

# Wave Equation Simulation Using a Compressed Modeler

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## ABSTRACT

Repeated simulations of large scale wave propagation problems are prevalent in many fields. In oil exploration earth imaging problems, the use of full wave simulations is becoming routine and it is only hampered by the extreme computational resources required. In this contribution, we explore the feasibility of employing reduced-order modeling techniques in an attempt to significantly decrease the cost of these calculations. We consider the acoustic wave equation in two-dimensions for simplicity, but the extension to three-dimensions and to elastic or even anisotropic problems is clear. We use the proper orthogonal decomposition approach to model order reduction and describe two algorithms: the traditional one using the SVD of the matrix of snapshots and a more economical and flexible one using a progressive QR decomposition. We include also two a posteriori error estimation procedures and extensive testing and validation is presented that indicates the promise of the approach.

**Keywords:** Wave Equation; POD; Model Order Reduction; Earth Imaging

## 1. Introduction

We consider the problem of the repeated simulation of acoustic wave propagation in complex media. This is the basic problem of the seismic method for hydrocarbon exploration and for many other applications. In the past 20 years or so there has been considerable activity in this area and many different approaches have been proposed. A limitation of these methods in the oil exploration industry today, when large scale 3D regions are in question, is the cost of the forward simulations, *i.e.*, the repeated solution of the acoustic or elastic wave equation in a large 3D mesh. This cost is exacerbated if one considers that a routine 3D survey involves thousands of sources and corresponding receivers, producing millions of seismograms in terabyte size multi-dimensional arrays.

**Model Order Reduction (MOR)** (compressed modeling) refers to a collection of techniques to reduce the number of degrees of freedom of the very large scale dynamical systems that result after space discretization of time-dependent partial differential equations. Some of these techniques have been successfully employed in the simulation of VLSI circuits, computational fluid mechanics, real-time control, heat conduction, flow in porous media, and other problems [1-3]. In comparison, not much has been done for wave propagation in the time domain, although it does not seem that there are fundamental difficulties for its application and activity in this area is

picking up [4-8]. The method is particularly attractive for linear problems.

Even with the use of large scale clusters, techniques that use full fidelity wave solvers are daunting for many imaging and data processing tasks. In [7], we have shown how to obtain very large reductions in the cost of solving the acoustic wave equation many times for related problems by using the method of **Proper Orthogonal Decomposition (POD)**, a MOR technique that basically projects a large dimensional space discretized dynamical system into a much smaller one, using orthogonal basis derived from a set of snapshots of one or at most a few full simulations. In this paper we enhance the solvers used in [7] by adding absorbing boundary conditions that are essential to simulate problems of real interest, where sources and receivers are generally on the free surface.

We propose to use the full fidelity and reduced order codes in several applications that require the repeated simulation of closely related problems, such as those in which only the position of the source or its frequency changes, or in the case of full wave form inversion, when the material properties are slowly varying in an optimization loop. The purpose of this contribution is to show the feasibility of using this approach to reduce considerably the demand in computer resources, while still preserving enough accuracy to make them useful in these real life applications. In separate publications we shall apply these

techniques to problems in seismic imaging.

We show results of an implementation for the two dimensional acoustic wave equation based on a high order finite difference method. There are two codes associated with it: a full fidelity forward solver (used to generate the snapshots and for error control) and a reduced order solver.

There are also two versions of the MOR code: in the first one we use the Singular Value Decomposition of the matrix of snapshots in order to trim those that are unnecessary and to generate an orthogonal basis. In the second and new version, we consider instead a progressive QR algorithm (within the HF code), which helps select a well conditioned basis of snapshots and that ends automatically with an orthogonal basis (the Q part). The second code is faster and produces as good or even a better basis as the first one, so it will be preferred in further developments.

