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MONOMIALITY, ORTHOGONAL AND PSEUDO-ORTHOGONAL POLYNOMIALS

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Abstract

We reconsider some families of orthogonal polynomials, within the framework of the so called monomiality principle. We show that the associated operational formalism allows the framing of the polynomial orthogonality using an algebraic point of view. Within such a framework, we introduce families of pseudo-orthogonal polynomials, namely polynomials, not orthogonal under the ordinary definition, but providing series expansions, which can be obtained from the ordinary series using the monomiality correspondence.

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1 Introduction

It has been shown that the two variable Laguerre polynomials can be introduced as (see ref. [1])

$$L_n(x,y) = (y - D_x^{-1})^n \tag{1}$$

where D_x^{-1} is the negative derivative operator, defined in such a way that

$$D_x^{-1} f(x) = \int_0^x f(\xi) d\xi \,. \tag{2}$$

With this assumption we get from eq. (1)

$$L_n(x,y) = \sum_{s=0}^{n} \binom{n}{s} (-1)^s y^{n-s} D_x^{-s} = n! \sum_{s=0}^{n} \frac{(-1)^s y^{n-s} x^s}{(n-s)!(s!)^2} \,, \tag{3}$$

obtained on account of the fact that, when the negative derivative operator is acting on unity, we find

$$D_x^{-n} = \frac{x^n}{n!} \,. \tag{4}$$

The ordinary Laguerre polynomials $L_n(x)$ are obtained from eq. (3) by setting $y = 1$.

The above definition, based on the negative derivative operator formalism, allows a fairly straightforward derivation of old and new relations involving expansions in terms of Laguerre polynomials. A very natural example is offered by eq. (4) itself, which, rewritten as

$$x^n = n!(y - (y - D_x^{-1}))^n \tag{5}$$

yields, according to eqs. (1)-(3),

$$x^n = n! \sum_{s=0}^{n} \binom{n}{s} (-1)^s y^{n-s} L_s(x,y) \,, \tag{6}$$

i.e. the expansion of an ordinary monomial in terms of Laguerre polynomials.

It must be stressed that the derivation of the identity (6) is purely algebraic and does not imply any assumption on the orthogonality of the Laguerre polynomials (see e.g. ref. [2] for an orthodox derivation). The extension of eq. (6) to non integer values of n is easily obtained too, using the properties of the Newton binomial.

We want to emphasize that such an algebraic point of view to the expansion in terms of Laguerre polynomial may be a fairly interesting tool to be further investigated for its usefulness in calculations and for possible extensions of the orthogonality concept it may offer.

2 Operational methods and expansion in Laguerre polynomial series

Let us go back to the definition (1) and note that can we use the following operational definition to introduce the Laguerre polynomials

$$L_n(x,y) = \exp\left(-D_x^{-1}\frac{\partial}{\partial y}\right) y^n \tag{7}$$

the exponential operator, which acts on any function of y as a shift operator, can be further understood as follows

$$\exp\left(-D_x^{-1}\frac{\partial}{\partial y}\right) = \sum_{s=0}^{\infty}\frac{(-1)^s}{s!}D_x^{-s}\left(\frac{\partial}{\partial y}\right)^s = \sum_{s=0}^{\infty}\frac{(-1)^s}{(s!)^2}x^s\left(\frac{\partial}{\partial y}\right)^s \tag{8}$$

which can cast in a closed form as follows

$$\exp\left(-D_x^{-1}\frac{\partial}{\partial y}\right) = C_0\left(x\frac{\partial}{\partial y}\right) \tag{9}$$

where we have exploited the 0^{th} order Tricomi function linked to the ordinary cylindrical Bessel function by

$$C_n(x) = x^{-n/2}J_n(2\sqrt{x})\,. \tag{10}$$

According to the previous relations we can also conclude that

$$\begin{aligned} L_n(x,y) &= C_0\left(x\frac{\partial}{\partial y}\right) y^n \\ y^n &= \exp\left(D_x^{-1}\frac{\partial}{\partial y}\right) L_n(x,y)\,. \end{aligned} \tag{11}$$

