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Numerical Simulation of Fluid Structure Interaction Benchmark Problem with BFGS Method

Ibrahima Mbaye

Department of Mathematics, Thies University, Thies, Senegal

Mamadou Diop

Thies University, Thies, Senegal

Abstract

The aim of this paper is to propose a numerical simulation of fluid structure interaction problem with BFGS method. Thus, fluid flow is governed by the incompressible newtonian steady stokes equations and we consider an elastic structure from the static theory of linear elasticity. Moreover, the finite element method is used for the fluid and the structure. Thereby, this paper shows that the benchmark problems can be simulated using an optimization method according the numerical results.

Mathematics Subject Classification: 64F10, 65N06, 65N30, 65N12

Keywords: Fluid-structure interaction ; Stokes; linear elasticity equations, finite element methods; BFGS method

1 Introduction

Problems involving fluid structure interaction occurs in a wide vatiety of engineering problems and therefore have attracted the interest of many investigations from different engineering disciplines. As a results, much efforts have

gone into the development of general computational methods for fluid structure systems [1, 2, 3, 4, 5, 8, 9].
Amongst the computational methods for fluid structure interaction problems, we cite the fixed point method, the Newton method, the Quasi-Newton method, the fictitious domain method. In effect, the fluid structure interaction problems occur in biomedical fluids areas for example blood flow interaction with elastic veins. In our previous works, our computational domain was a pipe with borders represented by an elastic structure[1, 2, 3]. Several methods are used to solve numerically the fluid structure interaction through a pipe. The aim of this paper is to show that we can use an optimization method to solve numerically the fluid structure interaction problems with a new computational domain see Figure 1. We assume that the structure is divided in two parts, Ω^r a rigid part and Ω^s an elastic part immersed in a fluid. On the one hand, we use a polynomial appriximation for the structure stress tensors acting on the fluid-structure interface, On the other hand, we use least squares method to define the objective function J to minimize by BFGS algorithm. In our last work [3, 4], we showed that if J goes to zero the optimization problem will be equivalent to the fluid structure interaction problem. In addition, the fluid flow is modelled by two dimensional Stokes equations for steady flow and the elastic structure by equations from a linear elasticity. The weak formulations are presented for the equations of fluid and structure.

2 Position of the problem

In this section, we present the mathematical model of fluid structure interaction problem and the weak formulation we consider.

2.1 Structure equations

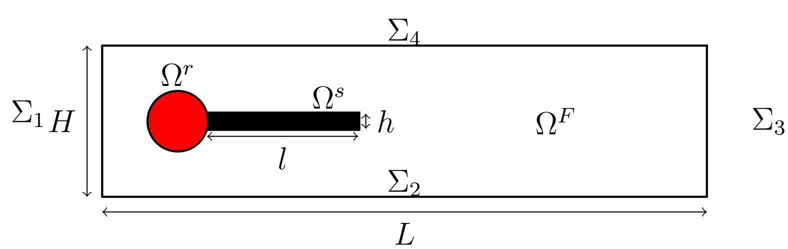

Figure 1: The fluid-structure-solid domain.

- Ω^r is the solid part of the structure.

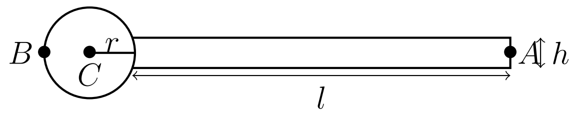

Figure 2: Details of the solid-elastic domain

- Ω^s is the elastic part of the structure.
- l is the structure elastic length.
- h is the structure elastic thickness.

In the case of small displacement of the elastic structure, we consider an isotropic elastic structure $\Omega^s \subset \mathbb{R}^2$. Let $u = (u_1, u_2)$ be the displacement and f^s the body forces. Thereby, in the static theory of linear elasticity, the equation satisfied by the displacement $u = (u_1, u_2)$ is:

$$-div(\sigma^s(u)) = f^s, \quad \text{in}\,\Omega^s \tag{1}$$

where the stress tensor $\sigma^s(u)$ is defined by:

$$\sigma^s(u) = 2\mu^s \varepsilon(u) + \lambda^s tr(\varepsilon(u))I.$$

The positive constants μ^s and λ^s are called the Lam constant and I is the 2×2 identity matrix. We also define the strain tensor $\varepsilon(u)$ by:

$$\varepsilon(u) = \frac{1}{2}(\nabla(u) + (\nabla u)^T).$$

Where, $\varepsilon(u)$ is a symmetric 2×2 matrix which has the components:

$$\varepsilon_{ij}(u) = \frac{1}{2}(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}).$$

where $(i, j) \in \{1, 2\} \times \{1, 2\}$.