As a preliminary step we study the projection error as the model is perturbed from the base one, from which the snapshots are taken. It is shown that if the basis is well chosen, the distance of the High Fidelity (HF) solution to the base subspace is small and grows slowly, *i.e.*, as hoped for, most of the action occurs close to this low dimensional subspace, even for substantial departures from the baseline model. This assesment uses the HF solution and therefore is not practical in estimating the error of actual MOR solutions. For this purpose we develop and test an *a posteriori* computable approximation error analysis. We emphasize that what is generated is an error estimate, not an error bound. In fact, if this estimate is sufficiently good, it can be used to correct the solution and increase its accuracy, as we demonstrate.

In [4], the authors derive a priori error bounds for POD. They present several numerical examples showing the sharpness of their estimates. In [9], an a-posteriori error bound is derived for some control problems using POD. The closest treatment to the one presented here is in [10], where the authors study carefully the *a posteriori* errors for nonlinear dynamical ordinary differential equations including interesting numerical examples. They concentrate on bounds rather than estimates as we do here. We illustrate the techniques on various examples of increasing complexity. The larger 2D examples approach in size those that would be useful in 3D when localization techniques are used for solving large problems in parallel.

## 2. Wave Equation in 2D

We consider the acoustic wave equation in 2D in second order form and in inhomogeneous media. We also add absorbing boundary conditions [11], to allow the solution of more realistic problems than in [7]. The equation to be solved numerically is:

$$w_{tt} = v^2(x, z)\Delta w + Bu(t) - 2\varepsilon(x, z)w_t - \varepsilon^2(x, z)w, \quad (1)$$

where the last two terms account for the absorbing boundary conditions on every side of the computational box (assumed to be  $[0, x_{\max}] \times [0, z_{\max}]$ ) with the exception of the top (free surface). We define

$$\varepsilon(x, z) = U_0 / \cosh(\alpha v)^2,$$

where  $U_0, \alpha$  are parameters and  $v$  is the distance to the boundary in number of mesh points. Obviously,  $\varepsilon(x, z)$  decays rapidly away from the artificial boundaries and it has a maximum value of  $U_0$ . This has the effect of damping spurious reflections caused by the artificial boundaries. As boundary conditions we leave the top free, while at the artificial boundaries we use the absorbing boundary conditions described above. If we discretize in space Equation (1) we obtain:

$$w_{tt} = Aw + Bu(t) - 2D(\varepsilon)w_t. \quad (2)$$

The matrix  $A$  of the spatial discretization, accounting for the terms  $v^2\Delta w - \varepsilon^2(x, z)w$ , is generated in sparse format. We use an 8th order discretization in space and solve the resulting large system of ODE's by the off-the-shelf integrator Runge, Kutta, Fehlberg RKF45.  $D(\varepsilon)$  is a diagonal matrix that contains the value of  $\varepsilon$  at each grid point.

The 8th order discretization of the Laplacian requires 17 mesh points. For each coordinate direction we use the weights given in [12] for an eight order approximation to the second derivative. Due to the length of the 8th order stencil, we use a 5-point second order approximation to the Laplacian for points in the three mesh lines adjacent to the boundary. We then solve one or a few of these problems using the HF code in order to produce a number of snapshots, *i.e.*, values of the field variable  $w$  at all the spatial mesh points, for selected times. In the absence of additional information, we will choose these snapshots evenly spaced in the time integration interval. Assuming a linear ordering for the mesh points ( $z$  faster than  $x$ , say), we form a matrix  $\Phi(n \times l)$ , where  $n$  is the number of mesh points and  $l$  is the number of snapshots. If  $\Phi = U_l \Sigma_l V_l^T$  is the singular value decomposition (SVD) of this matrix, we threshold the singular values to eliminate the columns of  $U$  that are associated with small SVs, and call  $U(n \times k)$  the resulting matrix, with  $k \leq l$ . As we show later, this potentially expensive step can be replaced by a progressive QR algorithm that, as a bonus, provides dynamical adaptivity in the selection of snapshots. In the next step we make the Ansatz for the solution of

$$w^*(x, t) = Ua^*(t),$$

where the  $k$  time-dependent coefficients  $a^*(t)$  are to be determined. By introducing this Ansatz in (2), we

obtain a Galerkin projection method, resulting in a reduced order model that approximates the original dynamical system:

$$a_u^* = U^T A U a^* + U^T B u(t) - 2U^T D(\epsilon) U a_t^*.$$

In this step we have used that  $U$  is an orthogonal matrix. Observe that this method can be extended to three dimensions and to elastic or even anisotropic equations. The difficulty there will be the problem size and complexity.

