Variational Homotopy Perturbation Method for

Solving Lane-Emden Initial Value Problems

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Abstract

This paper is devoted to studying the numerical solution of Lane-Emden type initial value problems. A Variational Homotopy Perturbation Method is applied. Some numerical examples are discussed to demonstrate the efficiency and accuracy of the proposed algorithm.

Keywords: ordinary differential equations, Variational Homotopy Perturbation Method, Lane-Emden initial value problems

1.   Introduction

Lane-Emden type initial value problems are well known ordinary differential equations (ODE's), used in the modeling of Physics and Astrophysics problems. These equations play important role in many disciplines of science and engineering. Therefore, these types of problems received special attention from scientists and researchers. It is named after the astrophysicists Jonathan H. Lane and Robert Emden [13], as it was first studied by them. The general form of Lane-Emden type of equations is:

$$v''(x) + \frac{\alpha}{x} v'(x) + g(v) = 0 \qquad (1.1)$$

subject to the initial conditions $v(0) = a$ and $v'(0) = b$. Here a and b are constants, $g(v)$ is a real-valued continuous function.

The equation has been numerically processed in many papers [2, 5, 14, 15, 16, 17, 19, 22]. The authors in [4, 18, 23, 24] have studied the Homotopy Perturbation Method (HPM) for solving Lane-Emden differential equation, in [20] the Variational Iteration Method (VIM) was applied. In [10] the authors were used modified homotopy analysis method. In [1] Lane-Emden differential equation was solved by combining homotopy perturbation and reproducing kernel Hilbert space methods. Optimal homotopy asymptotic method was applied for singular Lane–Emden type equation in [21].

In this paper, we will present a method based on combining the VIM and HPM, the Variational Homotopy Perturbation Method (VHPM) is proposed to solve many kinds of differential equations, for example, Bratu-Type Problems [1], Riccati Differential Problems [12] and Stiff Systems of ODEs [11]. Also, VHPM was applied on partial differential equations such as Fisher's Equation [21], Burgers' equations [9], Klein-Gordon and sine-Gordon equations [25]. The results reveal that the proposed method is very effective and simple.

In this paper we study the VHPM for solving the Lane-Emden type initial value problems. We organize this paper as follows. In Section 2, we introduce analysis of the method. In Section 3, we present some numerical results to illustrate the efficiency of the presented method.

2.   Analysis of the method

We start this method by applying HPM in following nonlinear differential equation

$$L(v) + N(v) - f(x) = 0 \tag{2.1}$$

where L is a linear operator, N is a nonlinear operator and $f(x)$ is a known analytical function [7].
Construct the following homotopy

$$\Psi(v;q) = (1-q)[L(v) - L(u_0)] + q[L(v) - N(v) - f(x)] = 0, \quad x \in \Omega, \quad q \in [0,1] \tag{2.2}$$

Then

$$L(v) - L(u_0) + q[L(v_0) - N(v) - f(x)] = 0, \tag{2.3}$$

The solution can be written as a power series in q

$$v(x) = \sum_{i=0}^{\infty} q^i v_i(x) \tag{2.4}$$

The correct function for VIM can be written down as follows [9]

$$v_{n+1} = v_n + q \int_0^x \lambda(\tau)[L(v_n) + N(\widehat{v_n}) - f(\tau)] d\tau \tag{2.5}$$

Now we using correction functional in Eq.(2.3) to get

$$v_{n+1} = v_n + \int_0^x \lambda(\tau)\{L(v_n) - L(u_0) + q[L(u_0) + N(\widehat{v_n}) - f(\tau)]\} d\tau \tag{2.6}$$

To solve Eq.(1.1) using VHPM, we apply the formula in Eq.(2.6) and we get

$$v_{n+1} = v_n + \int_0^x \lambda(\tau)\{L(v_n) - L(u_0) + q[L(u_0) + N(\hat{v}_n) - f(\tau)]\}d\tau \tag{2.7}$$

Taking the variation of Eq.(2.7) with respect to the independent variable v_n we find

$$\delta v_{n+1} = \delta v_n + \delta \int_0^x \lambda(\tau)\{L(v_n) - L(u_0) + q[L(u_0) + N(\hat{v}_n) - f(\tau)]\}d\tau \tag{2.8}$$

where $\hat{v}_n$ is a restricted value that means it behaves as a constant, hence $\delta\hat{v}_n = 0$, where δ is the Variational derivative.