- On $\partial\Omega^r$,we impose the displacement boundary condition $u = 0$.
- On $\partial\Omega^s$, we impose the traction boundary condition (where n^s is the unit outer normal vector) $\sigma^s(u).n^s = g$.

We have the following problem, find $u : \overline{\Omega^s} \to \mathbb{R}^2$ such as:

$$\begin{aligned}
-\text{div}(\sigma^s(u)) &= f^s, \quad \text{in} \quad \Omega^s & (2)\\
\sigma^s(u) &= 2\mu^s\varepsilon(u) + \lambda^s(tr(\varepsilon(u))I), \quad \text{in} \quad \Omega^s & (3)\\
u &= 0, \quad \text{on} \quad \partial\Omega^r & (4)\\
\sigma^s.n^s &= g, \quad \text{on} \quad \partial\Omega^s & (5)
\end{aligned}$$

Therefore, we have the following weak formulation of (2) through (5).
Find $u \in V = \{v \in (H^1(\Omega^s))^2,\, v = 0 \,\text{on}\, \partial\Omega^r\}$ such as

$$a^s(u, v^s) = l^s(v^s) \tag{6}$$

forall $v^s \in V$, where

$$a^s(u, v^s) = 2\mu^s \int_{\Omega^s} \varepsilon(u) : \varepsilon(v^s) dx + \lambda^s \int_{\Omega^s} \text{div}(u)\text{div}(v^s) dx. \tag{7}$$

and

$$l^s(v^s) = \int_{\Omega^s} f.v^s\, dx + \int_{\partial\Omega^s} g.v^s\, ds. \tag{8}$$

Assume that $f \in H^{-1}$, $g \in L^2(\partial\Omega^s)$, and $meas(\partial\Omega^r) > 0$. Then the variational problem (6) has a unique solution.([7])

2.2 Fluid flow equations

The Stokes equations for steady flow are considered to discribe the fluid flow. The Stokes equations can be written in a non-conservative form as:

$$\begin{aligned} -2\mu^f \text{div}(\varepsilon(v)) + \nabla p &= f^F, \,\text{in}\, \Omega^F && (9)\\ \text{div}(v) &= 0, \text{ in } \Omega^F && (10) \end{aligned}$$

subject to the boundary conditions

$$\begin{aligned} v &= G, \;\text{ on } \Sigma_1 && (11)\\ v &= 0, \,\text{on}\, \Sigma_2 \cup \Sigma_4 \cup \partial\Omega^s \cup \partial\Omega^r && (12)\\ \sigma^f.n^f &= 0, \,\text{on}\, \Sigma_3 && (13) \end{aligned}$$

Where,

$$\sigma^f = -pI + \mu^f(\bigtriangledown v + (\bigtriangledown v)^T) \tag{14}$$

is the stress tensor, μ^f the viscosity, v the velocity, p the pressure, G are the prescribed velocity on Σ_1, f^F the body forces and n^f is the unit outer normal vector to the boundary surface. Therefore, we consider the following mixed variational problem to find $v = (v_1, v_2) \in W$ and $p \in Q$ such as:

$$\begin{aligned} a(v, w) + b(w, p) &= l(w), \quad \forall w \in W && (15)\\ b(v, q) &= 0, \quad \forall q \in Q && (16) \end{aligned}$$

where,

$$Q = L^2(\Omega^F),$$

$$W = \{w \in H^1(\Omega^F),\, w = 0\, sur\, \Sigma_1 \cup \Sigma_2 \cup \Sigma_4\},$$

and,

$$a(v,w) = 2\mu \int_{\Omega^F} \varepsilon(v) : \varepsilon(w) dx$$

$$b(w,p) = -\int_{\Omega^F} p\nabla \cdot w dx$$

$$l(w) = \int_{\Omega^F} f^F \cdot w dx$$

2.3 Coupling conditions

Of course the coupling between the fluid and the structure must satisfy the equalities of the stress tensors and the velocity at the fluid-structure interface as:

$$\sigma^f.n^f = \sigma^s.n^s \qquad \text{and} \qquad v = 0$$

2.4 The coupled problem

Using the variational formulations for the structure elastic equations and the fluid flow equations, the coupled problem can be written in compact form as: Find $(u, v, p) \in V \times W \times Q$ such as:

$$a^s(u, v^s) = l^s(v^s), \quad \forall v^s \in V \tag{17}$$

$$a(v,w) + b(w,p) = l(w), \quad \forall w \in W \tag{18}$$

$$b(v,q) = 0, \quad \forall q \in Q \tag{19}$$

$$\sigma^f.n^f = \sigma^s.n^s \tag{20}$$

forall $(v^s, w, q) \in V \times W \times Q$.

