i $(1 \le i \le M)$ and batch size follows geometric distribution with mean $1/\sigma$, where $\sigma \in (0, 1]$. Assumed parameters for numerical analysis are tabulated in Table 2.

The values of L_I, L_W, L_V, L_ψ and L_d are calculated by varying arrival rate λ and service rate μ and presented in Tables 3 and 4. The values increase monotonically for increasing values of λ and decrease when μ increases.

Figs. 2 to 7 depict the behaviour of I_0, I, W, V, L_q and L_s against η and σ. From the figures it is clear that when $\sigma = 0.2$, increase in η, increases I_0 and decreases I, L_q and L_s. When $\sigma > 0.2$ change in η hardly affects I_0, I, L_q and L_s.

Figs. 2 to 5 indicate that increase in σ increases I_0, W and V but decreases I. Figs. 6 and 7 imply that W and V are independent of η. From Figs. 6 and 7, it is observed that for larger values of σ, the graphs corresponding to L_s and L_q move closer to x axis. This agrees with the intuitive expectation that when batch size decreases the values of L_s and L_q become negligible. The effect of arrival rate λ and retrial rate η on I_0, I, W, V, L_s and L_q are displayed in Figs. 8 to 13 respectively.

Table 2 Input Parameters

	Varying parameter	λ	η	M	q_i	ν_i	μ_i	β_i	σ
Table 4.3	λ	0.2 (0.2) 1.4	25	4	[0.8 0.5 0.4]	[0.3 0.3 0.3 0.2]	[35 30 25 20]	[10 10 20 20]	0.2
Table 4.4	η	1.2	20 (5) 50	4	[0.8 0.5 0.4]	[0.3 0.3 0.3 0.2]	[35 30 25 20]	[10 10 20 20]	0.2
Table 4.5	$\mu(=\mu_i)$	1.2	25	4	[0.8 0.5 0.4]	[0.3 0.3 0.3 0.2]	25 (5) 50	[10 10 20 20]	0.2
Table 4.6	$q(=q_i)$	1.2	25	4	0 (0.2) 0.8	[0.3 0.3 0.3 0.2]	[35 30 25 20]	[10 10 20 20]	0.2

Table 4.7	$\nu(=\nu_i)$	1.2	25	4	[0.8 0.5 0.4]	0 (0.2) 1	[35 30 25 20]	[10 10 20 20]	0.2
Figs. 4.2-4.7	σ and η	1.2	20 (5) 50	4	[0.8 0.5 0.4]	[0.3 0.3 0.3 0.2]	[35 30 25 20]	[10 10 20 20]	0.2, 0.4, 0.5, 0.7, 0.9
Figs. 4.8-4.13	λ and η	1 (0.2) 2	25 (5) 100	4	[0.8 0.5 0.4]	[0.3 0.3 0.3 0.2]	[35 30 25 20]	[10 10 20 20]	0.5

Table 3 Values of L_I, L_W, L_V, L_ψ and L_d by varying λ

λ	L_I	L_W	L_V	L_ψ	L_d
0.2	0.2608	0.3634	0.1609	0.5814	5.9256
0.4	0.7619	1.5987	0.4663	1.6382	8.0313
0.6	1.7085	4.2107	0.9981	3.5744	11.3118
0.8	3.57	9.3497	1.9426	7.3637	16.8948
1	7.7012	20.3284	3.8367	15.9698	28.027
1.2	20.2646	52.2028	9.1202	43.5616	59.4614
1.4	239.1529	588.3436	96.1255	562.3503	584.839

Table 4 Values of L_I, L_W, L_V, L_ψ and L_d by varying μ

μ	L_I	L_W	L_V	L_ψ	L_d
25	53.1356	163.2333	21.9794	131.8003	151.9319
30	19.6018	52.1371	8.8364	42.9822	57.4192
35	12.7818	30.0182	6.1587	26.3181	38.1151
40	9.8716	20.7712	5.0142	19.5723	29.8426
45	8.2661	15.76 14	4.3819	15.9871	25.2614
50	7.2519	12.6453	3.9821	13.7847	22.3574

