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# Multiple-Scales Method and Numerical Simulation of Singularly Perturbed Boundary Layer Problems

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**Abstract:** In this paper, Multiple-scales method is presented for solving second and third order singularly perturbed problems with the boundary layer at one end either left or right. The original second and third order ordinary differential equations are transformed to partial differential equations. These problems have been solved efficiently by using multiple-scales method and Numerical simulations are performed on standard test examples to justify the robustness of the proposed method.

**Keywords:** Singular perturbation problems, Multiple-scales method, Boundary layer problems.

## 1 Introduction

Differential equations with a small parameter multiplying the highest order derivative terms are said to be singularly perturbed which are often found in mathematical problems arising in sciences and engineering. Numerical solution of singularly perturbed boundary value problems in ordinary differential equations is a well known research area. Many numerical methods have been developed for solving singularly perturbed problems. For detail analysis of this type of problems we refer [1]-[33].

The analysis of boundary layer problems and multiple scale phenomena which have been generalized under the notion of singular perturbation problems played a significant role in applied mathematics and theoretical physics [1,15]. Regular perturbation theory is often not applicable to various problems due to resonance effects or the cancellation of degree of freedom. In order to obtain a uniformly valid asymptotic expansion of a solution for these singular perturbed problems a variety of methods has been developed such as boundary layer expansions, multiple scales methods, asymptotic matching, stretched coordinates, averaging and WKB expansions.

A physical system often involves multiple temporal or spatial scales on which characteristics of the system change. In some cases the long time behavior of the system can depend on slowly changing time scales which have to be identified in order to apply multiple scale theory. The choice of the slow or fast changing scales is a nontrivial task. A naive expansion in a power series of the small parameter is often prevented by the appearance of resonant terms in higher orders. These terms have to be compensated by the introduction of counter terms.

Boundary layers are also a common feature of singular perturbed systems. In these cases higher order derivatives disappear in the unperturbed equations which lead to the cancellation of degree of freedom of the system and finally in small regions where the system changes rapidly. Boundary layer theory is a collection of perturbation methods for obtaining an asymptotic approximation to the solution of a differential equation whose highest derivative is multiplied by a small parameter $\varepsilon$. Solutions to such equations usually develop regions of rapid variation as $\varepsilon \to 0$. If the thickness of these regions approaches 0 as $\varepsilon \to 0$, they are called boundary layer and boundary layer theory may be used to approximate solution. These rapid changes cannot be handled by slow scales, but they can be handled by fast or magnified or stretched scales.

In boundary layer theory we treat the solution of the differential equation as a function of two independent variables $x$ and $\varepsilon$ i.e. $y(x;\varepsilon)$. But the main target of this analysis is to obtain a global approximation to solution as a function of $x$, this is achieved by introducing the stretched scale $\xi = x/\varepsilon$, which in this case is the same as the inner variable $x_0 = x$, which in this case is the outer variable. The uniform expansion of the solution of a singular perturbation problem cannot be expressed in

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terms of a single scale i.e. a single combination of $x$ and $\varepsilon$ and , such as $x$ or $x/\varepsilon$ or $x/\varepsilon^{1/2}$ or $x/\varepsilon^2$, making it an ideal problem for application of the method of multiple scales [11].

The idea behind introducing multiple scales is to keep expansions well ordered, minimize the error of the approximation and avoid sometimes the appearance of secular terms. An expansion of a function that depends on an independent variable and a small parameter, such as $y(x;\varepsilon)$ depends strongly on the scale being used. In many situations, there exist multiple-boundary layers at one side, for which multiple calculations of inner and outer solutions and their asymptotic matching have to be made in different separated regions to obtain a uniformly valid solution. Again it turns out that the multiple scales method manages to produce the solution without any matching needed.

Author explores in [9] to solve the boundary layer second-order differential equations by using the interpolation perturbation method. An approximate boundary layer solution is presented in [7] for an axially moving beam with small flexural stiffness. The method of multiple scales is applied to the problem and the composite expansion including two inner solutions and one outer solution is found. In [19] the existence of the exponential series solution to the boundary value problem describing the boundary layer flows of Newtonian fluids has been described. The Crane's solution is generalized for stretching walls with a power law stretching velocity.

In [20] authors study a boundary value problem for a third order differential equation which arises in the study of self-similar solutions of the steady free convection problem for a vertical heated impermeable flat plate embedded in a porous medium. Jong-Shenq Guo et al. considered the structure of solutions of the initial value problem for this third order differential equation. R.A. Khan [21] used the generalized approximation method (GAM) to investigate the temperature field associated with the Falkner-Skan boundary-layer problem. Authors applied in [22] the modified variational iteration method (MVIM) for boundary layer equation in an unbounded domain and Pade approximants had been employed in order to make the work more concise and for the better understanding of the solution behavior.

