Contemporary Engineering Sciences, Vol. 11, 2018, no. 48, 2349 - 2356
HIKARI Ltd, www.m-hikari.com
https://doi.org/10.12988/ces.2018.85243

Radially Symmetric Solutions of a Non-Linear Problem with Neumann Boundary Condition in One Dimension Like Self Similar Solutions via Finite Differences

A. M. Marín, R. D. Ortíz and Alfredo-Yerman Cortes-Verbel

Universidad de Cartagena, ONDAS Research Group
Cartagena de Indias, Colombia

Abstract

In this paper it will be study the solution of a non-linear elliptic differential equation with Neumann boundary condition, studied by Marin and Ortiz. We write the problem in finite differences and then we find the solution points and with these we obtain the Vandermonde polynomial corresponding to the solution. Also with Scilab's help we could see the numerical solution and the Vandermonde approximation polynomial.

Keywords: Finite Differences, Maximum principle, Symmetric

1 Introduction

Now, on the other hand, we have the importance of the Principle of the Maximum that, as is known, provides valuable information on the characteristics of the solution of the differential equation of type elliptical without knowing the solution explicitly, so it allows us to obtain the uniqueness of the solution certain problems with Dirichlet and Neumann type boundary conditions [2],[3],[4],[6],[7],[8],[9],[10],[11],[12].

This principle is nothing more than the generalization of the following elementary fact of calculus: Given any function f which satisfies the inequality

$f'' > 0$ over an interval (a, b) that reaches its maximum value at the extremes of the interval.

In order to achieve the stated objective, we first consider some preliminaries such as the principle of Maximum with the boundary condition of Neumann. Finally the problem will be solved numerically; that is, the central finite differences will be applied to the problem

$$\begin{cases} a(x)u''(x) + b(x)u'(x) + c(x)u(x) = f(u) \\ u'(1) = -u'(-1) \end{cases} \tag{1}$$

2 Maximum Priciple

Let $u \in C^2((-1,1)) \cup C^0([-1,1])$ be a solution of

$$L(u) = a(x)u''(x) + b(x)u'(x) + c(x)u(x) = f(u(x))$$

on $(-1,1)$ with $u'(1) = -u'(-1)$; where $a, c : [-1,1] \to \mathbb{R}$ are bounded functions and symmetric with respect to the origin such that $a(x) > 0$ and $c(x) \leqslant 0$ for all $x \in [-1,1]$, and also

$$L(u) > 0, \forall x \in [-1,1] \tag{2}$$

u can not reach its maximum value at a point inside the interval $[-1,1]$; that is, if equation (2) is verified, then the function u reaches its maximum at the extremes of the interval, because if we assume that the function u reaches its maximum value in $d \in (-1,1)$, then we have by the nullification of the derivative at an inner end that $u'(d) = 0$ and also by the criterion of the second derivative for extremes in a critical point remains that $u''(d) \leqslant 0$; but it not fulfil the inequality (2). In the following theorem it was studied the symmetric solutions for a non-linear elliptic equation with Neumann boundary condition.

Theorem 2.1. *[3] Let $u \in C^2((-1,1)) \cup C^0([-1,1])$ be a non negative solution of the non-linear elliptic problem with Neumann condition boundary*

$$\begin{cases} a(x)u''(x) + b(x)u'(x) + c(x)u(x) = f(u(x)) \\ u'(1) = -u'(-1) \end{cases} \tag{3}$$

Where $a, c : [-1,1] \to \mathbb{R}$ are bound functions and symmetric with respect to the origin such that $a(x) > 0$ and $c(x) \leq 0$ for all x in the interval $[-1,1]$. $b : [-1,1] \to \mathbb{R}$ is a bounded function and odd. Supposing that f is strictly increasing, then u is radially symmetric with respect to the origin.