The last of eq. (11) is just a consequence of the fact that

$$\exp\left(-D_x^{-1}\frac{\partial}{\partial y}\right)\exp\left(D_x^{-1}\frac{\partial}{\partial y}\right) = 1 \tag{12}$$

which does not imply that

$$C_0\left(D_x^{-1}\frac{\partial}{\partial y}\right)C_0\left(-D_x^{-1}\frac{\partial}{\partial y}\right) = 1\,. \tag{13}$$

We can use the previous relations to get further series expansion in terms of Laguerre polynomials, without any explicit use of their orthogonality. We start from the definition of the exponential function in terms of the Taylor expansion

$$\exp(ty) = \sum_{n=0}^{\infty} \frac{t^n y^n}{n!} \tag{14}$$

and note that

$$C_0\left(x\frac{\partial}{\partial y}\right)\exp(y) = \exp(y)C_0(x) \tag{15}$$

thus getting, from eq. (11) and (14)

$$\exp(ty)C_0(tx) = \sum_{n=0}^{\infty} \frac{t^n}{n!}\, L_n(x,y)\,, \tag{16}$$

which is one of the well known generating functions of the Laguerre polynomials (ref. [2]).

Let us note that we can exploit other well known expansions like

$$\sum_{n=0}^{\infty} t^n y^n = \frac{1}{1-yt}\,, \qquad |\, yt \,| < 1\,, \tag{17}$$

to obtain, according to the same procedure as before, (see ref. [1])

$$\sum_{n=0}^{\infty} t^n L_n(x,y) = \frac{1}{1-t(y-D_x^{-1})} = \frac{1}{1-yt}\exp\left(-\frac{xt}{1-yt}\right)\,, \tag{18}$$

which is the ordinary generating function of Laguerre polynomials (ref. [2]).

The following well known expansion can be derived from eq. (18)

$$\exp(-ay) = \frac{1}{1+a}\sum_{n=0}^{\infty}\left(\frac{a}{1+a}\right)^n L_n(y)\,, \qquad a > -\frac{1}{2}\,. \tag{19}$$

The use of eq. (15) and of the identity

$$C_0\left(x\frac{\partial}{\partial y}\right)L_n(y) = {}_L L_n(x,y) = n!\sum_{r=0}^{n}\frac{(-1)^r L_r(x,y)}{(n-r)!(r!)^2}\,, \tag{20}$$

leads to

$$C_0(-ax) = \frac{\exp(ay)}{1+a}\sum_{n=0}^{\infty}\left(\frac{a}{1+a}\right)^n ({}_L L_n(x,y))\,. \tag{21}$$

This last relation can be viewed, on account of eq. (10), as a new form of expansion of Bessel functions, which has been obtained using the polynomials, given in eq. (20), without any explicit use of orthogonality properties and without knowing whether they belong to any orthogonal set.

In the forthcoming section we will see how the above point of view can be extended to other polynomial families and show that the concept of polynomial orthogonality can be framed within a more general algebraic context.

3 The pseudo-orthogonality

We have introduced the polynomials ${}_L L_n(x,y)$, which can be exploited to derive other expansions directly inspired by those of ordinary Laguerre, we note indeed that we obtain from eqs. (6), (7) the relation

$$C_0\left(x\frac{\partial}{\partial y}\right)y^n = n!\sum_{s=0}^{n}\binom{n}{s}(-1)^s C_0\left(x\frac{\partial}{\partial y}\right)L_s(y) \tag{22}$$

which eventually yields

$$L_n(x,y) = n!\sum_{s=0}^{n}\binom{n}{s}(-1)^s \; {}_L L_s(x,y)\,. \tag{23}$$

In this case too we have not exploited any concept associated with the orthogonality properties of the above family of polynomials.