3 The numerical resolution

In this section, we present the technics used to simulate the fluid strcture interaction problem. For the structure the triangular finite element $\mathbb{P}_2$ was used and for the fluid the triangular finite element $\mathbb{P}_2 - \mathbb{P}_1$ was used.

3.1 The numerical method of the coupled problem

We propose a numerical method of the coupled problem based on the BFGS[3, 4] algorithm. Thus, we introduce an objective function J to minimize. Hence, we approximate the forces applied by the fluid on the elastic structure interface. The stress tensors equality on the interface lead:

$$\sigma_{11}^f = 2\mu^s \frac{\partial u_1}{\partial x_1} + \lambda div(u) \tag{21}$$

$$\sigma_{22}^f = 2\mu \frac{\partial u_2}{\partial x_2} + \lambda div(u) \tag{22}$$

$$\sigma_{12}^f = \mu^s (\frac{\partial u_1}{\partial x_2} + \frac{\partial u_1}{\partial x_1}) \tag{23}$$

For the approximation, we assume that:

$$u_1(x,y) = a_0 + a_1 x_1 + a_2 x_2 + a_3 x_1^2 + a_4 x_2^2 + a_5 x_1 x_2 \tag{24}$$

$$u_2(x,y) = b_0 + b_1 x_1 + b_2 x_2 + b_3 x_1^2 + b_4 x_2^2 + b_5 x_1 x_2 \tag{25}$$

From the form of u_1 and u_2, we obtain:

$$\sigma_{11}^f = f_1(\alpha) \tag{26}$$

$$\sigma_{22}^f = f_2(\alpha) \tag{27}$$

$$\sigma_{12}^f = f_3(\alpha) \tag{28}$$

where, $\alpha = \{a_i, b_j\}$ for all $0 \leq i, j \leq 5$ and

$$\mu^s \frac{\partial u_1}{\partial x_1} + \lambda^s div(u) = f_1(\alpha) \tag{29}$$

$$2\mu^s \frac{\partial u_2}{\partial x_2} + \lambda^s div(u) = f_2(\alpha) \tag{30}$$

$$\mu^s (\frac{\partial u_1}{\partial x_2} + \frac{\partial u_1}{\partial x_1}) = f_3(\alpha) \tag{31}$$

In order to find the displacement $u = (u_1, u_2)$, the velocity v and the pressure p, we will solve the following unconstrained optimization problem:
Find α and $(v,p) \in W \times Q$ such as:

$$Inf J(\alpha) \tag{32}$$

$$a^s(u, v^s) = l^s(v^s, \alpha), \quad \forall v^s \in V \tag{33}$$

$$a(v,w) + b(w,p) = l(w), \quad \forall w \in W \tag{34}$$

$$b(v,q) = 0, \quad \forall q \in Q \tag{35}$$

$$\tag{36}$$

forall $(v^s, w, q) \in V \times W \times Q$, where

$$l^s(v^s, \alpha) = \int_{\Omega^s} f.v^s\, dx + \int_{\partial\Omega^s} g(\alpha).v^s\, ds \tag{37}$$

and

$$J(\alpha) = \int_{\partial\Omega^s} [(\sigma^f_{11} - f_1)^2 + (\sigma^f_{12} - f_3)^2 + (\sigma^f_{22} - f_2)^2] d\sigma \tag{38}$$

Remark
if J goes to zero, the coupled problem will become equivalent to the unconstrained optimization problem [3].

3.2 Numerical results

The Finite differences method gives the components of the gradient ∇J as $\frac{\partial J}{\partial \alpha_k}(\alpha) \approx \frac{J(\alpha+\Delta\alpha_k e_k)-J(\alpha)}{\Delta\alpha_k}$ where e_k is the k^{th} standard basis vector of $\mathbb{R}^n$ and the real $\Delta\alpha_k > 0$ tends to zero.
Parameters [8] of simulation are: the length $L = 2.5m$ and the height $H = 0.41m$ of the fluid domain, the radius and the center of the solid structure Ω^r are $C(0.2, 0.2)$ and $r = 0.05m$, the length and the thickness of the elastic structure Ω^s are $l = 0.35m$ and $h = 0.02m$. FreeFem++[6] is used for simulations.