Classical methods find the inner solution and outer solution separately and match the two solutions using physical constraints. The final solution is a composite expansion including the inner and outer solutions. On the other hand, using the method of multiple scales, the composite expansion can be retrieved at once using a single expansion [3,7]. In this article, the method of multiple scales is applied to construct the solution of second and third order boundary layer problems. It is shown that the method of multiple scales provides the exact solution for second order singularly perturbed boundary layer problems and approximate solution for third order singularly perturbed boundary layer problems. In [8],[23]-[25] approximate solution has been obtained while our proposed method gives the exact solution for second order singularly perturbed boundary layer problems. The original second and third order ordinary differential equations are transformed to partial differential equations. These problems have been solved efficiently by using multiple scale method and numerical simulations are performed on standard test examples to justify the robustness of the proposed method.

The paper is organized as follows: Multiple scales method for the second and third order boundary layer problems is described in Section 2. Numerical example of second and third order boundary layer problems solved by Multiple scales method are presented in Section 3. In Section 4, concluding discussion is briefly mentioned.

## 2 Multiple Scales Method for Second and Third order Boundary Layers Problems

In this section, description of multiple scales method for general second and third order boundary layer problems has been presented.

### 2.1 Second Order Boundary Layer Problem

To explain multiple scales method we consider the general second order singular perturbed equation

$$\varepsilon y'' + a(x)y' + b(x)y = 0, \quad 0 < x < 1 \tag{1}$$

and boundary conditions

$$y(0) = \alpha, \quad y(1) = \beta \tag{2}$$

Here, $\alpha, \beta$ are constants and $\varepsilon$ is the perturbation parameter. It is well known that in equation (1) if $a(x) > 0$, then the boundary layer is at $x = 0$ and if $a(x) < 0$, then the boundary layer is at $x = 1$.

The point of interest is to solve problem (1) with boundary conditions (2) by using multiple scales method. Due to the presence of the boundary layer in the problem (1) we consider two scales; the outer scale at $x_0 = x$ and an inner or boundary layer scale at $\xi = x/\varepsilon$.

Then, using chain rule, the derivatives are defined as follows

$$\frac{d}{dx} = \frac{1}{\varepsilon}\frac{\partial}{\partial \xi} + \frac{\partial}{\partial x_0} \tag{3}$$

$$\frac{d^2}{dx^2} = \frac{1}{\varepsilon^2}\frac{\partial^2}{\partial \xi^2} + \frac{2}{\varepsilon}\frac{\partial^2}{\partial \xi \partial x_0} + \frac{\partial^2}{\partial x_0^2} \tag{4}$$

$$\frac{d^3}{dx^3} = \frac{1}{\varepsilon^3}\frac{\partial^3}{\partial \xi^3} + \frac{\partial^3}{\partial x_0^3} + \frac{3}{\varepsilon^2}\frac{\partial^3}{\partial \xi^2 \partial x_0} + \frac{3}{\varepsilon}\frac{\partial^3}{\partial \xi \partial x_0^2} \tag{5}$$

For the existence of the two scales, we assume the following multi-scale expansion for the solution

$$y = y_0(\xi, x_0) + \varepsilon y_1(\xi, x_0) + \varepsilon^2 y_2(\xi, x_0) + \ldots \tag{6}$$

Substituting (3), (4) and (6) into original equation (1) we obtain

$$\varepsilon\left(\frac{1}{\varepsilon^2}\frac{\partial^2}{\partial\xi^2}+\frac{2}{\varepsilon}\frac{\partial^2}{\partial\xi\partial x_0}+\frac{\partial^2}{\partial x_0^2}\right)(y_0+\varepsilon y_1+\varepsilon^2 y_2+\ldots)$$
$$+a(x)\left(\frac{1}{\varepsilon}\frac{\partial}{\partial\xi}+\frac{\partial}{\partial x_0}\right)(y_0+\varepsilon y_1+\varepsilon^2 y_2+\ldots)$$
$$+b(x)(y_0+\varepsilon y_1+\varepsilon^2 y_2+\ldots)=0 \quad (7)$$

Thus, the original ordinary differential equation (1) is transformed into the partial differential equation (7). Separating the coefficients of each order of $\varepsilon$, one obtain the set of equations

$$\bigcirc(1/\varepsilon):\frac{\partial^2 y_0}{\partial\xi^2}+a\frac{\partial y_0}{\partial\xi}=0 \quad (8)$$

$$\bigcirc(\varepsilon^0):\frac{\partial^2 y_1}{\partial\xi^2}+a\frac{\partial y_1}{\partial\xi}=-2\frac{\partial^2 y_0}{\partial\xi\partial x_0}-a\frac{\partial y_0}{\partial x_0}-by_0 \quad (9)$$

$$\bigcirc(\varepsilon^1):\frac{\partial^2 y_2}{\partial\xi^2}+a\frac{\partial y_2}{\partial\xi}=-2\frac{\partial^2 y_1}{\partial\xi\partial x_0}-\frac{\partial^2 y_0}{\partial x_0^2}-a\frac{\partial y_1}{\partial x_0}-by_1 \quad (10)$$