**Implementation**

We implement the above method in two separate codes for simplicity.

1) *HFWave* is the High Fidelity solver that is capable of running multiple problems, varying either the position or the frequency of the source, or the velocity.

For each run a seismic section (*i.e.*, a collection of time histories at selected output points), and also a separate time history, are output for comparison and verification purposes.

2) *MORWave* solves the reduced problem, with the same capabilities of running multiple problems as *HFWave*. For each problem it outputs a seismic section and trace as *HFWave*, for comparison and verification.

**3. Numerical Examples of 2D Acoustic Wave Propagation**

**3.1. Example 1**

For the first comparison of the High Fidelity (HF) and Reduced Order (MOR) solvers performance, we use a simple model that consists of three flat layers with constant velocities; see **Table 1**.

The other parameters for this run are:

- #points in  $x$ -direction: 351
- #points in  $z$ -direction: 351
- $dx = dy = 10$ ;  $dt = 0.005$
- #snapshots: 400
- $\alpha = 0.18$
- $U_0 = 200$ .

The source is a 5 Hz Ricker wavelet placed at the mesh point ( $x = 1600, z = 50$ ). We collect time histories at 101 equally spaced receivers starting at  $x = 700$ . Observe that  $dt$  is the snapshot time spacing. The Runge-Kutta-Fehlberg ODE integrator RKF45 chooses its own timestep dynamically to satisfy stability and accuracy

**Table 1. Model vel3l.**

Depth (m)	Velocity (m/sec)
0 - 1500	1000
1500 - 2500	2000
2500 - 3000	3000

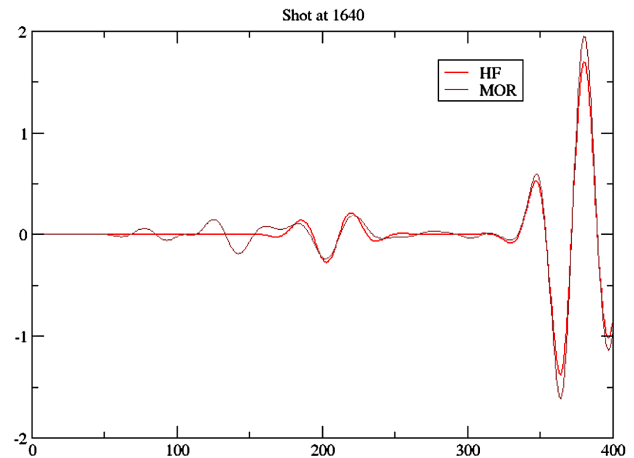
requirements (for a  $10^{-4}$  absolute and relative errors). We run five shots starting at  $x = 1600$ , spaced at 20 m. 400 equally spaced HF snapshots are used from the first and last shot. The rank of the projection is 33.

Results are shown in **Table 2**. In **Figure 1**, we show the cross-plots of the HF and MOR solutions for a shot at the end point of the shot interval, while in **Table 3** we list the Residual Mean Squares for every shot. The RMS is calculated over the whole section. We observe that the errors, as expected, are smaller at the end shots and increase towards the middle of the interval; also, even in the worst case the phases are very accurate, with the inaccuracies concentrated in the amplitudes. This is a good sign, since in most imaging tasks the phases (arrival times) are more important than the amplitudes (only relative amplitudes of events are meaningful).

**3.2. Example 2**

For the second example we consider a problem similar to the BP model of [7] that we call vel4pc. Instead of a fully inhomogeneous velocity, as in that example, we have binned the velocity and used an average velocity to obtain a piecewise constant model that preserves most of the geological complexity of the original one (see **Figure 2**).

