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The Odd Generalized Exponential Generalized Linear Exponential Distribution

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Abstract: In this article, a new generalization of the Generalized Linear Exponential distribution called the odd generalized exponential generalized linear exponential distribution is proposed. The mathematical properties, including moments and order statistics, have been derived. An application of the model to real data sets revealed that the new model can be used to provide a better fit than its sub-models.

Keywords: Odd, Generalized linear exponential, quantile, moments, hazard.

1 Introduction

Statistical distributions play an important role in modeling real world phenomenon. Because of this, umpteen of statistical distributions have been used in different branches of applied sciences (medicine, engineering and finance, amongst others) to model and analyze lifetime data. However, there still remain many vital problems where most of the existing distributions do not provide a good fit to these data sets. This creates room for researchers to continue to develop new distributions to model lifetime data. For instance, [12] developed another weighted Weibull from the Azzalinis family and [6] developed the Weibull Rayleigh distribution.

The Linear Exponential (LE) distribution, having exponential and Rayleigh distributions as sub-models, is a well-known distribution for modeling lifetime data and for modeling phenomenon with linearly increasing failure rates. A random variable X is said to have LE distribution with two parameters $a > 0$ and $b > 0$, if it has the Cumulative Distribution Function (CDF):

$$F(x; a, b) = 1 - e^{-\left(ax+\frac{bx^2}{2}\right)}, x > 0, \tag{1}$$

and the corresponding Probability Density Function (PDF) given by:

$$f(x; a, b) = (a + bx)\, e^{-\left(ax+\frac{bx^2}{2}\right)}. \tag{2}$$

However, the LE distribution does not provide a good fit for modeling phenomena with decreasing, non-linear increasing, or non-monotonic failure rates such as the bathtub shape, which usually occurs in firmware reliability modeling and biological studies; see for example [4]. Several different parametric families of distributions suitable for modeling non-monotonic failure rates have been proposed in statistical literature. Among these are; the generalized linear exponential distribution, which was proposed by [10] and the exponentiated generalized linear exponential distribution which was proposed by [2]. Also, [3] proposed another generalized linear failure rate distribution and [14] developed a new generalized linear exponential distribution.

Recently, [7]developed an odd family of univariate continuous distributions called the Odd Generalized Exponential

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(OGE) family and studied the OGE-Weibull (OGE-W) distribution, the OGE-Frchet (OGE-Fr) distribution and the OGE-Normal (OGE-N) distribution. According to [7], the CDF of the OGE family of distribution is defined by:

$$F\,(x;\alpha,\lambda,\boldsymbol{\xi}) = \left(1 - e^{-\lambda \frac{G(x,\boldsymbol{\xi})}{\bar{G}(x,\boldsymbol{\xi})}}\right)^{\alpha} \tag{3}$$

where $\alpha > 0$ and $\lambda > 0$ are two additional parameters. The Probability Density Function (PDF) corresponding to (3) is given by:

$$f\,(x;\alpha,\lambda,\boldsymbol{\xi}) = \frac{\lambda\alpha \mathrm{g}\,(x,\boldsymbol{\xi})}{\bar{G}\,(x,\boldsymbol{\xi})^2} e^{-\lambda \frac{G(x,\boldsymbol{\xi})}{\bar{G}(x,\boldsymbol{\xi})}} \left(1 - e^{-\lambda \frac{G(x,\boldsymbol{\xi})}{\bar{G}(x,\boldsymbol{\xi})}}\right)^{\alpha-1} \tag{4}$$

where $\mathrm{g}(x,\boldsymbol{\xi})$ is the baseline PDF and $\bar{G}(x,\boldsymbol{\xi}) = 1 - G(x,\boldsymbol{\xi})$ is the baseline survival function. These models are flexible in nature because of the behavior of their hazard function: increasing, decreasing, bathtub and upside down bathtub shapes. Using the CDF defined in (3), [1] developed the OGE-Rayleigh (OGE-R) distribution and studied its statistical properties. In addition, [8] developed the OGE-Log Logistic (OGE-LL) distribution while [9] developed and studied the OGE-Gompertz (OGE-G) distribution. In this paper, a new generalization of the Generalized Linear Exponential (GLE) distribution, called the Odd Generalized Exponential Generalized Linear Exponential (OGE-GLE) distribution is proposed, and its properties studied.

The rest of the paper is organized as follows: In section 2, we define the OGE-GLE distribution, discuss some special sub-models and provide its CDF, PDF and hazard function. A formula for generating OGE-GLE random samples from the OGE-GLE distribution is given in section 2. Section 3 discusses some statistical properties of the OGE-GLE distribution such as the quantile, median, moments, and moment generating function. The distribution of its order statistic are derived in section 4. The maximum likelihood estimators of the parameters are established in section 5 and the application of the new distribution to real data sets, demonstrated in section 6.