The general solution of (8) is given by

$$y_0=A(x_0)+B(x_0)e^{-a\xi} \quad (11)$$

where $A$ and $B$ are undetermined at this level of approximation; they are determined at this next level of approximation by imposing the solvability conditions. Putting $y_0$ in (9) gives;

$$\frac{\partial^2 y_1}{\partial\xi^2}+a\frac{\partial y_1}{\partial\xi}=2aB'e^{-a\xi}-aA'-aB'e^{-a\xi}-bA-bBe^{-a\xi} \quad (12)$$

or

$$\frac{\partial^2 y_1}{\partial\xi^2}+a\frac{\partial y_1}{\partial\xi}=(aB'-bB)e^{-a\xi}-(aA'+bA) \quad (13)$$

A particular solution of (13) is

$$y_{1p}=-\frac{(aB'-bB)}{a}\xi e^{-a\xi}-\frac{(aA'+bA)}{a}\xi \quad (14)$$

We seek a solution which is uniformly valid for $0\le x\le 1$ and $0\le\xi\le\varepsilon^{-1}$, but the latter implies $\xi\to\infty$ (as $\varepsilon\to 0^+$); thus we require, for uniformity the coefficients of $\xi$ and $\xi\exp(-a\xi)$ in (14) must vanish independently.

The result is

$$aB'-bB=0,\quad aA'+bA=0. \quad (15)$$

Solving equation (15), find the value of $A$ and $B$. Then substituting these values in (11) and imposing the boundary conditions from (2), we find the values of $a$ and $b$ to obtain the approximate solution. This method has been illustrated more precisely by considering a specific example of second order singularly perturbed boundary layer problem in section 3.

### *2.2 Third Order Boundary Layer Problem.*

To discuss the method of multiple scales method we consider the general third order singular perturbed equation of the form;

$$\varepsilon y'''+a(x)y''+b(x)y'+c(x)y=0,\qquad 0<x<1 \quad (16)$$

and initial conditions are

$$y(0)=\alpha,\quad y'(0)=\beta\quad\text{and}\quad y''(1)=\gamma. \quad (17)$$

Here, $\alpha,\beta,\gamma$ are arbitrary constants and $\varepsilon$ is the perturbation parameter.

Substituting (3), (4), (5) and (6) into the original equation (16) then we have;

$$\varepsilon\left(\frac{1}{\varepsilon^3}\frac{\partial^3}{\partial\xi^3}+\frac{\partial^3}{\partial x_0^3}+\frac{3}{\varepsilon^2}\frac{\partial^3}{\partial\xi^2\partial x_0}+\frac{3}{\varepsilon}\frac{\partial^3}{\partial\xi\partial x_0^2}\right)$$
$$\times(y_0+\varepsilon y_1+\varepsilon^2 y_2+\ldots)$$
$$+a(x)\left(\frac{1}{\varepsilon^2}\frac{\partial^2}{\partial\xi^2}+\frac{2}{\varepsilon}\frac{\partial^2}{\partial\xi\partial x_0}+\frac{\partial^2}{\partial x_0^2}\right)$$
$$\times(y_0+\varepsilon y_1+\varepsilon^2 y_2+\ldots)$$
$$+b(x)\left(\frac{1}{\varepsilon}\frac{\partial}{\partial\xi}+\frac{\partial}{\partial x_0}\right)(y_0+\varepsilon y_1+\varepsilon^2 y_2+\ldots)$$
$$+c(x)(y_0+\varepsilon y_1+\varepsilon^2 y_2+\ldots)=0 \quad (18)$$

Equating the coefficients of each power of $\varepsilon$ to zero of equation (18), we can write

$$\bigcirc(1/\varepsilon^2):\frac{\partial^3 y_0}{\partial\xi^3}+a\frac{\partial^2 y_0}{\partial\xi^2}=0 \quad (19)$$

$$\bigcirc(1/\varepsilon):\frac{\partial^3 y_1}{\partial\xi^3}+a\frac{\partial^2 y_1}{\partial\xi^2}=-3\frac{\partial^3 y_0}{\partial\xi^2\partial x_0}-2a\frac{\partial^2 y_0}{\partial\xi\partial x_0}-b\frac{\partial y_0}{\partial\xi} \quad (20)$$

$$\bigcirc(\varepsilon^0):\frac{\partial^3 y_2}{\partial\xi^3}+a\frac{\partial^2 y_2}{\partial\xi^2}=-3\frac{\partial^3 y_1}{\partial\xi^2\partial x_0}-3\frac{\partial^3 y_0}{\partial\xi\partial x_0^2}-2a\frac{\partial^2 y_1}{\partial\xi\partial x_0}$$
$$-a\frac{\partial^2 y_0}{\partial x_0^2}-b\frac{\partial y_1}{\partial\xi}-b\frac{\partial y_0}{\partial x_0}-cy_0 \quad (21)$$

The general solution of equation (19) is

$$y_0=A(x_0)+B(x_0)\xi+C(x_0)e^{-a\xi}, \quad (22)$$

where $A,B$ and $C$ are undetermined at this level of approximation; they are to be determined at the next level of approximation by imposing the solvability conditions.