3 Numerical Development

We have finite differences are:

$$\left.\frac{du(x)}{dx}\right|_{x_i} = \frac{u_{i+1}-u_{i-1}}{2\Delta}$$
$$\left.\frac{d^2u(x)}{dx^2}\right|_{x_i} = \frac{u_{i+1}-2u_i+u_{i-1}}{\Delta^2}$$

replacing in the first equation (1) we have

$$a(x_i)\left[\frac{u_{i+1}-2u_i+u_{i-1}}{\Delta^2}\right]+b(x_i)\left[\frac{u_{i+1}-u_{i-1}}{2\Delta}\right]+c(x_i)u_i=f(u_i)$$
$$[2a(x_i)-b(x_i)\Delta]\,u_{i-1}+\left[2\Delta^2c(x_i)-4a(x_i)\right]u_i+[2a(x_i)+b(x_i)\Delta]\,u_{i+1}$$
$$=2\Delta^2f(u_i)$$

Let

$$A_{1i} = 2a(x_i)-b(x_i)\Delta$$
$$A_{2i} = 2\Delta^2c(x_i)-4a(x_i)$$
$$A_{3i} = 2a(x_i)+b(x_i)\Delta$$

Then the previous equation becomes:

$$A_{1i}u_{i-1}+A_{2i}u_i+A_{3i}u_{i+1}=2\Delta^2f(u_i) \tag{4}$$

For different values of $i=1,2,3,...,n$ where n is the number of partitions of the interval $(-1,1)$ we have that:

$$\begin{array}{lllll} i & = & 1, & A_{11}u_0+A_{21}u_1+A_{31}u_2 & = & 2\Delta^2f(u_1) \\ i & = & 2, & A_{12}u_1+A_{22}u_2+A_{32}u_3 & = & 2\Delta^2f(u_2) \\ i & = & 3, & A_{13}u_2+A_{23}u_3+A_{33}u_4 & = & 2\Delta^2f(u_3) \\ i & = & 4, & A_{14}u_3+A_{24}u_4+A_{34}u_5 & = & 2\Delta^2f(u_4) \\ \vdots & = & \vdots & \vdots & = & \vdots \\ i & = & n-1, & A_{1(n-1)}u_{n-2}+A_{2(n-1)}u_{n-1}+A_{3(n-1)}u_n & = & 2\Delta^2f(u_{n-1}) \\ i & = & n, & A_{1n}u_{(n-1)}+A_{2n}u_n+A_{3n}u_{(n+1)} & = & 2\Delta^2f(u_n) \end{array} \tag{5}$$

In the first equation for $i=1$ we do not know the value of u_0 but we know its first derivative. We know that:

$$\begin{aligned} \frac{u_1 - u_{-1}}{2\Delta} &= u'(-1) \\ u_{-1} &= u_1 - 2\Delta u'(-1) \end{aligned} \tag{6}$$

Adding a new equation. For $i = 0$ in the equation (4) we have get

$$A_{10}u_{-1} + A_{20}u_0 + A_{30}u_1 = 2\Delta^2 f(u_0) \tag{7}$$

Substituting the equation (6) in the equation (7) we obtain

$$\begin{aligned} A_{10}\left[u_1 - 2\Delta u'(-1)\right] + A_{20}u_0 + A_{30}u_1 &= 2\Delta^2 f(u_0) \\ A_{20}u_0 + (A_{10} + A_{30})u_1 &= 2\Delta^2 f(u_0) + 2\Delta u'(-1)A_{10} \end{aligned} \tag{8}$$

Analogously for $i = n$

$$\begin{aligned} \frac{u_{n+1} - u_{n-1}}{2\Delta} &= u'(1) \\ u_{n+1} &= u_{n-1} + 2\Delta u'(1) \end{aligned} \tag{9}$$