2 The Odd Generalized Exponential Generalized Linear Exponential Distribution

Using the definition of [7], the CDF of of a non-negative random variable X having OGE-GLE distribution denoted as OGE-GLE(α,a,b,c,λ) is given by:

$$F_{OGE-GLE}(x;\alpha,a,b,c,\lambda) = \left[1 - e^{-\lambda\left(e^{\left(a+\frac{bx^2}{2}\right)^c}-1\right)}\right]^{\alpha}, x \geqslant 0 \tag{5}$$

The parameters $\lambda > 0, a > 0$ and $b > 0$ are scale parameters while $\alpha > 0$ and $c > 0$ are shape parameters. The corresponding PDF of the OGE-GLE(α,a,b,c,λ) distribution is given by:

$$f_{OGE-GLE}(x;\alpha,a,b,c,\lambda) = \alpha\lambda c\,(a+bx)\left(ax+\frac{bx^2}{2}\right)^{c-1} e^{\left(ax+\frac{bx^2}{2}\right)^c} e^{-\lambda\left(e^{\left(a+\frac{bx^2}{2}\right)^c}-1\right)} \left[1 - e^{-\lambda\left(e^{\left(a+\frac{bx^2}{2}\right)^c}-1\right)}\right]^{\alpha-1}, x \geqslant 0 \tag{6}$$

One advantage of the OGE-GLE distribution is that, it has a closed form CDF, which enables us to generate random numbers from it by using the relation:

$$X = \frac{-a + \sqrt{a^2 + 2b\left[\ln\left(\frac{\lambda - \ln\left(1-u^{\frac{1}{\alpha}}\right)}{\lambda}\right)\right]^{\frac{1}{c}}}}{b}, \tag{7}$$

where U is a uniformly distributed random variable on the $(0,1)$ interval. The relation can be used to generate random samples from a wide set of sub-models of the OGE-GLE distribution, such as the OGE-Exponential (OGE-E), OGE-Rayleigh (OGE-R), OGE-LE, and OGE-Weibull (OGE-W). Table 1 displays the sub-models of the OGE-GLE distribution.

The PDF of the OGE-GLE(α,a,b,c,λ) can be written in terms of the cumulative hazard and the hazard functions of LE(a,b) as:

$$f_{OGE-GLE}(x;\alpha,a,b,c,\lambda) = \alpha\lambda c h_{LE}\,(x)\,[H_{LE}\,(x)]^{c-1}\, e^{H_{LE}(x)^c}\, e^{-\lambda\left(e^{[H_{LE}]^c}-1\right)} \left[1 - e^{-\lambda\left(e^{[H_{LE}(x)]^c}-1\right)}\right]^{\alpha-1}, x \geqslant 0 \tag{8}$$

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Table 1: Sub-models of OGE-GLE

Model	α	a	b	c	λ
OGE-E	-	-	0	1	-
OGE-R	-	0	-	1	-
OGE-LE	-	-	-	1	-
OGE-W	-	-	0	-	-

where $h_{LE}(x) = h_{LE}(x; a, b) = a + bx$, and $H_{LE}(x) = ax + \frac{bx^2}{2}$, are the hazard and the cumulative hazard functions of the LE distribution respectively. Using both binomial and Taylor series expansion, the PDF of the OGE-GLE distribution can be written as:

$$f_{OGE-GLE}(x; \alpha, a, b, c, \lambda) = \alpha\lambda c\,(a + bx)\left(ax + \frac{bx^2}{2}\right)^{c-1} \sum_{j,k=0}^{i=0} \frac{(-1)^{i+j+K}\Gamma(\alpha)\Gamma(j+1)[\lambda(k+1)]^j}{i!J!K!\Gamma(\alpha-k)\Gamma(j-i+1)} e^{-(i-j-1)\left(ax+\frac{bx^2}{2}\right)^c}, x \geqslant 0 \tag{9}$$

The hazard function of the OGE-GLE$(\alpha, a, b, c, \lambda)$ is given by:

$$h_{OGE-GLE}(x; \alpha, a, b, c, \lambda) = \frac{\alpha\lambda c\,(a + bx)\left(ax + \frac{bx^2}{2}\right)^{c-1} e^{\left(ax+\frac{bx^2}{2}\right)^c} e^{-\lambda\left(e^{\left(a+\frac{bx^2}{2}\right)^c}-1\right)} \left[1 - e^{-\lambda\left(e^{\left(a+\frac{bx^2}{2}\right)^c}-1\right)}\right]^{\alpha-1}}{1 - \left[1 - e^{-\lambda\left(e^{\left(a+\frac{bx^2}{2}\right)^c}-1\right)}\right]^{\alpha}}, x \geqslant 0 \tag{10}$$

The PDF and hazard function of the OGE-GLE distribution, for different parameter values, are displayed in Figure 1 and Figure 2 respectively. From the figures, it is clear that the PDF can be decreasing or unimodal and the hazard function can be increasing or bathtubed in nature.