Substitute $y_0$ from (22) in equation (20), we can write (20) in view of

$$\frac{\partial^3 y_1}{\partial\xi^3}+a\frac{\partial^2 y_1}{\partial\xi^2}=-3a^2C'e^{-a\xi}-2a(B'-C'ae^{-a\xi})$$
$$-b(B-aCe^{-a\xi}) \quad (23)$$

or

$$\frac{\partial^3 y_1}{\partial \xi^3} + a\frac{\partial^2 y_1}{\partial \xi^2} = -(a^2C' - abC)e^{-a\xi} - (2aB' + bB) \quad (24)$$

The particular solution of (24) is

$$y_{1p} = -\frac{(aC' - bC)}{a}\xi e^{-a\xi} - \frac{(2aB' + bB)}{2a}\xi^2 \quad (25)$$

which makes $\varepsilon y_1$, much larger than $y_0$ as $\xi \to \infty$ . Hence, for a uniform expansion, the coefficients of $\xi\exp(-a\xi)$ and $\xi^2$ in (25) should vanish independently. So, we have

$$aC' - bC = 0, \quad 2aB' + bB = 0. \quad (26)$$

In the forthcoming section, we apply this procedure to find the approximate solution of a particular example of third order singular perturbed boundary layer problem.

## 3 Numerical Simulation

To demonstrate the applicability and robustness of the Multiple scales method, we consider three linear singular perturbation problems; one with left-end boundary layer and two with right-end boundary layer. These examples have been chosen because they have been widely discussed in literature and exact solutions are available for comparison.

**Example 1:** Consider the following homogenous singular perturbation problem

$$\varepsilon y''(x) + (1+\varepsilon)y'(x) + y(x) = 0; \quad x \in [0,1] \quad (27)$$

with

$$y(0) = 0 \quad \text{and} \quad y(1) = 1. \quad (28)$$

As this problem has a boundary layer at $x = 0$ i.e., at the left end of the underlying interval.

Substituting (3), (4) and (6) into the original equation (27), we get

$$\varepsilon\left(\frac{1}{\varepsilon^2}\frac{\partial^2}{\partial\xi^2} + \frac{2}{\varepsilon}\frac{\partial^2}{\partial\xi\partial x_0} + \frac{\partial^2}{\partial x_0^2}\right) \times (y_0 + \varepsilon y_1 + \varepsilon^2 y_2 + \ldots) + (1+\varepsilon)\left(\frac{1}{\varepsilon}\frac{\partial}{\partial\xi} + \frac{\partial}{\partial x_0}\right)(y_0 + \varepsilon y_1 + \varepsilon^2 y_2 + \ldots) + (y_0 + \varepsilon y_1 + \varepsilon^2 y_2 + \ldots) = 0. \quad (29)$$

Equating coefficients of each power of $\varepsilon$ to zero of equation (29), we have

$$\bigcirc(1/\varepsilon): \frac{\partial^2 y_0}{\partial\xi^2} + \frac{\partial y_0}{\partial\xi} = 0 \quad (30)$$

$$\bigcirc(\varepsilon^0): \frac{\partial^2 y_1}{\partial\xi^2} + \frac{\partial y_1}{\partial\xi} = -2\frac{\partial^2 y_0}{\partial\xi\partial x_0} - \frac{\partial y_0}{\partial\xi} - \frac{\partial y_0}{\partial x_0} - y_0 \quad (31)$$

$$\bigcirc(\varepsilon^1): \frac{\partial^2 y_2}{\partial\xi^2} + \frac{\partial y_2}{\partial\xi} = -2\frac{\partial^2 y_1}{\partial\xi\partial x_0} - \frac{\partial^2 y_0}{\partial x_0^2} - \frac{\partial y_1}{\partial\xi} - \frac{\partial y_1}{\partial x_0} - \frac{\partial y_0}{\partial x_0} - y_1 \quad (32)$$

The general solution of (30) is

$$y_0 = A(x_0) + B(x_0)e^{-\xi}. \quad (33)$$

Substitute the value of $y_0$ from (33) in equation (31), we obtain

$$\frac{\partial^2 y_1}{\partial\xi^2} + \frac{\partial y_1}{\partial\xi} = -2(-B'e^{-\xi}) + Be^{-\xi} - (A' + B'e^{-\xi}) - (A + Be^{-\xi}) \quad (34)$$

or

$$\frac{\partial^2 y_1}{\partial\xi^2} + \frac{\partial y_1}{\partial\xi} = B'e^{-\xi} - (A' + A). \quad (35)$$

A particular solution of (35) is

$$y_{1p} = -B'\xi e^{-\xi} - (A' + A)\xi, \quad (36)$$

which makes $\varepsilon y_1$, much bigger than $y_0$ as $\xi \to \infty$. Hence, for a uniform expansion, the coefficients of $\xi$ and $\xi\exp(-\xi)$ in (36) must vanish independently. The result is

$$B' = 0, \quad A' + A = 0. \quad (37)$$

The solution of (37) is

$$A = ae^{-x_0}, \quad B = b \quad (38)$$

where $a$ and $b$ are arbitrary constants.