Crossplot for model vel3l. Traces at  $x=1000$ .



**Figure 1. Crossplot of one trace: HF vs MOR. Problem vel3l.**

**Table 2. Results for model vel3l. 5 shots.**

vel3l	# Equations.	#Timest.	Pre-proc.	Integ.
HF	246,402	8850	-	168.17''
MOR	66	4004	548.23''	4.0''
Ratio	3733	2.2	-	42.04''

**Table 3. RMS for the various shots. Model vel3l.**

Shot	1	2	3	4	5
RMS	0.03	0.2	0.23	0.14	0.03

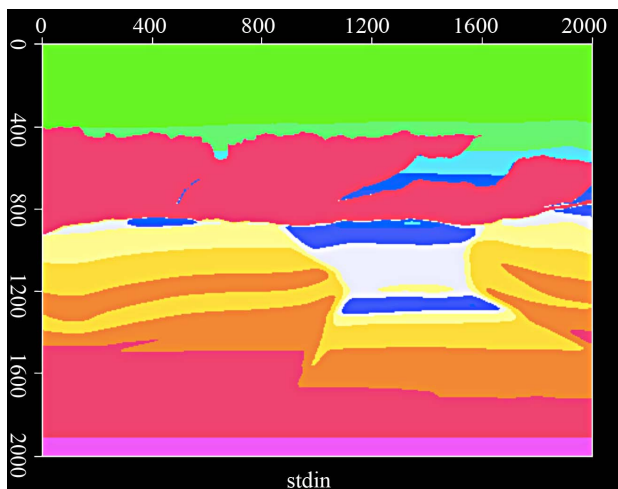


Figure 2. Velocity for vel4pc model.

The parameters for this run are:

#points in  $x$ -direction: 2001

#points in  $z$ -direction: 2001

$dx = dy = 6.25$ ;  $dt = 0.025$

#snapshots: 200

First source at  $x = 8075$ ,  $z = 25$ ; 5 Hz Ricker wavelet.

Boundary absorber:  $U_0 = 200$ ,  $\alpha = 0.5$ .

There are 7 different regions with velocities listed in **Table 4**. In **Table 5**, we report the comparative performance and level of data compression for these calculations. We see that the level of data compression is much higher than the gain in computing time, again because we have included in MOR the time to generate the snapshots and the orthogonal basis. In **Figure 3**, we show a crossplot of one trace for the high fidelity and MOR codes. The overhead of the SVD is incurred only once and then the orthogonal basis can be used for all close by simulations.

### 3.3. Example 3

The model for this example is also of a salt region, but in contrast to Example 2 the velocity is fully inhomogeneous. The parameters for the comparison are:

BP1 model (velocity in **Figure 4**):

Steps in  $x$  direction: 640

Steps in  $z$  direction: 400

Time snapshots: 400

$dx = dy = 25$  m;  $dt = 0.02$  sec;

Size:  $16,000 \times 10,000$ ; Integration time: 10 sec.

First we run the High Fidelity (HF) code for five shots separated by 50 m with a first shot located at  $x = 8075$ ,  $z = 25$ . We use 400 snapshots at 20 ms intervals from the shots at both ends of the shot interval. The ordinary differential equations that result from the space discretization are solved by the off-the-shelf integrator RKF45, which regulates its internal time step to achieve a prescribed tolerance set to  $10^{-4}$ . In the second stage we run

Table 4. Velocities for model vel4pc.

Region	Velocity
1	1621.28
2	2248.42
3	2776.11
4	3365.47
5	3842.13
6	4071.14
7	4596.12

Table 5. Performance and compression for problem vel4pc. 4 shots.