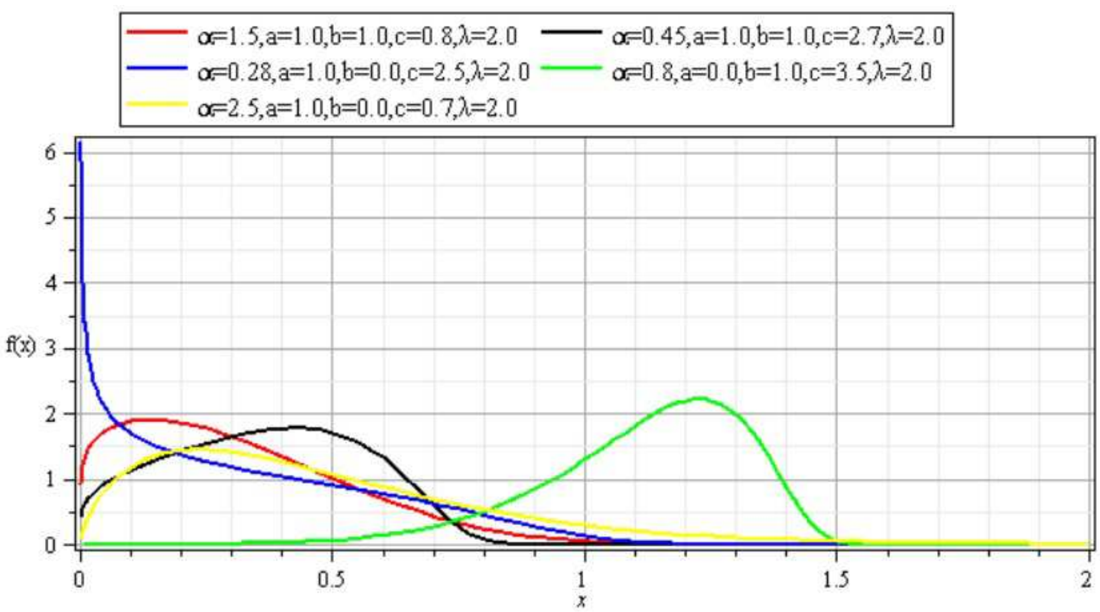

Fig. 1: PDF plot of the OGE-GLE distribution

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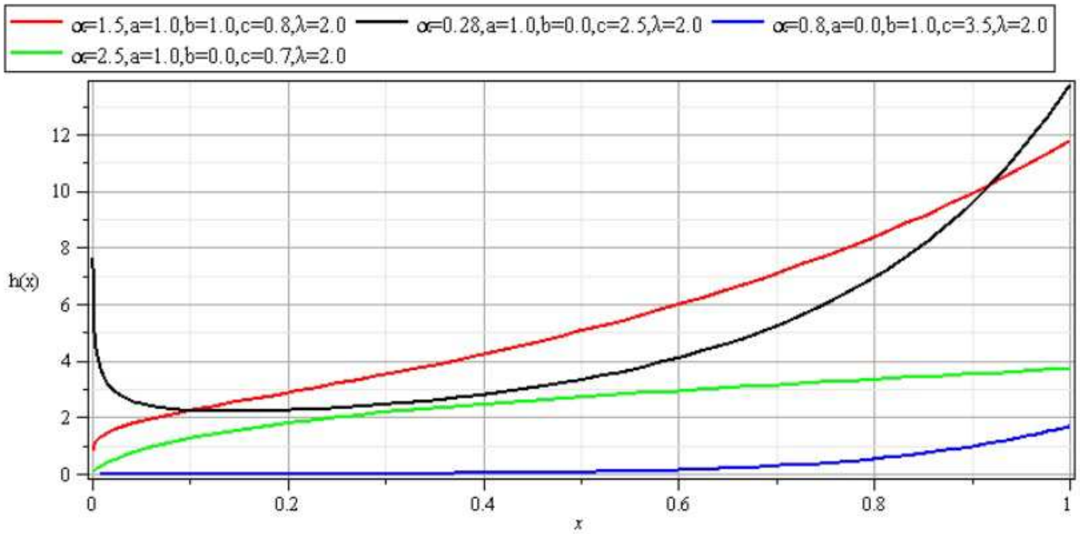

Fig. 2: Hazard plot of the OGE-GLE distribution

3 Statistical Properties

3.1 Quantile and Median

The OGE-GLE$(\alpha, a, b, c, \lambda)$ quantile function, say $Q(u) = F^{-1}(u)$, is straight forward and is computed by inverting (5); we have

$$Q(u) = \frac{-a + \sqrt{a^2 + 2b\left[\ln\left(\frac{\lambda - \ln\left(1 - u^{\frac{1}{\alpha}}\right)}{\lambda}\right)\right]^{\frac{1}{c}}}}{b}, \tag{11}$$

where U is a uniformly distributed random variable on the $(0,1)$ interval. Using (11), the median of the OGE-GLE$(\alpha, a, b, c, \lambda)$ can be obtained as

$$Q(0.5) = \frac{-a + \sqrt{a^2 + 2b\left[\ln\left(\frac{\lambda - \ln\left(1 - 0.5^{\frac{1}{\alpha}}\right)}{\lambda}\right)\right]^{\frac{1}{c}}}}{b}, \tag{12}$$