Putting the values $A$ and $B$ from (38) in equation (33) gives

$$y_0 = ae^{-x_0} + be^{-\xi} \quad (39)$$

or, in terms of the original variable is

$$y_0 = ae^{-x} + be^{-x/\varepsilon}. \quad (40)$$

Substituting the value of $y_0$ in equation (6) gives

$$y = ae^{-x} + be^{-x/\varepsilon} + \ldots \quad (41)$$

Imposing the boundary conditions from (28) in (41) yields

$$a + b = 0 \quad \text{and} \quad ae^{-1} + be^{-1/\varepsilon} = 1. \quad (42)$$

Solving the equation (42) for $a$ and $b$, we obtain

$$a = -\frac{1}{(e^{-1/\varepsilon} - e^{-1})} \quad \text{and} \quad b = \frac{1}{(e^{-1/\varepsilon} - e^{-1})} \quad (43)$$

Putting these values in (41), we obtain the final solution

$$y = -\frac{e^{-x}}{(e^{-1/\varepsilon} - e^{-1})} + \frac{e^{-x/\varepsilon}}{(e^{-1/\varepsilon} - e^{-1})} \quad (44)$$

$$y = \frac{(e^{-x/\varepsilon} - e^{-x})}{(e^{-1/\varepsilon} - e^{-1})} \tag{45}$$

Equation (45) is the exact solution of equation (27) which is given in [8]

**Example 2:**Consider the following homogeneous singular perturbation problem

$$\varepsilon y''(x) - y'(x) - (1+\varepsilon)y(x) = 0; \quad x \in [0,1] \tag{46}$$

with

$$y(0) = 1 + \exp(-(1+\varepsilon)/\varepsilon) \quad \text{and} \quad y(1) = 1 + 1/e \tag{47}$$

which has a boundary layer at $x = 1$ because the coefficient of $y'$ is negative i.e., the boundary layer will be situated at the right end of the underlying interval. For the solution near $x = 1$, we introduce the two scales, $\xi = (x-1)/\varepsilon$ the inner scale and the outer scale $x_0 = x$.

The derivatives can be defined in terms of these scales as

$$\frac{d}{dx} = -\frac{1}{\varepsilon}\frac{\partial}{\partial \xi} + \frac{\partial}{\partial x_0} \tag{48}$$

$$\frac{d^2}{dx^2} = \frac{1}{\varepsilon^2}\frac{\partial^2}{\partial \xi^2} - \frac{2}{\varepsilon}\frac{\partial^2}{\partial \xi \partial x_0} + \frac{\partial^2}{\partial x_0^2} \tag{49}$$

Putting (48), (49) and (6) into the original equation (46) it becomes

$$\begin{aligned}\varepsilon\left(\frac{1}{\varepsilon^2}\frac{\partial^2}{\partial \xi^2} - \frac{2}{\varepsilon}\frac{\partial^2}{\partial \xi \partial x_0} + \frac{\partial^2}{\partial x_0^2}\right)(y_0 + \varepsilon y_1 + \varepsilon^2 y_2 + \dots) \\ -\left(-\frac{1}{\varepsilon}\frac{\partial}{\partial \xi} + \frac{\partial}{\partial x_0}\right)(y_0 + \varepsilon y_1 + \varepsilon^2 y_2 + \dots) \\ -(1+\varepsilon)(y_0 + \varepsilon y_1 + \varepsilon^2 y_2 + \dots) = 0.\end{aligned} \tag{50}$$

Equating coefficients of each power of $\varepsilon$ to zero of (50), we have

$$\bigcirc(1/\varepsilon): \frac{\partial^2 y_0}{\partial \xi^2} + \frac{\partial y_0}{\partial \xi} = 0 \tag{51}$$

$$\bigcirc(\varepsilon^0): \frac{\partial^2 y_1}{\partial \xi^2} + \frac{\partial y_1}{\partial \xi} = 2\frac{\partial^2 y_0}{\partial \xi \partial x_0} + \frac{\partial y_0}{\partial x_0} + y_0 \tag{52}$$

$$\bigcirc(\varepsilon): \frac{\partial^2 y_2}{\partial \xi^2} + \frac{\partial y_2}{\partial \xi} = 2\frac{\partial^2 y_1}{\partial \xi \partial x_0} - \frac{\partial^2 y_0}{\partial x_0^2} + \frac{\partial y_1}{\partial x_0} + y_1 + y_0 \tag{53}$$