The ordinary Laguerre polynomials $L_n(y)$ are bi-orthogonal (ref. [3]) to

$$\phi_n(y) = \exp(-y)L_n(y)\,, \tag{24}$$

we can therefore introduce, along with the polynomials (20), (24), their pseudo-orthogonal partners defined as

$$\Phi_n(x,y) = C_0\left(x\frac{\partial}{\partial y}\right)\phi_n(y) = \phi_n(y - D_x^{-1}) \tag{25}$$

which explicitly writes

$$\begin{aligned}
\Phi_n(x,y) &= \exp(-y)A_n(x,y)\,, \\
A_n(x,y) &= n!\sum_{s=0}^{n}\frac{(-1)^s T_s(x,y)}{(s!)^2(n-s)!} \\
T_s(x,y) &= \sum_{r=0}^{s}\binom{s}{r}y^{r-s}(-1)^s x^s C_s(-x)
\end{aligned} \tag{26}$$

obtained after using the identity

$$D_x^{-s}C_0(x) = x^s C_s(x)\,. \tag{27}$$

The polynomials (20) and the functions (26) are isospectral to the ordinary Laguerre polynomials and to their bi-orthogonal partners but they are by no means orthogonal, in the strict sense of the word. Notwithstanding they can be exploited to define series expansions, like those given in eq. (21) of algebraic nature and not implying any orthogonality condition. *We will say therefore that* $\Phi_n(x,y)$ *and* ${}_L L_n(x,y)$ *form a pseudo-orthogonal set.* This concept will be further discussed in the forthcoming sections.

4 The Hermite polynomials: an algebraic point of view to their series expansion

The Hermite polynomials are defined within the framework of the monomiality treatment by means of the operational rule (ref. [1])

$$H_n(x,y) = \exp\left(y\frac{\partial^2}{\partial x^2}\right) x^n = n! \sum_{r=0}^{\left[\frac{n}{2}\right]} \frac{x^{n-2r} y^r}{r!(n-2r)!} \tag{28}$$

These polynomials belong to the Hermite-Kampé de Fériet family (see [4]) and reduce to the ordinary Hermite for

$$\begin{aligned} &H_n(2x,-1) = H_n(x) \\ &H_n\left(x,-\frac{1}{2}\right) = He_n(x)\,. \end{aligned} \tag{29}$$

It is evident from eq. (28) that we can invert the above definition to get

$$x^n = \exp\left(-y\frac{\partial^2}{\partial x^2}\right) H_n(x,y) = n! \sum_{r=0}^{\left[\frac{n}{2}\right]} \frac{(-y)^r H_{n-2r}(x,y)}{r!(n-2r)!} \tag{30}$$

which represents a kind of expansion of the ordinary monomials in terms of a Hermite-like family of polynomials, which can be proved to belong to an orthogonal set for negative values of y only.

Let us consider a given function $f(x)$ to be expanded in series of ordinary Hermite polynomials, namely

$$f(x) = \sum_{n=0}^{\infty} c_n He_n(x) \tag{31}$$

according to eq. (28) we can also write

$$\exp\left(\frac{1}{2}\frac{\partial^2}{\partial x^2}\right) f(x) = \sum_{n=0}^{\infty} c_n x^n\,. \tag{32}$$

The use of the Gauss transform, namely

$$\exp\left(a\frac{\partial^2}{\partial x^2}\right) f(x) = \frac{1}{2\sqrt{\pi a}} \int_{-\infty}^{\infty} \exp\left(-\frac{(x-\xi)^2}{4a}\right) f(\xi)d\xi \tag{33}$$

allows the following conclusion.

If the function

$$\Pi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left(-\frac{(x-\xi)^2}{2}\right) f(\xi)d\xi \tag{34}$$

can be expanded as

$$\Pi(x) = \sum_{n=0}^{\infty} c_n x^n \tag{35}$$

then the expansion (31) in terms of Hermite polynomials hold, we find indeed from eq. (34) that

$$\Pi(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left(-\frac{\xi^2}{2}\right) He_n(\xi) f(\xi) d\xi \,, \tag{36}$$

which yields the general form of the c_n coefficient and the orthogonality of the polynomials $He_n(x)$.

According to the previous results the orthogonality of Hermite polynomials is one of the consequences of their operational definition (28) and of the associated Gauss transform.

Just to give a further example we note that for $f(x) = \exp(2bx)$ we find

$$\Pi(x) = \exp(2b^2 + 2bx) \tag{37}$$

which leads to

$$c_n = \exp(2b^2) \frac{(2b)^n}{n!} \tag{38}$$

and thus to a well known expansion in terms of Hermite polynomials.