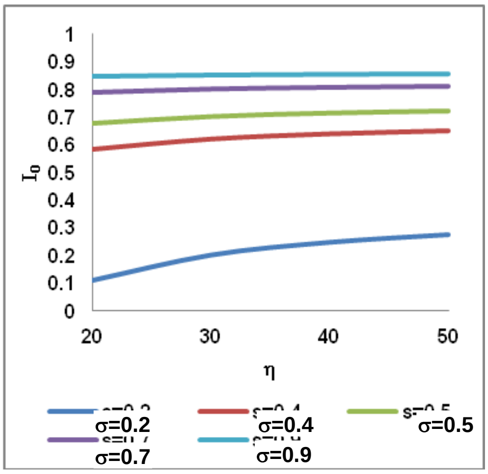

Fig. 2 Effect of σ and η on I_0

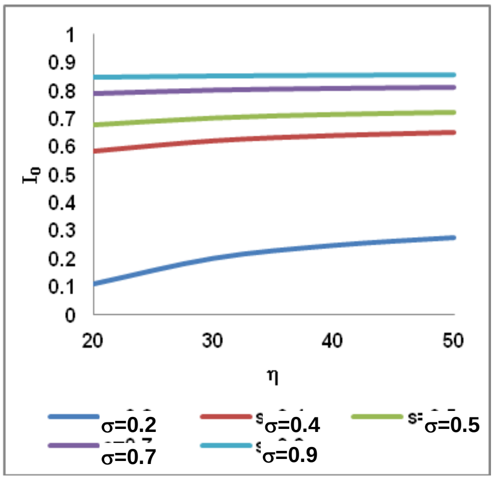

Fig. 3 Effect of σ and η on I

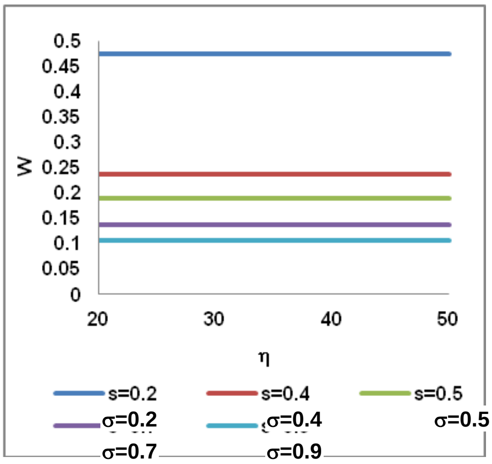

Fig. 4 Effect of σ and η on W

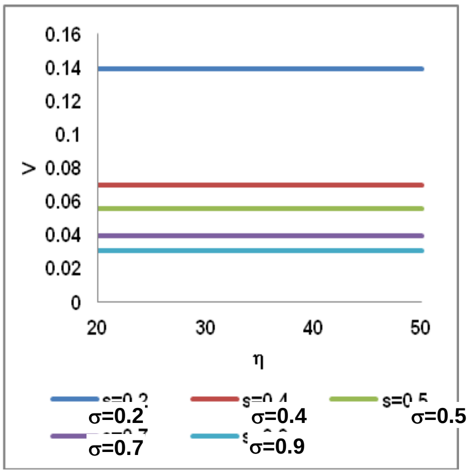

Fig. 5 Effect of σ and η on V

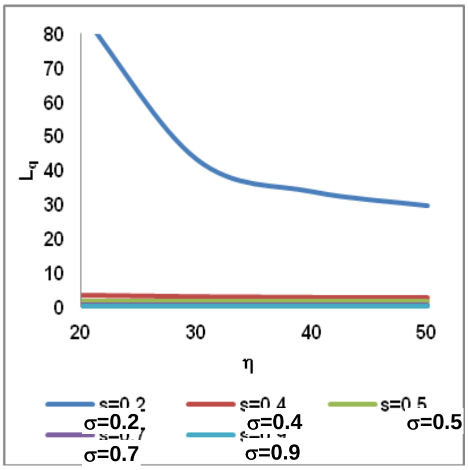

Fig. 6 Effect of σ and η on L_q

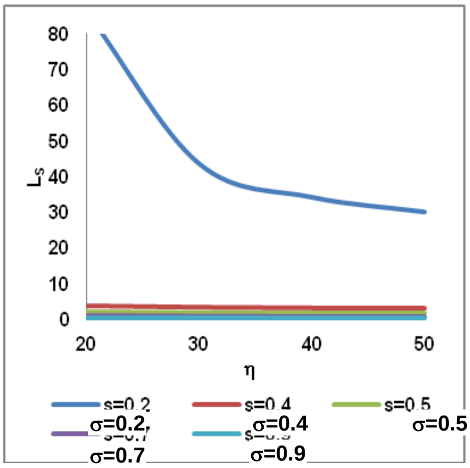

Fig. 7 Effect of σ and η on L_S

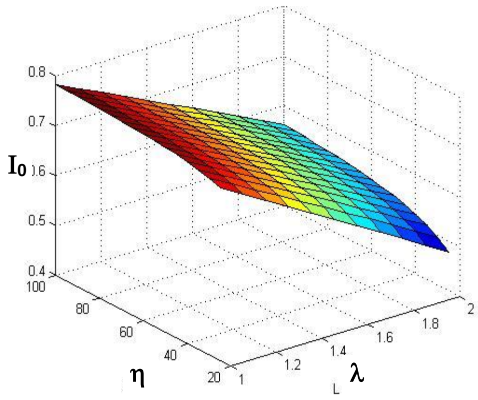