The general solution of equation (51) is

$$y_0 = A(x_0) + B(x_0)e^{-\xi} \tag{54}$$

Substituting the value of $y_0$ from (54) in equation (52), we obtain

$$\frac{\partial^2 y_1}{\partial \xi^2} + \frac{\partial y_1}{\partial \xi} = -2B'e^{-\xi} + A' + B'e^{-\xi} + A + Be^{-\xi} \tag{55}$$

or

$$\frac{\partial^2 y_1}{\partial \xi^2} + \frac{\partial y_1}{\partial \xi} = (-B' + B)e^{-\xi} + (A' + A) \tag{56}$$

A particular solution of (56) is

$$y_{1p} = (B' - B)\xi e^{-\xi} + (A' + A)\xi \tag{57}$$

which makes $\varepsilon y_1$, much bigger than $y_0$ as $\xi \to \infty$ . Hence, for a uniform expansion, the coefficients of $\xi$ and $\xi \exp(-\xi)$ in equation (57) must vanish independently. Then the result is

$$B' - B = 0, \quad A' + A = 0 \tag{58}$$

The solution of (58) is

$$A = ae^{-x_0}, \quad B = be^{x_0}, \tag{59}$$

where $a$ and $b$ are arbitrary constants.

Putting the values $A$ and $B$ from (59) in (54) and we get

$$y = ae^{-x_0} + (be^{x_0})e^{-\xi} \tag{60}$$

or, in terms of the original variable is

$$y_0 = ae^{-x} + be^{x}e^{(x-1)/\varepsilon} \tag{61}$$

Putting the value of $y_0$ in equation (6), we get

$$y = ae^{-x} + be^{x}e^{(x-1)/\varepsilon} + \dots \tag{62}$$

Imposing the boundary conditions from (47) then equation (62) yields

$$a + be^{-1/\varepsilon} = 1 + \exp\left(-\frac{(1+\varepsilon)}{\varepsilon}\right) \quad \text{and} \quad a + be^2 = e + 1 \tag{63}$$

Solving these equations we obtain

$$a = 1 \quad \text{and} \quad b = 1/e \tag{64}$$

Putting these values in (62) and considering first two terms, we obtain

$$y = e^{-x} + \frac{1}{e}e^{x(x-1)/\varepsilon} \tag{65}$$

or

$$y = e^{-x} + e^{(1+\varepsilon)(x-1)/\varepsilon} \tag{66}$$

Equation (66) represents the exact solution of equation (46) which is given in [8], [23]-[25]. In these papers authors applied different numerical methods for solving second order singular perturbed two point boundary value problems with the boundary layers and obtained the numerical solution, but our proposed multiple scales method gives directly exact solution to these problems.

**Example 3:** Let us consider the following initial value problem [6]

$$\varepsilon^{3/2}y''' + (\varepsilon^{1/2} + \varepsilon + \varepsilon^{3/2})y'' + (1 + \varepsilon^{1/2} + \varepsilon)y' + y = 0 \tag{67}$$

with initial conditions

$$y(0) = 3, \quad y'(0) = -1 - \varepsilon^{-1/2} - \varepsilon^{-1} \quad \text{and} \quad y''(0) = 1 + \varepsilon^{-1} + \varepsilon^{-2} \tag{68}$$

The coefficient of $y'$ is positive, the boundary layer will be situated at the left-hand edge of the domain i.e. near $x = 0$. Pretending we do not know how to solve it, we resort to conventional singular perturbation methods. It turns out that the conventional perturbation calculation is very tedious and rather challenging. But our proposed multiple scales method successfully finds the approximate solution without any matching by starting only with the thinnest or innermost boundary layer by rescaling $x = \varepsilon\xi$ and expanding $y(x)$ in terms of $\varepsilon$.

Substituting (3), (4), (5) and (6) into the equation (67) we have;

$$\begin{aligned}
&\varepsilon^{3/2}\left(\frac{1}{\varepsilon^3}\frac{\partial^3}{\partial\xi^3} + \frac{\partial^3}{\partial x_0^3} + \frac{3}{\varepsilon^2}\frac{\partial^3}{\partial\xi^2\partial x_0} + \frac{3}{\varepsilon}\frac{\partial^3}{\partial\xi\partial x_0^2}\right) \times (y_0 + \varepsilon y_1 + \ldots) \\
&+(\varepsilon^{1/2} + \varepsilon + \varepsilon^{3/2})\left(\frac{1}{\varepsilon^2}\frac{\partial^2}{\partial\xi^2} + \frac{2}{\varepsilon}\frac{\partial^2}{\partial\xi\partial x_0} + \frac{\partial^2}{\partial x_0^2}\right) \times (y_0 + \varepsilon y_1 + \ldots) \\
&+(1 + \varepsilon^{1/2} + \varepsilon)\left(\frac{1}{\varepsilon}\frac{\partial}{\partial\xi} + \frac{\partial}{\partial x_0}\right) \times (y_0 + \varepsilon y_1 + \ldots) \\
&+(y_0 + \varepsilon y_1 + \ldots) = 0.
\end{aligned} \tag{69}$$