In the case of the polynomials $H_n(x)$ we can formulate our statement in analogous terms, by noting indeed that

$$H_n(x) = \exp\left(-\frac{1}{4}\frac{\partial^2}{\partial x^2}\right)(2x)^n \tag{39}$$

we have to replace the function $\Pi(x)$ with

$$\overline{\Pi}(x) = \exp\left(\frac{1}{4}\frac{\partial^2}{\partial x^2}\right) f(x) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \exp\left(-(x-\xi)^2\right) f(\xi) d\xi \tag{40}$$

and c_n with

$$\bar{c}_n = \frac{c_n}{2^n} \,. \tag{41}$$

In the case of $f(x) = \exp(-a^2x^2)$, $a^2 > -1$, we find

$$\overline{\Pi}(x) = \frac{1}{\sqrt{1+a^2}} \exp\left(-\frac{a^2x^2}{1+a^2}\right) . \tag{42}$$

Thus yielding

$$c_n = \frac{(-1)^n}{2^{2n} n!} \frac{a^{2n}}{(1+a^2)^{n+\frac{1}{2}}} \tag{43}$$

and the corresponding, well known, expansion in terms of Hermite polynomials.

For further extensions of the above point of view to other polynomial sets see the concluding section.

We must underline that, albeit we have not done any explicit request of orthogonality, the present reformulation of the series expansion in terms of Hermite polynomials is perfectly equivalent to the ordinary case and the conditions underlying the expansions are left unchanged.

It is worth stressing that not all the Hermite-Kampé de Fériet provide orthogonal sets, in the strict sense, they are indeed limited to negative values of the y variable only.

Notwithstanding, according to eqs. (28)-(33), we can obtain expansions in terms of Hermite-Kampé de Fériet polynomials even when orthogonality in the strict sense is not guaranteed. We find indeed

$$\begin{aligned} \frac{1}{\sqrt{1+4a}} \exp\left(-\frac{x^2}{1+4a}\right) &= \exp\left(a\frac{\partial^2}{\partial x^2}\right)\exp(-x^2) = \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} H_{2n}(x,a)\,, \qquad |\, a \,| < \frac{1}{4}\,. \end{aligned} \tag{44}$$

In the forthcoming section we will see how the above algebraic point of view can be extended to other families of polynomials.

We can exploit the previous results to discuss the concept of pseudo-orthogonality from a different point of view.

We consider indeed the polynomials $G_n(x,y)$ defined as

$$G_n(x,y) = n! \sum_{r=0}^{\left[\frac{n}{2}\right]} \frac{x^{n-2r} y^r}{[(n-2r)!]^2 r!} \tag{45}$$

which provide a family of polynomials discussed in ref. [1] and share some analogies with the Hermite polynomials in the sense that

$$G_n(x,y) = H_n(\hat{D}_x^{-1}, y)\,. \tag{46}$$

It is therefore worth noting that they can be defined through the operational rule

$$G_n(x,y) = \exp\left(y\left(\frac{\partial}{\partial x} x \frac{\partial}{\partial x}\right)^2\right)\left(\frac{x^n}{n!}\right)\,, \tag{47}$$

which can be exploited in perfect analogy with the analogous rule defining the Hermite polynomials to decide on their orthogonality properties.

To this aim we will try to establish the existence of the analogous of the Gauss transform for the action of the operator $\exp\left[\alpha(\frac{\partial}{\partial x} x \frac{\partial}{\partial x})^2\right]$ on a given function. To this aim we note that in operational terms, using the correspondence

$$x \to \hat{D}_x^{-1} \tag{48}$$

we get also (see ref. [1])

$$\frac{\partial}{\partial \hat{D}_x^{-1}} = \frac{\partial}{\partial x} x \frac{\partial}{\partial x} \,. \tag{49}$$

Let us now consider any function $f(x)$ admitting the series expansion

$$f(x) = \sum_{r=0}^{\infty} \frac{a_r}{r!} x^r \tag{50}$$

and assume that, in correspondence of this, the function

$$g(x) = \sum_{r=0}^{\infty} a_r x^r \tag{51}$$

linked to $f(x)$ by the Laplace-type transform

$$g(x) = \int_0^{+\infty} f(xt) \exp(-t) dt \tag{52}$$

exists. We can therefore state the following identity

$$\exp\left(\alpha \left(\frac{\partial}{\partial x} x \frac{\partial}{\partial x}\right)^2\right) f(x) = \exp\left(\alpha \left(\frac{\partial}{\partial \hat{D}_x^{-1}}\right)^2\right) g(\hat{D}_x^{-1}) \,. \tag{53}$$