Code	time	Speedup	size	Compress.
H. Def.	220,065	1	8,008,002	1
MOR	0.98	3708	88	90,909

Comparison of time histories at  $x=250$ ,  $z=625$

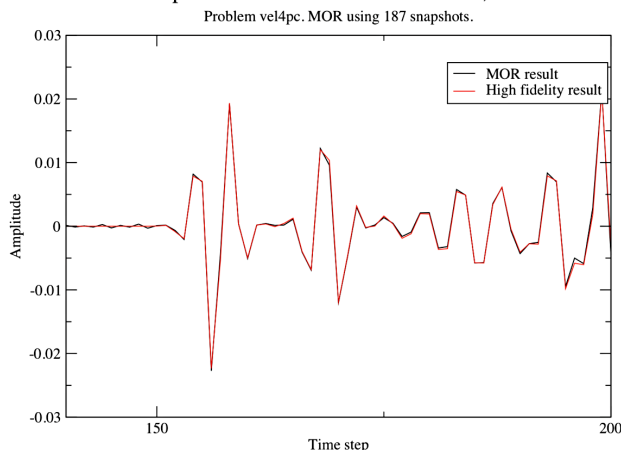


Figure 3. Crossplots of seismograms at  $x = 250$ ,  $z = 625$ .

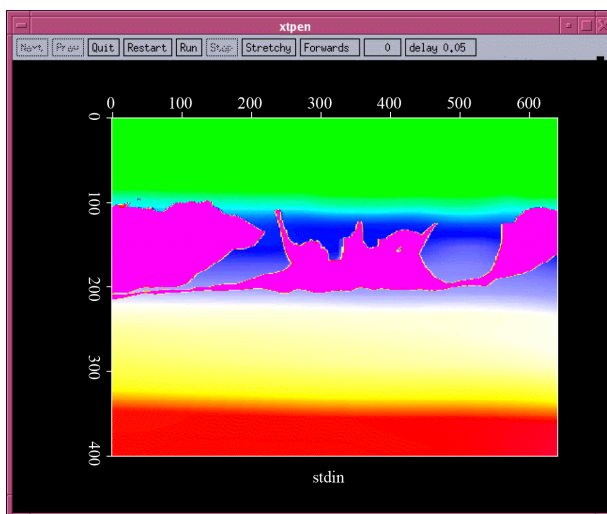


Figure 4. Velocity for model BP1.

the Model Order Reduced code for the same five shots. In both cases we produce one section with 360 receivers spaced by 25 m, starting at  $x = 4975$ . The statistics are given in **Table 6**.

In **Figures 5-7**, we show cross-plots of a trace at  $x = 4975$  for HF and MOR, corresponding to three shots.

### 3.4. Example 4

For this example we consider the inhomogeneous version of model BP (see Example 2) that we call vel4 (**Figure 8**). The size of the mesh is the same as before, but now we have:  $dx = dy = 6.25$   $dt = 0.025$  #snapshots: 400 Source at  $x = 6250, z = 125$ ; 5 Hz Ricker wavelet. The results are shown in **Figure 9** and **Table 7**.

## 4. Applications

The results of the previous section show that we can reproduce accurately the solution of the wave equation using projections on a small space of snapshots with considerable savings in computing time. Reverse time migration of seismic data [13] requires simulating the

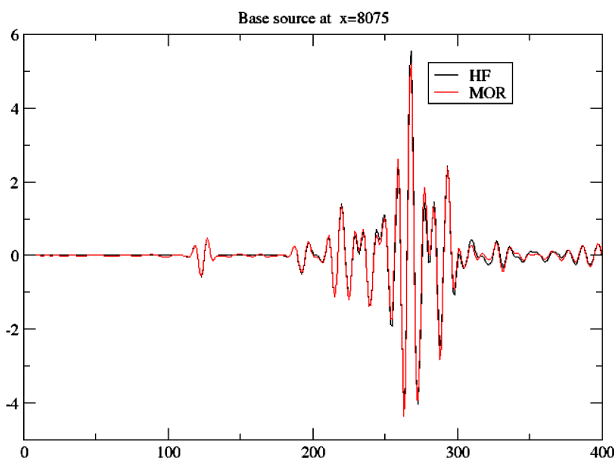
**Table 6. Results for model BP1.**

BP1	# Equations.	#Timest.	Pre-process	Total time
HF	514,082	23,746	-	1936.5
MOR	574	12,147	1826"	16.6
Ratio	896	1.95	-	117