3.2 Moment

It is customary to derive the moments when a new distribution is proposed. Moments play an important role in any statistical analysis, especially in applications. They are used for finding measures of central tendency, dispersion, skewness and kurtosis among others. Using (9), the $r^t h$ non-central moment of the OGE-GLE distribution is:

$$\mu_r^{'} = \int_0^{\infty} f_{OGE-GLE}(x; \alpha, a, b, c, \lambda) = \omega_{ijk} \int_0^{\infty} x^r \alpha\lambda c\,(a + bx)\left(ax + \frac{bx^2}{2}\right)^{c-1} e^{-(i-j-1)\left(ax + \frac{bx^2}{2}\right)^c} dx \tag{13}$$

where

$$\omega_{ijk} = \sum_{j,k=0}^{i=0} \frac{(-1)^{i+j+K}\Gamma(\alpha)\Gamma(j+1)[\lambda(k+1)]^j}{i!J!K!\Gamma(\alpha-k)\Gamma(j-i+1)}$$

Now, if we define the substitution:

$$y = (i - j - 1)\left(ax + \frac{bx^2}{2}\right)^c \Rightarrow c(a + bx)\left(ax + \frac{bx^2}{2}\right)^{c-1} dx = \frac{dy}{(i - j - 1)}$$

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Clearly,

$$x = \frac{-a + \sqrt{a^2 + 2b\left[\frac{y}{(i-j-1)}\right]^{\frac{1}{c}}}}{b}$$

Thus, expanding binomially, this yields:

$$\mu_r^{'} = \omega_{ijk}\alpha\lambda\int_0^\infty \left[\frac{-a + \sqrt{a^2 + 2b\left[\frac{y}{(i-j-1)}\right]^{\frac{1}{c}}}}{b}\right]^r e^{-y}\frac{dy}{(i-j-1)}$$

$$= \omega_{ijk}\alpha\lambda\int_0^\infty \sum_{s=0}^{r}\binom{r}{s}\frac{(-1)^s(a)^s}{b^{\frac{r+s}{2}}(i-j-1)^{\frac{r-s+2c}{2c}}}\left[1 + \frac{a^2}{2b\left(\frac{y}{i-j-1}\right)^{\frac{1}{c}}}\right]^{\frac{r-s}{2}} 2^{\frac{r-s}{2}}y^{\frac{r-s}{2c}}e^{-y}dy$$

It is easy to verify that $|\frac{a^2(i-j-1)^{\frac{1}{c}}}{2by^{\frac{1}{c}}}| < 1$ when $y > 0$; again expanding binomially results in:

$$\mu_r^{'} = \omega_{ijk}\alpha\lambda\int_0^\infty \sum_{m=0}^{\infty}\sum_{s=0}^{r}\binom{r}{s}\binom{\frac{r-s}{2}}{m}\frac{(-1)^s 2^{\frac{r-s-2m}{2}}a^{2m+s}}{b^{\frac{r+s+2m}{2}}(i-j-1)^{\frac{r-s+2c(1-m)}{2c}}}y^{\frac{r-s-2m}{2c}}e^{-y}dy$$

$$= \omega_{ijk}\alpha\lambda\sum_{m=0}^{\infty}\sum_{s=0}^{r}\binom{r}{s}\binom{\frac{r-s}{2}}{m}\frac{(-1)^s 2^{\frac{r-s-2m}{2}}a^{2m+s}}{b^{\frac{r+s+2m}{2}}(i-j-1)^{\frac{r-s+2c(1-m)}{2c}}}\Gamma\left(1 + \frac{r-s-2m}{2c}\right)$$

This implies:

$$\mu_r^{'} = \varpi_{ijkms}\frac{2^{\frac{r-s-2m}{2}}a^{2m+s}a^{2m+s}\Gamma\left(1 + \frac{r-s-2m}{2c}\right)}{b^{\frac{r+s+2m}{2}}(i-j-1)^{\frac{r-s+2c(1-m)}{2c}}} \tag{14}$$

where;

$$\varpi_{ijkms} = \sum_{j,k,m=0}^{\infty}\sum_{s=0}^{r}\sum_{i=0}^{j}\frac{(-1)^{i+j+k+s}\Gamma(\alpha+1)\Gamma(j+1)\lambda^{j+1}(k+1)^j}{i!j!K!\Gamma(\alpha-k)\Gamma(j-i+1)}$$

for $r = 1, 2, \ldots$, where $\Gamma(.)$ is the gamma function.