Accordingly we can write that

$$\exp\left(\alpha \left(\frac{\partial}{\partial x} x \frac{\partial}{\partial x}\right)^2\right) f(x) = \frac{1}{2\sqrt{\pi\alpha}} \int_{-\infty}^{\infty} \exp\left(-\frac{(\hat{D}_x^{-1} - \xi)^2}{4\alpha}\right) g(\xi) d\xi \,. \tag{54}$$

Let us now consider the expansion of the function $f(x, y)$, where $y > 0$ denotes a fixed parameter, in terms of the polynomials $G_n(x, y)$, namely

$$f(x, y) = \sum_{n=0}^{\infty} c_n G_n(x, y) \,. \tag{55}$$

By applying the same considerations, we used in the case of Hermite polynomials, we find that

$$\Pi(x, y) := \exp\left(y \left(\frac{\partial}{\partial x} x \frac{\partial}{\partial x}\right)^2\right) f(x, y) = \sum_{n=0}^{\infty} c_n(y) \frac{x^n}{n!} \tag{56}$$

and, according to eq. (54), we and up with

$$\Pi(x,y)=\frac{1}{2\sqrt{\pi y}}\int_{-\infty}^{\infty}\exp\left(-\frac{(\hat{D}_x^{-1}-\xi)^2}{4y}\right)g(\xi)d\xi=\sum_{n=0}^{\infty}c_n(y)\frac{x^n}{n!}\,, \tag{57}$$

thus getting

$$c_n(y)=\frac{1}{2\sqrt{\pi y}}\int_{-\infty}^{\infty}\exp\left(-\frac{\xi^2}{2}\right)H_n\left(\frac{\xi}{2y},-\frac{1}{4y}\right)g(\xi)d\xi\,. \tag{58}$$

The conclusion is that the polynomials $G_n(x,y)$ are not orthogonal in the strict sense, but they can be exploited to expand a function $f(x,y)$ using the coefficients of the expansion in terms of Hermite polynomials of the function $g(x)$. The polynomials $G_n(x,y)$ can therefore be considered pseudo-orthogonal and its partners, omitted for the sake of conciseness, have been discussed in ref. [5].

According to the discussion of this section, we can draw a general conclusion.

We consider a family of polynomials $p_n(x)$ defined by the operational rule

$$p_n(x)=\widehat{T}x^n\,, \tag{59}$$

where the operator $\widehat{T}$ admits an inverse. If the inverse function

$$\widehat{T}^{-1}f(x)=\Phi(x) \tag{60}$$

exists and can be expanded in Taylor series, then the function $f(x)$ can be expanded in series of the above polynomials with the same coefficients of the Taylor expansion of the function $\Phi(x)$.

5 The Legendre polynomials

It has been shown that the Legendre polynomials can be framed within the monomiality treatment and that they can be introduced using an operational definition in between the Hermite and the Laguerre case, we have indeed (ref. [1])

$$S_n(x,y)=C_0\left(x\frac{\partial^2}{\partial y^2}\right)y^n=n!\sum_{r=0}^{\left[\frac{n}{2}\right]}\frac{(-1)^r y^{n-2r}x^r}{(n-2r)!(r!)^2}\,. \tag{61}$$

The ordinary Legendre polynomials are obtained as a particular case of (33) as (for the ordinary formulation see refs. [1], [2])

$$P_n(y)=S_n\left(\frac{1-y^2}{4}\,,\,y\right)\,. \tag{62}$$

We can now exploit the same point of view as before and look for expansions in terms of Legendre polynomials with the only help of algebraic concepts.

We can extend the previous considerations relevant to the Hermite polynomials to the present family by noting that an alternative operational definition is

$$S_n(x, y) = H_n(y, -D_x^{-1}) \,. \tag{63}$$

In this case we can directly apply the formalism developed for Hermite polynomials in a fairly direct way.