**Table 7. Results for model vel4. 9 sources.**

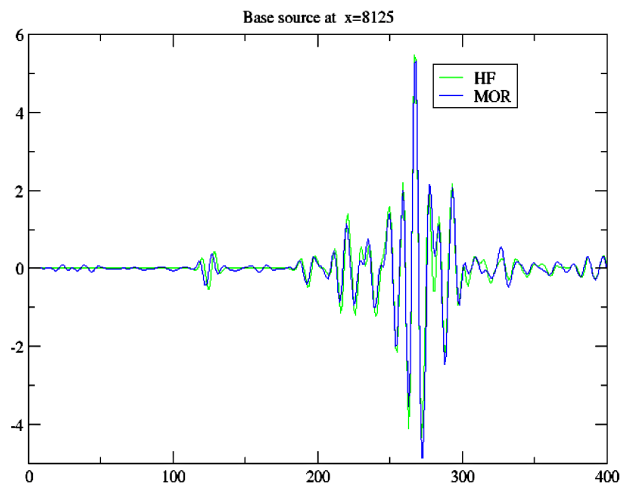
vel4	# Equations.	#Timest.	Pre-proc.	Total time
HF	8,008,002	117,694	-	130,275"
MOR	726	13,087	14,908"	68.4"
Ratio	11,030	9	-	1904"

Crossplot for model BP1. Trace at  $x=4975$ .



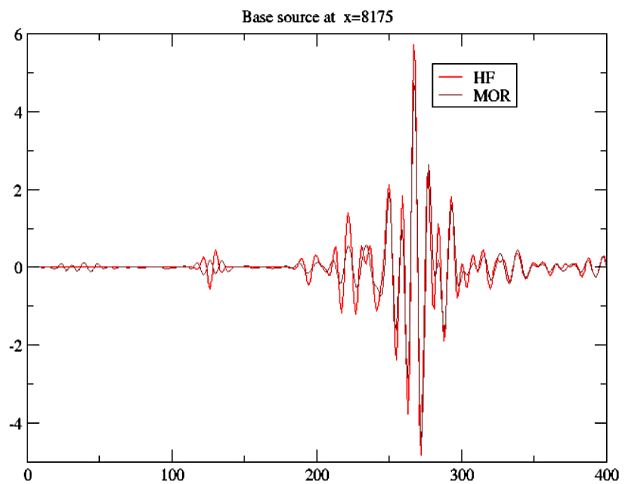
**Figure 5. Crossplot for first shot. Model BP1. HF vs MOR.**

Crossplot for model BP1. Trace at  $x=4975$ .