Table 1: Parameters [8]

Parameters	values
$\rho^s[10^3 \frac{kg}{m^3}]$	1
ν^s	0.4
$\mu^s[10^6 \frac{kg}{ms^2}]$	0.5
$\rho^f[10^3 \frac{kg}{m^3}]$	1
$\nu^f\ [10^{-3} \frac{m^2}{s}]$	1

A parabolic velocity profile is prescribed at Σ_1 as: $v_1(0, y) = 1.5\bar{U}\frac{y(H-y)}{(\frac{H}{2})^2}$ and $v_2 = 0$ where $\bar{U} = 0.2\frac{m}{s}$ [8].
The velocity and the pressure profile are presented in the following figures 3-4, the structure deformation is presented in figure 5.

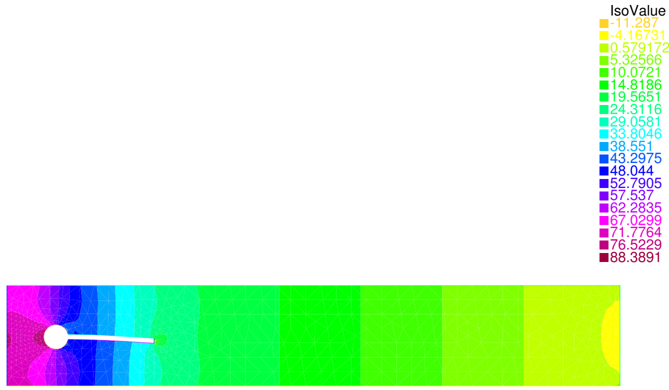

Figure 3: The pressure profile and the structure deformation

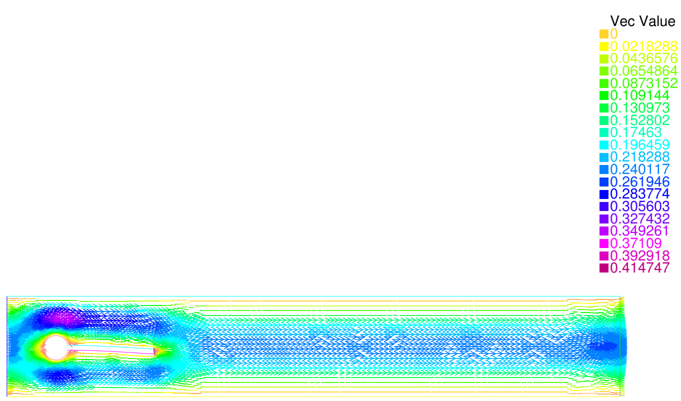

Figure 4: The velocity profile and the structure deformation

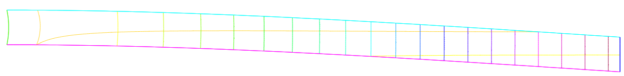

Figure 5: The structure deformation

In the following table, we present numerical values obtained after simulations.

Table 2: Numerical values

N	$J(\alpha_{ini})$	$J(\alpha_{op})$	$\|\nabla J(\alpha)\|_{\infty}$
20	25.2911	2.6259	2.007×10^{-2}

In this table, from initial value of $J = 25.2911$ for $\alpha = \alpha_{ini} = 0$, we reach about $N = 20$ iterations a final value of $J = 2.6259$ with $\|\nabla J(\alpha)\|_{\infty} = 2.007 \times 10^{-2}$. This table of values shows that the optimization method with BFGS algorithm is suitable for solving a fluid structure solid interaction problem. However, our goal is to bring J closer to zero and $\|\nabla J(\alpha)\|_{\infty} \leq \epsilon$ with ϵ very small. So, changing the shape of approximations functions and increasing the number of optimization variables, we can hope tending J to zero[4]

4 Conclusion

In this paper, we show that we can use an optimization method to solve fluid structure interaction benchmark problems. Our method is based mainly on the BFGS algorithm. The numerical results obtained appear good. In our future work, we will try to find a suitable approximation functions in order to tend J to zero and to apply this strategy on the unsteady benchmark problems.

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