Since $L(v_n) = v_n''(\tau) + \dfrac{\alpha}{x}v_n'(\tau)$, Eq.(2.8) becomes

$$\delta v_{n+1} = \delta v_n[1 - \lambda'(\tau) + \frac{\alpha}{x}\lambda(\tau)] + \delta\lambda(\tau)v_n(\tau) + \int_0^x[\lambda''(\tau) - 2\frac{\zeta\lambda'(\tau) - \lambda(\tau)}{\tau^2}]\delta v_n\, d\tau \tag{2.9}$$

The extremum condition of v_{n+1} requires that $\delta v_{n+1} = 0$. This means that the left-hand side of Eq.(2.9) is zero, and as a result, the right-hand side should be zero as well.

This yield the constant conditions:

$$1 - \lambda'(\tau)\big|_{\tau=x} + \frac{\alpha}{x}\lambda(\tau)\big|_{\tau=x} = 0, \qquad \lambda(\tau)\big|_{\tau=x} = 0, \qquad \lambda''(\tau)\big|_{\tau=x} = -2\frac{\tau\lambda'(\tau) - \lambda(\tau)}{\tau^2} = 0 \tag{2.10}$$

Thus, the Lagrange multiplier can be determined as $\lambda = \dfrac{\tau(\tau - x)}{x}$.

Finally, Eq.(2.7) can be given as

$$v_{n+1} = u_0 + q\int_0^x \lambda(\tau)[L(u_0) + N(v_n) - f(\tau)]d\tau \tag{2.11}$$

Now we can rewrite Eq.(2.11) in the form

$$\sum_{i=0}^{\infty} q^i v_i = u_0 + q\int_0^x \lambda(\tau)[L(u_0) + N(\sum_{i=0}^{\infty} q^i v_i) - f(\tau)]d\tau \tag{2.12}$$

As we see, the procedure is formulated by the coupling of VIM and HPM, a comparison of like powers of q give the solutions of the various orders.

3. Numerical Examples

Example 1.
  Consider the nonlinear Lane-Emden equation [21]

$$v''(x) + \frac{2}{x}v'(x) + 4(2e^{v(x)} + e^{\frac{v(x)}{2}}) = 0 \tag{3.1}$$

subject to the initial conditions $v(0) = v'(0) = 0$.

The exact solution of above equation is $v(x) = -2\ln(1 + x^2)$.

The Taylor expansion of $v(x)$ about $x = 0$ gives

$$v(x) = -2x^2 + x^4 - \frac{2}{3}x^6 + \frac{1}{2}x^8 - \frac{2}{5}x^{10} + \frac{1}{3}x^{12} - \frac{2}{7}x^{14} + \cdots$$

Suppose that the initial approximation is $u_0(x) = 0$.

In order to solve this equation by using the VHPM, using the formula in Eq.(2.12) leads to

$$\sum_{i=0}^{\infty} q^i v_i = u_0 + q \int_0^x \lambda(\tau)[u_0'' + 4(2(\sum_{n=0}^{\infty} \frac{(\sum_{i=0}^{\infty} q^i v_i)^n}{n!}) + \sum_{n=0}^{\infty}(\frac{(\frac{1}{2}\sum_{i=0}^{\infty} q^i v_i)^n}{n!}))]d\tau \qquad (3.2)$$

Substituting $\lambda = \frac{\tau(\tau-x)}{x}$ give

$$\sum_{i=0}^{\infty} q^i v_i = u_0 + q \int_0^x \frac{\tau(\tau-x)}{x}[u_0'' + 4(2(\sum_{n=0}^{\infty} \frac{(\sum_{i=0}^{\infty} q^i v_i)^n}{n!}) + \sum_{n=0}^{\infty}(\frac{(\frac{1}{2}\sum_{i=0}^{\infty} q^i v_i)^n}{n!}))]d\tau \qquad (3.3)$$

By comparing the coefficient of like powers of q, we get

$q^0:$ $\quad v_0(x) = u_0(x) = 0$

$q^1:$ $\quad v_1(x) = 4\int_0^x \frac{\tau(\tau-x)}{x}[2\sum_{n=0}^{\infty} \frac{(v_0)^n}{n!}) + \sum_{n=0}^{\infty} \frac{(\frac{1}{2}v_0)^n}{n!}]d\tau = -2x^2$

$q^2:$ $\quad v_2(x) = 4\int_0^x \frac{\tau(\tau-x)}{x}[2\sum_{n=0}^{\infty} \frac{(v_0)^n v_1}{n!}) + \sum_{n=0}^{\infty} \frac{(\frac{1}{2}v_0)^n (\frac{1}{2}v_1)}{n!}]d\tau = x^4$ $\qquad (3.4)$