Replacing the equation (9) in the last of the equations of (5)

$$\begin{aligned} A_{1n}u_{(n-1)} + A_{2n}u_n + A_{3n}u_{(n+1)} &= 2\Delta^2 f(u_n) \\ A_{1n}u_{(n-1)} + A_{2n}u_n + A_{3n}\left[u_{n-1} + 2\Delta u'(1)\right] &= 2\Delta^2 f(u_n) \\ \left[A_{1n} + A_{3n}\right]u_{(n-1)} + A_{2n}u_n &= 2\Delta^2 f(u_n) - 2\Delta u'(1)A_{3n} \end{aligned} \tag{10}$$

Now from the equations (5), (8) and (10) we have the following system of non-linear equations

$$\begin{aligned} A_{20}u_0 + \left[A_{10} + A_{30}\right]u_1 &= 2\Delta^2 f(u_0) + 2\Delta u'(-1)A_{10} \\ A_{11}u_0 + A_{21}u_1 + A_{31}u_2 &= 2\Delta^2 f(u_1) \\ A_{12}u_1 + A_{22}u_2 + A_{32}u_3 &= 2\Delta^2 f(u_2) \\ \vdots &= \vdots \\ \left[A_{1n} + A_{3n}\right]u_{(n-1)} + A_{2n}u_n &= 2\Delta^2 f(u_n) - 2\Delta u'(1)A_{3n} \end{aligned}$$

If we denote by

$$A = \begin{bmatrix} A_{20} & [A_{10}+A_{30}] & 0 & 0 & \cdots & 0 \\ A_{11} & A_{21} & A_{31} & 0 & \cdots & 0 \\ 0 & A_{12} & A_{22} & A_{32} & \cdots & 0 \\ 0 & 0 & A_{13} & A_{23} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \left[A_{1(n)}+A_{3(n)}\right] & A_{2(n)} \end{bmatrix}$$

and by $\Phi(u) : R^{n+1} \longrightarrow R^{n+1}$ such that

$$\Phi(u_i) = \begin{bmatrix} f(u_0) + \dfrac{u'(-1)A_{10}}{\Delta} \\ f(u_1) \\ f(u_2) \\ \vdots \\ f(u_n) - \dfrac{u'(1)A_{3(n)}}{\Delta} \end{bmatrix}$$

We reduce the problem of Neumann boundary conditions (1) to a system of non-linear equations:

$$Au = 2\Delta^2 \Phi(u) \tag{11}$$

with $u = (u_0, u_1, u_2, ..., u_n)^t$. In general, a problem of boundary non-linear it can be expressed as follows

$$\begin{cases} g(x, u, u', u'') & = & 0 \\ u'(1) & = & -u'(-1) \end{cases}$$

Trough a process of discretization analogous to the above, it is possible to obtain a system of non-linear equations whose solution is also that of the boundary problem.

On the other hand we see that the equations of (11) can be expressed as

$$\begin{aligned} A_{20}u_0 + [A_{10}+A_{30}]\, u_1 - 2\Delta^2 f(u_0) - 2\Delta u'(-1)A_{10} &= 0 \\ A_{11}u_0 + A_{21}u_1 + A_{31}u_2 - 2\Delta^2 f(u_1) &= 0 \\ A_{12}u_1 + A_{22}u_2 + A_{31}u_3 - 2\Delta^2 f(u_2) &= 0 \\ \vdots &= \vdots \\ [A_{1n}+A_{3n}]\, u_{(n-1)} + A_{2n}u_n - 2\Delta^2 f(u_n) + 2\Delta u'(1)A_{3n} &= 0 \end{aligned}$$