3.3 Moment Generating Function

The moment generating function denoted by $M_X(t)$ is obtained by using the definition:

$$M_X(t) = E\left(e^{tx}\right) = \int_0^\infty e^{tx} f_{OGE-GLE}(x;\alpha,a,b,c,\lambda)dx \tag{15}$$

Using the Taylor series expansion of e^{tx}, (15) can be written as

$$M_X(t) = \sum_{r=0}^{\infty}\frac{t^r}{r!}\int_0^\infty x^r f_{OGE-GLE}(x;\alpha,a,b,c,\lambda)dx \tag{16}$$

$$= \sum_{r=0}^{\infty}\frac{t^r}{r!}\mu_r^{'}$$

$$= \varpi^*_{ijkmsr}\frac{2^{\frac{r-s-2m}{2}}a^{2m+s}a^{2m+s}\Gamma\left(1 + \frac{r-s-2m}{2c}\right)}{b^{\frac{r+s+2m}{2}}(i-j-1)^{\frac{r-s+2c(1-m)}{2c}}}$$

where;

$$\varpi^*_{ijkmsr} = \sum_{j,k,m,r=0}^{\infty}\sum_{s=0}^{r}\sum_{i=0}^{j}\frac{(-1)^{i+j+k+s}\Gamma(\alpha+1)\Gamma(j+1)\lambda^{j+1}(k+1)^j}{i!j!k!\Gamma(\alpha-k)\Gamma(j-i+1)}$$

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4 Distribution of Order Statistics

In this section, the PDF of the r_{th} order statistic is derived. Let $X_1, X_2, ..., X_n$ be a random sample from an OGE-GLE distribution and $X_{1:n} \le X_{2:n} \le ... \le X_{n:n}$ denote the corresponding order statistics obtained from the sample. Then the PDF, $f_{r:n}$, of the r^{th} order statistic $X_{r:n}$ is given by

$$f_{r:n}(x) = \frac{1}{B(r, n-r+1}[F(x)][1-F(x)]^{n-r}f(x)$$

where $F(x)$ and $f(x)$ are the CDF and PDF given by (5) and (6) respectively, and $B(.,.)$ is the Beta function. since $0 < F(x) < 1$ for $x > 0$, by using the binomial series expansion of $[1-F(x)]^{n-r}$, which is given by

$$[1-F(x)]^{n-r} = \sum_{\ell=0}^{n-r}(-1)^{\ell}\binom{n-r}{\ell}[F(x)]^{\ell}$$

we have

$$f_{r:n}(x) = \frac{1}{B(r, n-r+1}\sum_{\ell=0}^{n-r}(-1)^{\ell}\binom{n-r}{\ell}[F(x)]^{r+\ell-1}f(x) \tag{17}$$

Therefore, substituting (5) and (6) into (17) one gets

$$f_{r:n}(x) = \sum_{\ell=0}^{n-r}\frac{(-1)^{\ell}n!}{\ell!(r-1)!(n-r-\ell)!(r+\ell)}f_{OGE-GLE}(x; \alpha_{r+\ell}, a, b, c, \lambda) \tag{18}$$

where $f_{OGE-GLE}(x; \alpha_{r+\ell}, a, b, c, \lambda)$ is the PDF of the OGE-GLE distribution with parameters $\alpha_{r+\ell} = \alpha(r+\ell), a, b, c$ and λ. Relation (18) revealed that $f_{r:n}(x)$ is the weighted average of the OGE-GLE distribution with different shape parameters. Using (18), the p^{th} moment of the r^{th} order statistics $X_{r:n}$ is

$$\mu_p^{'(r:n)} = \varpi^{**}_{ijkmsr}\frac{2^{\frac{p-s-2m}{2}}a^{2m+s}\Gamma(1+\frac{p-s-2m}{2c})}{b^{\frac{p+s+2m}{2}}(i-j-1)^{\frac{p-s+2c(1-m)}{2c}}} \tag{19}$$

where;

$$\varpi^{**}_{ijkmsr} = \sum_{j,k,m=0}^{\infty}\sum_{s=0}^{p}\sum_{i=0}^{j}\sum_{\ell=0}^{n-r}\frac{(-1)^{i+j+k+s+\ell}\Gamma(n+1)\Gamma(\alpha_{r+\ell}+1)\Gamma(j+1)\lambda^{j+1}(k+1)^{j}}{\ell!i!j!k!(r-1)!(n-r-\ell)!(r+\ell)\Gamma(\alpha_{r+\ell}-k)\Gamma(j-i+1)}$$

5 Estimation and Inference

In this section, the maximum likelihood estimation approach was employed to estimate the parameters of the OGE-GLE distribution. Given a random sample, denoted as $X_1, X_2, ..., X_n$, with size n, upon using (6) the log-likelihood function for the vector of parameters $\boldsymbol{\Theta} = (\alpha, a, b, c, \lambda)'$ can be written as:

$$\ell = n\ln\alpha + n\ln\lambda + n\ln c + \sum_{i=1}^{n}\ln(a+bx_i) + (c-1)\sum_{i=1}^{n}\ln\left(ax_i + \frac{bx_i^2}{2}\right) + \sum_{i=1}^{n}\left(ax_i + \frac{bx_i^2}{2}\right)^c$$
$$-\lambda\sum_{i=1}^{n}\left[e^{\left(ax_i+\frac{ax_i^2}{2}\right)^c} - 1\right] + (\alpha-1)\sum_{i=1}^{n}\ln\left[1 - e^{-\lambda\left(e^{\left(ax_i+\frac{ax_i^2}{2}\right)^c}-1\right)}\right] \tag{20}$$