Fig. 8 Effect of λ and η on I_0

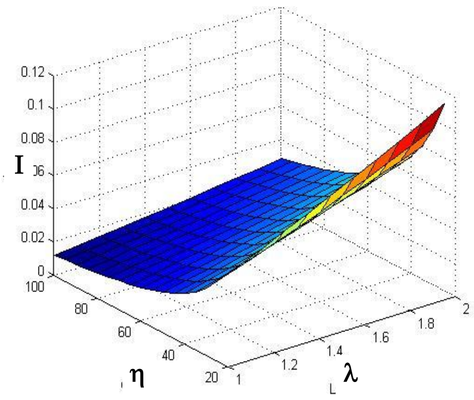

Fig. 9 Effect of λ and η on I

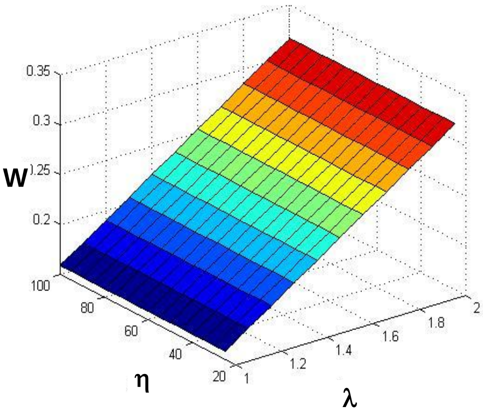

Fig. 10 Effect of λ and η on W

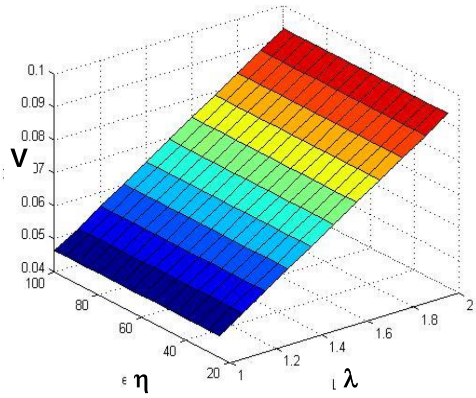

Fig. 11 Effect of λ and η on V

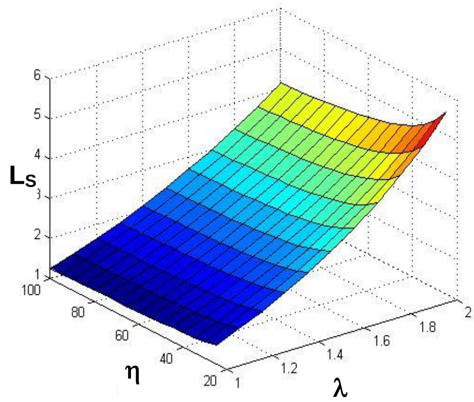

Fig. 12 Effect of λ and η on L_s

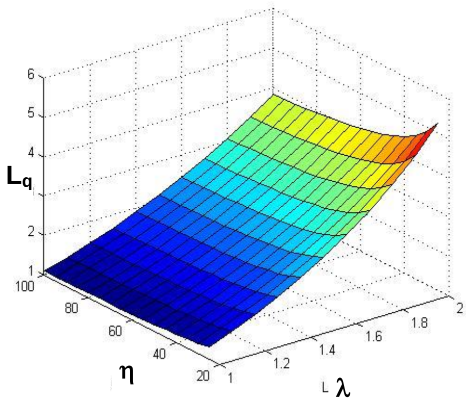

Fig. 13 Effect of λ and η on L_q

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