We can indeed expand the function $f(y) = \exp(2by)$ as follows

$$\exp(2by) = \sum_{n=0}^{\infty} \frac{(2b)^n}{n!} \exp(4b^2 D_x^{-1}) H_n(y, -D_x^{-1}) \,. \tag{64}$$

The above relation is of operational nature and can be written in a more explicit form by exploiting the following identity

$$\begin{aligned} H_n(y, -D_x^{-1}) \exp(4b^2 D_x^{-1}) &= n! \sum_{r=0}^{[\frac{n}{2}]} \frac{(-1)^r y^{n-2r} x^r}{(n-2r)! r!} C_r(-4b^2 x) = \\ &= B_n(y, -x; 4b^2 x) \,, \end{aligned} \tag{65}$$

thus getting

$$\exp(2by) = \sum_{n=0}^{\infty} \frac{(2b)^n}{n!} B_n(y, -x; 4b^2 x) \,. \tag{66}$$

We will comment on the nature of the function $B_n(x, y)$ in the concluding section.

By noting furthermore that

$$\begin{aligned} \exp(4b^2 D_x^{-1}) H_n(y, -D_x^{-1}) &= \sum_{r=0}^{\infty} \frac{(4bx)^r}{r!} S_n^r(x, y) \,, \\ S_n^r(x, y) &= n! \sum_{s=0}^{[\frac{n}{2}]} \frac{(-1)^s y^{n-2s} x^s}{(n-2s)!(r+s)! s!} \end{aligned} \tag{67}$$

we can obtain an alternative expansion in terms of the polynomials $S_n^r(x, y)$, whose link with the associated Legendre polynomials will be discussed in a forthcoming paper.

Before concluding this section we will see how the ordinary monomials can be expanded in Legendre series. The use of the operational analogy with the Hermite polynomials and eq. (30) yields

$$y^n = n! \sum_{r=0}^{[\frac{n}{2}]} \frac{D_x^{-r} H_n(y, -D_x^{-1})}{r!(n-2r)!} = n! \sum_{r=0}^{[\frac{n}{2}]} \frac{x^r S_{n-2r}^r(x, y)}{r!(n-2r)!} \tag{68}$$

again obtained without any assumption of orthogonality.

In the forthcoming concluding section we will clarify some points left open in the so far developed analysis.

6 Concluding remarks

In the previous section we have introduced the functions $B_n(a,b;x) = B_n^0(a,b;x)$, without any further specification. We will now comment on their nature and discuss an independent derivation in terms of the generating function

$$\begin{aligned} \exp(at)C_m(x-bt^2) &= \sum_{n=0}^{\infty} \frac{t^n}{n!}\, B_n^m(a,b;x)\,, \\ B_n^m(a,b;x) &= n! \sum_{r=0}^{[\frac{n}{2}]} \frac{b^r a^{n-2r}}{(n-2r)!r!}\, C_{m+r}(x) \end{aligned} \tag{69}$$

which can be easily derived after noting that

$$\begin{aligned} \exp(at)C_m(x-bt^2) &= \exp\left(at - bt^2 \frac{\partial}{\partial x}\right) C_m(x) = \\ &= \sum_{n=0}^{\infty} \frac{t^n}{n!} H_n\left(a, -b\frac{\partial}{\partial x}\right) C_m(x)\,, \\ (-1)^s \left(\frac{\partial}{\partial x}\right)^s C_m(x) &= C_{m+s}(x)\,. \end{aligned} \tag{70}$$

This family of functions satisfies the recurrences

$$\begin{aligned} \frac{\partial}{\partial a} B_n^m(a,b;x) &= n B_{n-1}^m(a,b;x) \\ \frac{\partial}{\partial b} B_n^m(a,b;x) &= n(n-1) B_{n-2}^{m+1}(a,b;x) \end{aligned} \tag{71}$$

which are a direct consequence of their generating function.