Differentiating (20) with respect to α, a, b, c and λ, respectively and equating to zero gives

$$\frac{n}{\alpha} + \sum_{i=1}^{n}\ln\left[1 - e^{-\lambda\left(e^{[H(x_i)]^c}-1\right)}\right] = 0 \tag{21}$$

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Table 2: Parameter estimates of first data set

Model	Parameter Estimates	ℓ	Λ_{H_0}	P-value
OGE-E	$\hat{a} = 0.022(0.002)$ $\hat{\alpha} = 2.555(1.274)$ $\hat{\lambda} = 1.078(0.105)$	-845.96	1431.580	0.000
OGE-R	$\hat{b} = 1.919 \times 10^{-4}(9.50 \times 10^{-6})$ $\hat{\alpha} = 2.016(4.935 \times 10^{-10})$ $\hat{\lambda} = 8.989 \times 10^{-1}(6.628 \times 10^{-10})$	-944.31	1628.280	0.000
OGE-LE	$\hat{a} = 5.207 \times 10^{-3}(7.687^{-4})$ $\hat{b} = 2.040 \times 10^{-4}(1.751 \times 10^{-5})$ $\hat{\alpha} = 1.989(4.218 \times 10^{-6})$ $\hat{\lambda} = 6.992 \times 10^{-1}(7.649 \times 10^{-7})$	-768.870	1277.400	0.000
OGE-W	$\hat{a} = 0.0014(0.001)$ $\hat{c} = 1.162(0.018)$ $\hat{\alpha} = 1.144(0.290)$ $\hat{\lambda} = 0.954(0.064)$	-571.00	881.660	0.000
OGE-GLE	$\hat{a} = 2.028^{-2}(6.500^{-3})$ $\hat{b} = 2.724 \times 10^{-5}(9.032^{-5})$ $\hat{c} = 1.014(4.754 \times 10^{-2})$ $\hat{\alpha} = 9.067 \times 10^{-2}(1.393 \times 10^{-1})$ $\hat{\lambda} = 1.003 \times 10^{-1}(7.776^{-3})$	-130.170	-	-

$$\sum_{i=1}^{n} \frac{1}{h(x_i)} + (c-1)\sum_{i=1}^{n} \frac{x_i}{H(x_i)} + c\sum_{i=1}^{n} x_i[H(x_i)]^{c-1} - \lambda c \sum_{i=1}^{n} x_i[H(x_i)]^{c-1} e^{[H(x_i)]^c} + \lambda c(\alpha-1)\sum_{i=1}^{n} \frac{x_i[H(x_i)]^{c-1} e^{-\lambda\left(e^{[H(x_i)]^c}-1\right)} e^{[H(x_i)]^c}}{1-e^{-\lambda\left(e^{[H(x_i)]^c}-1\right)}} = 0 \quad (22)$$

$$\sum_{i=1}^{n} \frac{x_i}{h(x_i)} + (c-1)\sum_{i=1}^{n} \frac{x_i^2}{2H(x_i)} + \frac{c}{2}\sum_{i=1}^{n} x_i^2[H(x_i)]^{c-1} - \frac{\lambda c}{2} \sum_{i=1}^{n} x_i^2[H(x_i)]^{c-1} e^{[H(x_i)]^c} + \lambda c(\alpha-1)\sum_{i=1}^{n} \frac{x_i^2[H(x_i)]^{c-1} e^{-\lambda\left(e^{[H(x_i)]^c}-1\right)} e^{[H(x_i)]^c}}{2\left[1-e^{-\lambda\left(e^{[H(x_i)]^c}-1\right)}\right]} = 0 \quad (23)$$

$$\frac{n}{c} + \sum_{i=1}^{n} \ln[H(x_i)] + \sum_{i=1}^{n}[H(x_i)]^c \ln[H(x_i)] - \lambda \sum_{i=1}^{n}[H(x_i)]^c e^{[H(x_i)]^c} \ln[H(x_i)] + \lambda c(\alpha-1)\sum_{i=1}^{n} \frac{[H(x_i)]^c e^{-\lambda\left(e^{[H(x_i)]^c}-1\right)} e^{[H(x_i)]^c} \ln[H(x_i)]}{1-e^{-\lambda\left(e^{[H(x_i)]^c}-1\right)}} = 0 \quad (24)$$