Equating the coefficients of each power of $\varepsilon$ to zero of (69), we get

$$\bigcirc(1/\varepsilon^{3/2}) : \frac{\partial^3 y_0}{\partial\xi^3} + \frac{\partial^2 y_0}{\partial\xi^2} = 0 \tag{70}$$

$$\bigcirc(1/\varepsilon^{1/2}) : \frac{\partial^3 y_1}{\partial\xi^3} + \frac{\partial^2 y_1}{\partial\xi^2} = -3\frac{\partial^3 y_0}{\partial\xi^2\partial x_0} - 2\frac{\partial^2 y_0}{\partial\xi\partial x_0} - \frac{\partial^2 y_0}{\partial\xi^2} - \frac{\partial y_0}{\partial\xi} \tag{71}$$

$$\begin{aligned}
\bigcirc(\varepsilon^{1/2}) : \frac{\partial^3 y_2}{\partial\xi^3} + \frac{\partial^2 y_2}{\partial\xi^2} &= -3\frac{\partial^3 y_0}{\partial\xi^2\partial x_0} - 3\frac{\partial^3 y_0}{\partial\xi\partial x_0^2} \\
&- 2\frac{\partial^2 y_1}{\partial\xi\partial x_0} - 2\frac{\partial^2 y_0}{\partial\xi\partial x_0} - \frac{\partial^2 y_0}{\partial x_0^2} \\
&- \frac{\partial y_1}{\partial\xi} - \frac{\partial y_0}{\partial x_0}.
\end{aligned} \tag{72}$$

The general solution of equation (70) is

$$y_0 = A(x_0) + B(x_0)\xi + C(x_0)e^{-\xi}, \tag{73}$$

where $A, B$ and $C$ are undetermined at this level of approximation. They are determined at the next level of approximation by imposing the solvability conditions.

Substituting the value of $y_0$ from (73) in equation (71), we obtain

$$\frac{\partial^3 y_1}{\partial\xi^3} + \frac{\partial^2 y_1}{\partial\xi^2} = -3C'e^{-\xi} - 2(B' - C'e^{-\xi}) - Ce^{-\xi} - B + Ce^{-\xi} \tag{74}$$

or

$$\frac{\partial^3 y_1}{\partial\xi^3} + \frac{\partial^2 y_1}{\partial\xi^2} = -C'e^{-\xi} - (2B' + B) \tag{75}$$

A particular solution of (75) is

$$y_{1p} = -C'\xi e^{-\xi} - \frac{(2B' + B)}{2}\xi^2 \tag{76}$$

which makes $\varepsilon y_1$, much bigger than $y_0$ as $\xi \to \infty$ . Hence, for a uniform expansion, the coefficients of $\xi\exp(-\xi)$ and $\xi^2$ in (76) must vanish independently.

The results are

$$2B' + B = 0, \quad C' = 0 \tag{77}$$

The solutions of (77) are

$$B = b_0 e^{-x_0/2}, \quad C = c_0, \tag{78}$$

where $b_0$ and $c_0$ are arbitrary constants.

Putting the values of $B$ and $C$ from (78) in equation (73), which gives

$$y_0 = A + b_0 e^{-x_0/2}\xi + c_0 e^{-\xi} \tag{79}$$

or, in terms of the original variable

$$y_0 = A + \frac{x}{\varepsilon}b_0 e^{-x/2} + c_0 e^{-x/\varepsilon} \tag{80}$$

Substituting $y_0$ in (6) we have

$$y = A + \frac{x}{\varepsilon}b_0 e^{-x/2} + c_0 e^{-x/\varepsilon} + \ldots \tag{81}$$

Imposing the boundary conditions (68) in (81), which yields

$$A + c_0 = 3, \quad b_0 - c_0 = -(1 + \varepsilon^{1/2} + \varepsilon) \quad \text{and} \quad -\varepsilon b_0 + c_0 = (1 + \varepsilon + \varepsilon^2) \tag{82}$$

Solving equation (82) for $A$, $b_0$ and $c_0$ we obtain

$$A = \frac{2 - 3\varepsilon + \varepsilon\sqrt{\varepsilon}}{(1 - \varepsilon)}, \quad b_0 = \frac{\varepsilon^2 - \sqrt{\varepsilon}}{(1 - \varepsilon)} \quad \text{and} \quad c_0 = \frac{1 - \varepsilon\sqrt{\varepsilon}}{(1 - \varepsilon)} \tag{83}$$

Then, putting these values in (81) and then equation (81) becomes

$$\begin{aligned}
y = &\left(\frac{2 - 3\varepsilon + \varepsilon\sqrt{\varepsilon}}{(1 - \varepsilon)}\right) + \frac{x}{\varepsilon}\left(\frac{\varepsilon^2 - \sqrt{\varepsilon}}{(1 - \varepsilon)}\right)e^{-x/2} \\
&+ \left(\frac{1 - \varepsilon\sqrt{\varepsilon}}{(1 - \varepsilon)}\right)e^{-x/\varepsilon},
\end{aligned} \tag{84}$$