In the previous sections we have introduced the Laguerre polynomials by using the operational definition (7), according to which the variable x is viewed as a parameter, on the other side we can use the identity (see ref. [1])

$$\exp\left(-y\frac{\partial}{\partial x} x \frac{\partial}{\partial x}\right)\left(\frac{(-1)^n x^n}{n!}\right) = L_n(x,y) \tag{72}$$

which is just a consequence of the fact that the Laguerre polynomials satisfy the p.d.e.

$$\begin{aligned} \frac{\partial}{\partial y}\, F(x,y) &= -\frac{\partial}{\partial x} x \frac{\partial}{\partial x}\, F(x,y) \\ F(x,0) &= \frac{(-x)^n}{n!}\,. \end{aligned} \tag{73}$$

More in general if $F(x,0) = f(x)$ the solution of the above equation can be written in terms of the following integral transform (ref. [5])

$$F(x,y) = \exp\left(\frac{x}{y}\right) \int_0^{+\infty} \exp(-s) C_0\left(\frac{x}{y}s\right) f(-ys)ds \tag{74}$$

which holds since the 0^{th} order Tricomi function is an eigenfunction of the operator $-\frac{\partial}{\partial x} x \frac{\partial}{\partial x}$.

It is evident that the previous transform plays a role analogous to the Gauss transform of the Hermite case, we can therefore formulate the following statement:

If the function

$$F(x) = \exp(-x) \int_0^{+\infty} \exp(-s) C_0(-xs) f(s) ds \tag{75}$$

admits the expansion

$$F(x) = \sum_{n=0}^{\infty} c_n \frac{(-x)^n}{n!} \tag{76}$$

then $f(x)$ can be expanded in terms of Laguerre polynomials as

$$f(x) = \sum_{n=0}^{\infty} c_n L_n(x)\,. \tag{77}$$

By expanding, indeed, the exponential and the Tricomi function in eq. (75), we get

$$\begin{aligned} F(x) &= \sum_{r=0}^{\infty} \frac{(-x)^r}{r!} \int_0^{+\infty} \exp(-s) \sum_{p=0}^{\infty} \frac{(xs)^p}{(p!)^2} f(s) ds = \\ &= \sum_{n=0}^{\infty} \frac{(-x)^n}{n!} \int_0^{+\infty} \exp(-s) f(s) L_n(s) ds\,, \end{aligned} \tag{78}$$

which is the proof of our statement.

In a forthcoming paper we will see how the present results can be extended to higher order and multidimensional Hermite polynomials.

Before concluding we want to discuss a further problem involving the Chebyshev polynomials of second kind $U_n(x)$, which are linked to the Hermite polynomials by means of the following transform (see ref. [1])

$$U_n(x) = \frac{1}{n!} \int_0^{+\infty} \exp(-s) s^{n/2} H_n(x\sqrt{s}) ds\,. \tag{79}$$

If a given function $f(x)$ can be expanded in terms of Chebyshev polynomials, we also have

$$f(x) = \sum_{n=0}^{\infty} \frac{c_n}{n!} \int_0^{+\infty} \exp(-s) s^{n/2} H_n(x\sqrt{s}) ds \tag{80}$$

if the sum on the r.h.s. of eq. (65) is such that

$$F(x,s) = \sum_{n=0}^{\infty} \frac{c_n}{n!} s^{n/2} H_n(x\sqrt{s}) \,, \tag{81}$$

we can conclude that all the functions of the type

$$f(x) = \int_0^{+\infty} \exp(-s) F(x,s) ds \tag{82}$$

can be expanded in terms of $U_n(x)$ polynomials, just exploiting the corresponding expansion in terms of ordinary Hermite.

The function $F(x) = \exp(-x)$ corresponds to $f(x) = 1/(1-x)$ and according to the previous relations and to eqs. (35)-(38) we find

$$\frac{1}{1-x} = \frac{4}{3} \sum_{n=0}^{+\infty} \left(\frac{1}{\sqrt{3}}\right)^n U_n\left(\frac{2}{\sqrt{3}} x\right) . \tag{83}$$

This procedure is fairly helpful but does not fully reply the concept of orthogonality of Chebyshev polynomials, since the family of functions (69) is only a family of functions admitting an expansion in terms of $U_n(x)$ polynomials. We will show elsewhere that the method can be extended to Legendre, Gegenbauer, ... polynomials and that their orthogonality properties can be framed within an algebraic context.

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