$$\frac{n}{\lambda} + (\alpha-1)\sum_{i=1}^{n} \frac{e^{-\lambda\left(e^{[H(x_i)]^c}-1\right)}\left(e^{[H(x_i)]^c}-1\right)}{1-e^{-\lambda\left(e^{[H(x_i)]^c}-1\right)}} + \sum_{i=1}^{n}\left(e^{[H(x_i)]^c}-1\right) = 0 \quad (25)$$

where;
$h(x_i) = a + bx_i$, and $H(x_i) = ax_i + \frac{bx_i^2}{2}$

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The maximum likelihood estimates of $\mathbf{\Theta} = (\alpha, a, b, c, \lambda)'$ say $\hat{\mathbf{\Theta}} = (\hat{\alpha}, \hat{a}, \hat{b}, \hat{c}, \hat{\lambda})'$, can be obtained by solving the non-linear equations (21), (22), (23), (24) and (25). The system of these five non-linear equations cannot be solved analytically and statistical is applied to get numerical solutions via iterative techniques. The Likelihood Ratio (LR) test can be used to compare the fit of the OGE-GLE distribution with its sub-models by computing the maximized unrestricted and restricted log-likelihood for a given data set. The LR test statistic is

$$\Lambda_{H_0} = 2[L_{H_a} - L_{H_0}]$$

The test statistic Λ_{H_0} is asymptotically ($n \to \infty$) distributed as χ^2_d, where d is the degrees of freedom. The LR rejects the null hypothesis if $\Lambda_{H_0} > \chi^2_{d,\gamma}$, where $\chi^2_{d,\gamma}$ denotes the upper 100γ

Table 3: Parameter estimates of second data set

Model	Parameter Estimates	ℓ	Λ_{H_0}	P-value
OGE-E	$\hat{a} = 0.171(0.074)$	-118.150	115.780	0.000
	$\hat{\alpha} = 8.915(5.594)$			
	$\hat{\lambda} = 9.657(1.255)$			
OGE-R	$\hat{b} = 0.106(0.082)$	-105.210	89.900	0.000
	$\hat{\alpha} = 2.733(13.475)$			
	$\hat{\lambda} = 4.953(12.173)$			
OGE-LE	$\hat{a} = 0.214(0.072)$	-79.740	38.960	0.000
	$\hat{b} = 0.004(0.013)$			
	$\hat{\alpha} = 4.924(0.608)$			
	$\hat{\lambda} = 4.806(0.972)$			
OGE-W	$\hat{a} = 4.450(0.586)$	-96.190	71.860	0.000
	$\hat{c} = 0.019(0.005)$			
	$\hat{\alpha} = 1.381(0.118)$			
	$\hat{\lambda} = 3.384(0.888)$			
OGE-GLE	$\hat{a} = 0.818(0.424)$	-60.260	-	-
	$\hat{b} = 0.381(0.749)$			
	$\hat{c} = 0.245(0.185)$			
	$\hat{\alpha} = 0.662(0.274)$			
	$\hat{\lambda} = 0.66742(0.301)$			

6 Applications

In this section, two real data sets were used to demonstrate the importance of the OGE-GLE distribution. The first data set represents the survival times of 121 patients with breast cancer obtained from a large hospital in a period from 1929 to 1938 [5]. The data examined by [11] are: 0.3, 0.3, 4.0, 5.0, 5.6, 6.2, 6.3, 6.6, 6.8, 7.4, 7.5, 8.4, 8.4, 10.3, 11.0, 11.8, 12.2, 12.3, 13.5, 14.4, 14.4, 14.8, 15.5, 15.7, 16.2, 16.3, 16.5, 16.8, 17.2, 17.3, 17.5, 17.9, 19.8, 20.4, 20.9, 21.0, 21.0, 21.1, 23.0, 23.4, 23.6, 24.0, 24.0, 27.9, 28.2, 29.1, 30.0, 31.0, 31.0, 32.0, 35.0, 35.0, 37.0, 37.0, 37.0, 38.0, 38.0, 38.0, 39.0, 39.0, 40.0, 40.0, 40.0, 41.0, 41.0, 41.0, 42.0, 43.0, 43.0, 43.0, 44.0, 45.0, 45.0, 46.0, 46.0, 47.0, 48.0, 49.0, 51.0, 51.0, 51.0, 52.0, 54.0, 55.0, 56.0, 57.0, 58.0, 59.0, 60.0, 60.0, 60.0, 61.0, 62.0, 65.0, 65.0, 67.0, 67.0, 68.0, 69.0, 78.0, 80.0, 83.0, 88.0, 89.0, 90.0, 93.0, 96.0, 103.0, 105.0, 109.0, 109.0, 111.0, 115.0, 117.0, 125.0, 126.0, 127.0, 129.0, 129.0, 139.0, 154.0. The second data set is the data studied by [13], which gives the times of failure and running times for a sample of devices from an eld-tracking study of a larger system. At a certain point in time, 30 units were installed in normal service conditions. Two causes of failure were observed for each unit that failed: the failure caused by an accumulation of randomly occurring damage from power-line voltage spikes during electric storms, and the failure caused by normal product wear. The times are: 2.75, 0.13, 1.47, 0.23, 1.81, 0.30, 0.65, 0.10, 3.00, 1.73, 1.06, 3.00, 3.00, 2.12, 3.00, 3.00, 3.00, 0.02, 2.61, 2.93, 0.88, 2.47, 0.28, 1.43, 3.00, 0.23, 3.00, 0.80, 2.45, 2.66. Table 2 and Table 3 display the maximum likelihood estimates for the first and second data sets respectively, with their corresponding standard errors in brackets,

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and the log-likelihood (ℓ) values for each model. The LR test in Table 2 and Table 3 also revealed that the OGE-GLE distribution provides a good fit than its sub-models as it has the least value for all the information criteria.