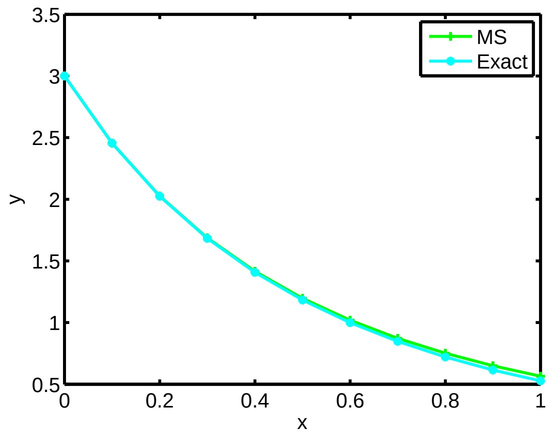


**Fig. 1:** $\varepsilon = 0.3$

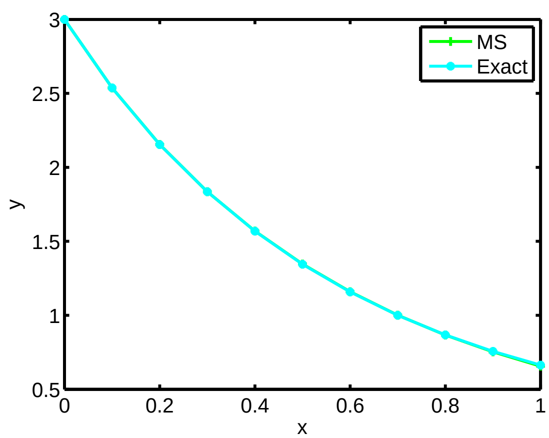


**Fig. 2:** $\varepsilon = 0.4$

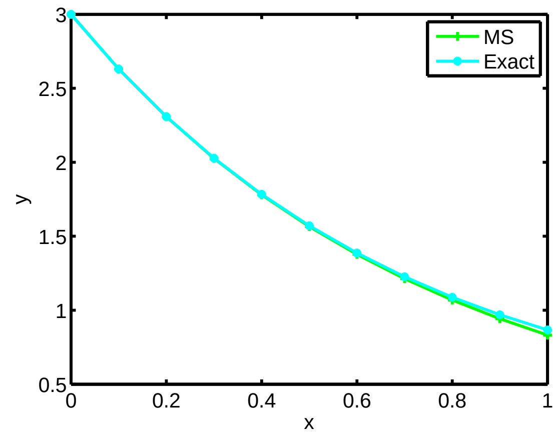


**Fig. 3:** $\varepsilon = 0.6$

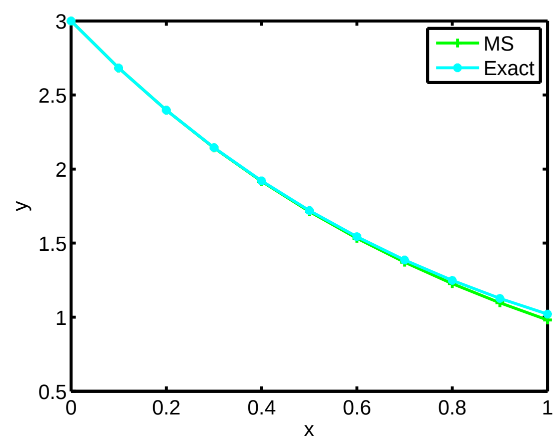


**Fig. 4:** $\varepsilon = 0.8$

then

$$y = \frac{1}{(1-\varepsilon)}\left\{(2-3\varepsilon+\varepsilon\sqrt{\varepsilon}) + \frac{(\varepsilon^2-\sqrt{\varepsilon})x}{\varepsilon}e^{-x/2} + (1-\varepsilon\sqrt{\varepsilon})e^{-x/\varepsilon}\right\}. \quad (85)$$

The equation (85) represents the final approximate solution of equation (67). Numerical simulations are performed varying $\varepsilon$ as it is shown in figure 1-4. The figures indicate that the solution of (85) is very close to the exact solution given in [6].

Figure 1-4 of the numerical solution (85) obtained by Multiple scales (MS) method with the exact solution given in [6].

## 4 Conclusion

We present a numerical method for solving second and third order singularly perturbed problems with the boundary layer at one end (left or right) by Multiple scales method. The original second and third order ordinary differential equations i.e. boundary value problems are transformed to partial differential equations. This method is very easy for implementation. Numerical results of standard examples chosen from the references are presented in support of the proposed theory. Using the method of Multiple scales, a single expansion is sufficient and requires no matching between the expansions. In references [8], [23]-[25] authors applied different numerical techniques to obtain the approximate solution of second order singularly perturbed boundary layer problems while in this article exact solution has been obtained by our proposed method.

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