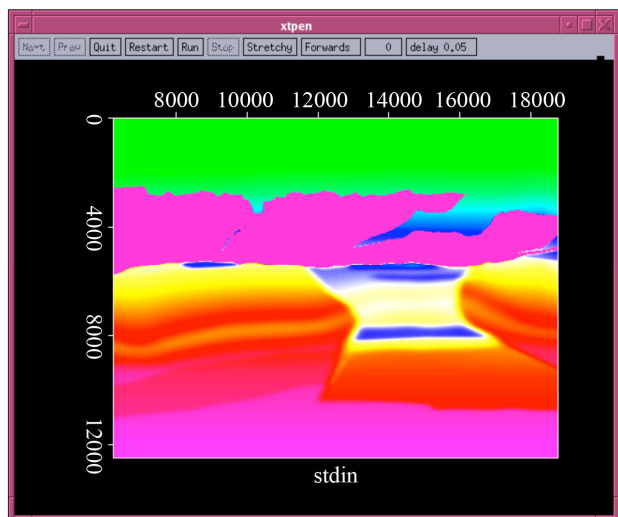


**Figure 6. Crossplot for shot 2.**

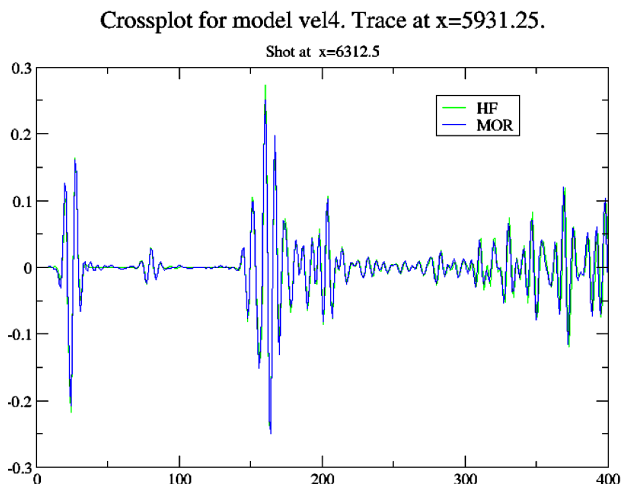
Crossplot for model BP1. Trace at  $x=4975$ .



**Figure 7. Crossplot for shot 3.**



**Figure 8. Velocity model for vel4.**



**Figure 9. Crossplot of trace at  $x = 5931.25$  for model vel4. HF vs MOR.**

wave equation forward from the source and backward from the receivers, correlating the images at each time step in order to construct a focused map in depth from seismograms collected in time.

Large scale 3D problems are solved using domain decomposition distributed in large scale computer clusters or some times, decomposing the problem in decoupled shots with a limited aperture. One of the current problems is the amount of network traffic that these correlations require, since the size of the data set that contains all the closely spaced time slices of the 3D domain is very large.

A possible idea is to perform one full solve from the source and instead of saving all the time snapshots at very closely spaced time steps, save only a sparse set that is then used to MOR-solve and produce the closely spaced slices on the fly instead of having to move them through the network. The reverse simulation from the receivers can also be fully performed with MOR-solves. Accuracy will have to be assessed in order to see that no artifacts are created in the imaging by the use of this approximation. Observe that we are hoping for savings in computer time measured in orders of magnitude, not in small percentages. Adjoint methods to calculate gradients with respect to parameters share some of the characteristics of the problem described above, in which an initial and end value time problem (for the adjoint variables) are coupled, so that a similar approach can be used. In the case of tomographic full wave form inversion, we start from an approximate knowledge of the velocity of propagation and would like to use the data to improve that knowledge. Usually this is done in the least squares sense, where data and simulations for one or several seismic cubes (source gathers) are compared and the sum of squares of the differences of each trace is minimized. This can be an essential part of the RTM and other mi-

gration processes, which require accurate velocity models to produce useful depth maps. There are two possible applications for MOR here.

One is to use a HF solve corresponding to a source in order to generate the projection basis and then use MOR for a number of nearby sources. The second one uses MOR with a fixed basis for a number of iterations of the inversion process in which the velocity is been updated.

The plausibility of these uses will be predicated on how accurate are the MOR solutions when we step away from the problem that generated the snapshots used in the projection basis and that is what we study now. In order to have a better idea about the error between the HF and MOR solutions with regards to perturbations to the base model we perform the following experiment. We introduce a parameter  $\alpha$  and multiply the velocity by  $1 + \alpha$ . We calculate the RMS of the difference between the high fidelity solution at the last time step  $T$  and its projection into the subspace of snapshots. This is a measure of what is the best approximation one can obtain with this reduced subspace:

$$e(\alpha) = \sqrt{\|(I - UU^T)HF^N(\alpha)\|_2^2 / M}.$$

For  $\alpha = 0, 0.1, \dots, 0.9$ , we calculate  $re(\alpha) = e(\alpha)/e(0)$ , so that  $re(0) = 1$ . In **Table 8** and **Figure 10**, we can see the behavior of  $re(\alpha)$  for the 3 layers model vel31.  $HF^N$  is the snapshot of the high fidelity solution for the last time step ( $T$ ); thus  $e(\alpha)$  contains the accumulated errors in the time integration (worse case scenario). As expected, the projection errors grow for increasing  $\alpha$ , but they grow slowly: to about one order of magnitude when doubling the velocity from the base model.