In order to compare the OGE-GLE distribution with its sub-models, the Akaike Information Criterion (AIC), the Corrected Akaike Information Criterion (AICC) and the -2ℓ criterion were used. The better distribution corresponds to the smaller AIC, AICC and -2ℓ values. Clearly, the results as indicated in Table 4 for the first data, and Table 5 for the second data, reveal that the OGE-GLE distribution provides a good fit than its sub-models.

Table 4: Criteria for Comparison for first data set

Model	-2ℓ	AIC	AICC
OGE-E	1691.911	1697.911	1698.116
OGE-R	1888.627	1894.627	1894.832
OGE-LE	1537.744	1545.744	1546.089
OGE-W	1142.199	1150.199	1150.544
OGE-GLE	260.324	270.342	270.864

Table 5: Criteria for Comparison for second data set

Model	-2ℓ	AIC	AICC
OGE-E	236.304	242.304	243.227
OGE-R	210.418	216.418	217.341
OGE-LE	159.484	167.484	169.084
OGE-W	192.383	200.383	201.983
OGE-GLE	120.520	130.520	133.020

7 Conclusion

In this article, a new model has been proposed, the so called Odd Generalized Exponential Generalized Linear Exponential (OGE-GLE) distribution which extends the Generalized Linear Exponential distribution in the analysis of data with real support. Various statistical properties of the new distribution such as quantile, moments and moment generating function have been derived. The estimation of parameters of this new distribution was approached by the method of maximum likelihood. An application of the OGE-GLE distribution to real data set revealed that the new distribution can be used quite effectively to provide better fits than its sub-models.

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References

[1] A. Luguterah, Odd Generalized Exponential Rayleigh Distribution, Advances and Applications in Statistics **48**,33-48 (2016).
[2] A. M. Sarhan, A. E. A. Ahmad, A. I. Alasbahi, Exponentiated generalized linear exponential distribution, Applied Mathematical Modeling **37**, 2838-2849 (2013).
[3] A. M. Sarhan,D. Kundu, Generalized linear failure rate distribution, Communications in Statistics: Theory and Methods **38**, 642-660 (2009).
[4] C. D. Lai, M. Xie, D. N. P. Murthy, Bathtub shaped failure rate distributions, in: N. Balakrishnan and C. R. Rao (Eds), Handbook in Reliability **30**, 69-104 (2001).
[5] E. T. Lee, Statistical methods for survival data analysis, John Wiley, New York, (1992).
[6] F. Merovci, I. Elbatal, Weibull-Rayleigh distribution: theory and applications, Applied Mathematics and Information Sciences **9**, 1-11 (2015).

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[7] H. M. Tahir, M. G. Cordeiro, M. Alizadeh, M. Mansoor, M. Zubair, G. G. Hamedani, The odd generalized exponential family of distributions with applications, Journal of Statistical Distributions and Applications **2**, 1-28 (2015).
[8] K. Rosaiah, S. G. Rao, D. C. U. Sivakumar, K. Kalyani, The Odd Generalized Exponential Log Logistic Distribution, International Journal of Mathematics and Statistics Invention **4**, 21-29 (2016).
[9] M. A. El-Damcese, A. Mustafa, B. S. El-Desouky, M. E. Mustafa, Generalized Exponential Gompertz Distribution, Applied Mathematics **6**, 2340-2353 (2015).
[10] M. A. W. Mahmoud, F. M. A. Alam, The generalized linear exponential distribution, Statistics and Probability Letters **80**, 1005-1014 (2010).
[11] M. W. A. Ramos, G. M. Cordeiro, P. R. D. Marinho, C. R. B. Dias, G. G. Hamedani, The Zografos-Balakrishnan log-logistic distribution: properties and applications, Journal of Statistical Theory and Applications **12**, 225-244 (2013).
[12] N. Suleman, Another Weighted Weibull distribution from Azzalini's family, European Scientific Journal **11**, 134-144 (2015).
[13] Q. W. Meeker, A. L. Escobar, Statistical methods in reliability data, John Wiley, New York, (1998).
[14] Y. Tian, M. Tian, Q. Zhu, A new generalized linear exponential distribution and its applications, Acta Mathematicae Applicatae Sinica, English Series **30**, 1049-1062 (2014).

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