If we do $u = (u_0, u_1, u_2, .., u_n)$ and $F(u) = (f_0, f_1, f_2, ..., f_n)$ with

$$\begin{aligned} f_0 &= A_{20}u_0 + [A_{10} + A_{30}]\, u_1 - 2\Delta^2 f(u_0) - 2\Delta u'(-1)A_{10} \\ f_1 &= A_{11}u_0 + A_{21}u_1 + A_{31}u_2 - 2\Delta^2 f(u_1) \\ f_2 &= A_{12}u_1 + A_{22}u_2 + A_{32}u_3 - 2\Delta^2 f(u_2) \\ \vdots &= \vdots \\ f_n &= [A_{1n} + A_{3n}]\, u_{n-1} + A_{2n}u_n - 2\Delta^2 f(u_n) + 2\Delta u'(1)A_{3n} \end{aligned}$$

we have (11) that is equivalent to finding u such that

$$F(u) = 0 \tag{12}$$

Once u in (12) is found from the solution of non-linear equations systems, we approximate the points by a polynomial function; for this we use the Vander-Monde interpolation.

4 Interpolation of VanderMonde

Given the points $(x_0, u_0), (x_1, u_1), (x_2, u_2), ..., (x_n, u_n)$ all with different abscissas (for how the x_i is chosen for the discretization) or n, these are different;

$$P(x) = a_0 + a_1 x + a_2 x^2 + ... + a_n x^n \tag{13}$$

the interpolating polynomial. Imposing the $n + 1$ conditions

$$P(x_j) = u_j, j = 0, 1, 2, ..., n,$$

It results in a system of $n+1$ linear equations in $n+1$ unknowns $a_0, a_1, a_2, ..., a_n$

$$\begin{cases} a_0 + a_1x_0 + a_2x_0^2 + ... + a_nx_0^n &= u_0 \\ a_0 + a_1x_1 + a_2x_1^2 + ... + a_nx_1^n &= u_1 \\ \vdots &= \vdots \\ a_0 + a_1x_n + a_2x_n^2 + ... + a_nx_n^n &= u_n \end{cases}$$

The system can be written matrix in the form

$$\begin{bmatrix} 1 & x_0 & x_0^2 & \cdots & x_0^n \\ 1 & x_1 & x_1^2 & \cdots & x_1^n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_n & x_n^2 & \cdots & x_n^n \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} u_0 \\ u_1 \\ \vdots \\ u_n \end{bmatrix}$$

After solving the previous system we have that the interpolation polynomial that approximates the solution of the problem (3) is the polynomial (13)

5 Some Results

Solution of $u''(x) = u(x), -u'(-1) = u'(1) = 1$ Exact Solution: $u(x) = c_1(e^x + e^{-x})$ where $c_1 = e/(e^2 - 1)$.

Vandermonde Interpolation Polynomial

$$\begin{aligned} u(x) &= 1.307717 + 1.1338713x + 1.0053805x^2 \\ &+ 0.9171049x^3 + 0.8655135x^4 + 0.8485427x^5 \\ &+ 0.8655135x^6 + 0.9171049x^7 + 1.0053805x^8 \\ &+ 1.1338713x^9 + 1.307717x^{10} \end{aligned}$$

The heat equation $\frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial x^2}$ [1], [5] with the self-similar variable $\eta = \frac{x}{\sqrt{t}}$, $T = u(\eta)$ we obtain $2u'' + \eta u' = 0$ whose solution is $u(\eta) = C_1 \int e^{-\eta^2/4} d\eta + C_2$ but this equation can be solved numerically when $u'(-1) = u'(1) = 1$ we can apply our numerical method in Scilab.

Vandermonde Interpolation Polynomial

$$\begin{aligned} u(x) = &-1.905122 - 1.695122x - 1.467622x^2 - 1.2260498x^3 - 0.9746176x^4 \\ &-0.7181059x^5 - 0.4615942x^6 - 0.2101619x^7 + 0.0314103x^8 \\ &+0.2589103x^9 + 0.4689103x^{10} \end{aligned}$$

6 Conclusion

We see that our algorithm can be applied not only to ordinary differential equations with symmetric solutions with respect to the origin but also to different problems of physics, mathematics and engineering.

Acknowledgements. The authors express their deep gratitude to Universidad de Cartagena for financial support.

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Received: June 4, 2018; Published: June 26, 2018