$q^3:$ $\quad v_3(x) = 4\int_0^x \frac{\tau(\tau-x)}{x}[2\sum_{n=0}^{\infty} \frac{(v_0)^n (v_1^2 + v_2)}{n!}) + \sum_{n=0}^{\infty} \frac{(\frac{1}{2}v_0)^n (\frac{1}{2}(v_1^2 + v_2))}{n!}]d\tau = -\frac{2}{3}x^6$

$\vdots$

The other components of the VHPM can be determined in similar way. Finally, the approximate solution of Eq.(3.1) is

$$\sum_{i=0}^{\infty} v_i(x) = -2x^2 + x^4 - \frac{2}{3}x^6 + \frac{1}{2}x^8 - \frac{2}{5}x^{10} + \frac{1}{3}x^{12} - \frac{2}{7}x^{14} + \cdots \qquad (3.5)$$

Which converge to the exact solution.

Example 2.

Consider the nonlinear Lane-Emden equation [21]

$$v''(x) + \frac{2}{x}v'(x) - 6v(x) - 4v(x)\ln v(x) = 0 \qquad (3.6)$$

subject to the initial conditions $v(0) = 1, v'(0) = 0$.

The exact solution is $v(x) = e^{x^2}$.

The Taylor expansion of $v(x)$ about $x = 0$ gives

$$v(x) = 1 + x^2 + \frac{1}{2}x^4 + \frac{1}{6}x^6 + \frac{1}{24}x^8 + \cdots$$

Suppose that the initial approximation is $u_0(x) = 1$.

In order to solve this equation by using the VHPM, applying the formula in Eq.(2.12) to obtain

$$\sum_{i=0}^{\infty} q^i v_i = u_0 + q \int_0^x \lambda(\zeta) [u_0'' - 6 \sum_{i=0}^{\infty} q^i v_i - 4 (\sum_{i=0}^{\infty} q^i v_i)(\sum_{k=1}^{\infty} \frac{(-1)^{k-1}(\sum_{i=0}^{\infty} q^i v_i - 1)^k}{k})] d\zeta \quad (3.7)$$

Substituting Lagrange multiplier $\lambda = \dfrac{\tau(\tau - x)}{x}$ give

$$\sum_{i=0}^{\infty} q^i v_i = u_0 + q \int_0^x \frac{\tau(\tau - x)}{x} [u_0'' - 6 \sum_{i=0}^{\infty} q^i v_i - 4 (\sum_{i=0}^{\infty} q^i v_i)(\sum_{k=1}^{\infty} \frac{(-1)^{k-1}(\sum_{i=0}^{\infty} q^i v_i - 1)^k}{k})] d\tau \quad (3.8)$$

By comparing the coefficient of like powers of q, we get

$q^0:$ $\quad v_0(x) = 1$

$q^1:$ $\quad v_1(x) = \int_0^x \dfrac{\tau(\tau - x)}{x} \left(-6v_0 - 4v_0 \sum_{n=1}^{\infty} \dfrac{(-1)^{n-1}(v_0 - 1)^n}{n} \right) d\tau = x^2$

$q^2:$ $\quad v_2(x) = \int_0^x \dfrac{\tau(\tau - x)}{x} \left(-6v_0 - 4(v_1 \sum_{n=1}^{\infty} \dfrac{(-1)^{n-1}(v_0 - 1)^n}{n} + v_0 \sum_{n=0}^{9} (-1)^{n-1}(v_0 - 1)^n v_1) \right) d\tau = \dfrac{x^4}{2}$

$\vdots$

$$(3.9)$$

The other components of the VHPM can be determined in similar way. Finally, the approximate solution of Eq.(3.6) is

$$\sum_{i=0}^{\infty} v_i(x) = 1 + x^2 + \frac{1}{2} x^4 + \frac{1}{6} x^6 + \frac{1}{24} x^8 + \cdots$$

Which converge to the exact solution.

4. Conclusion

In this paper, the VHPM has been successful in finding the solution of nonlinear Lane-Emden equations with initial conditions. A clear conclusion can be drawn from the numerical results that the VHPM produces a series that converges to the exact solution. It is noticed that the present scheme uses the full advantage of the variational iteration method and the homotopy perturbation method. Finally, we conclude that the VHPM may be considered as a good improvement in existing numerical techniques.

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