Since the solution amplitude itself is of order 1, then we see that at the far end of the perturbation the error has grown by a factor of 10.

**Table 8. Errors for vel31 with gradients in the layer.**

$\alpha$	$\text{Max}_x  w(x, T) $	Proj. Err.	$e(\alpha)/e(0)$
0.	0.9503644	1.631E-02	1.000
1.	0.9971880	9.333E-02	5.721
2.	0.8639822	9.538E-02	5.846
3.	0.9436362	0.1098	6.731
4.	0.9369125	0.1353	8.298
5.	0.9586850	0.1410	8.645
6.	0.8289407	0.1377	8.445
7.	0.9785962	0.1407	8.626
8.	1.009829	0.1507	9.237
9.	1.186977	0.1516	9.294

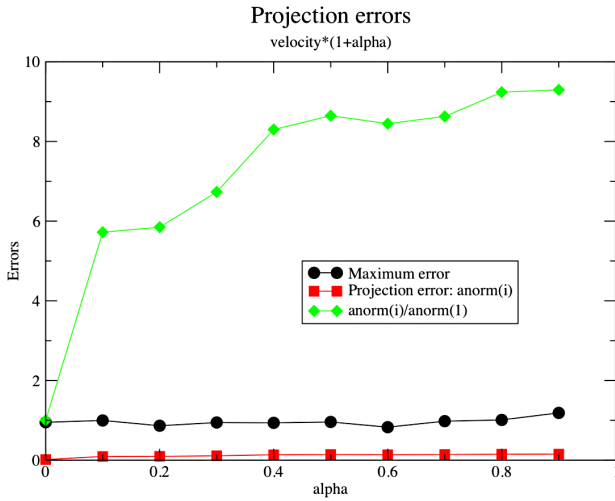


Figure 10. Projection errors.

## 5. Global Error Estimation-I

In the previous section we demonstrated the behavior of the error of MOR when perturbing the problem, using the exact high fidelity solution to calculate it. Of course, in practice, when we are solving a reduced problem for an equation different from the base one that generated the snapshots we will not have the HF solution. Thus, in this section we use ideas of deferred corrections [14-16] to produce a computable a posteriori error estimate that does not require the HF solution.

In [17], a different procedure for estimating a posteriori the projection errors in a Krilov algorithm for a nonlinear problem is described. Grepl and Patera [18] consider a posteriori error bounds for parabolic equations when using a reduced-basis method and they apply them to construct a basis by a greedy algorithm.

We start from the HF equation

$$w_u = Aw + Bu(t) - 2D(\epsilon)w_t \quad (3)$$

and consider also the reduced one

$$a_u^{(1)} = U^{(1)T}AU^{(1)}a^{(1)} + U^{(1)T}Bu(t) - 2U^{(1)T}D(\epsilon)U^{(1)}a_t^{(1)}, \quad (4)$$

where  $U^{(1)}$  stands for our initial selection of orthogonal basis vectors.

For simplicity we assume that the discretization errors are negligible compared to the approximation error, although they can also be easily accounted for. Putting  $w^{(1)} = U^{(1)}a^{(1)}$  and calling  $L = A - 2D(\epsilon)(\cdot)_t$  we get:

$$\begin{aligned} U^{(1)}a_u^{(1)} &= P^{(1)}[LU^{(1)}a^{(1)} + Bu(t)], \\ w_u^{(1)} &= P^{(1)}[Lw^{(1)} + Bu(t)], \end{aligned} \quad (5)$$

where  $P^{(1)} = U^{(1)}U^{(1)T}$  is the projector on the space generated by the columns of  $U^{(1)}$ . This equation tells us that the whole trajectory  $w^{(1)}(t)$  stays in the subspace

spanned by  $U^{(1)}$ . We will also use the projector on the orthogonal complement  $P^{(1)\perp} = I - U^{(1)}U^{(1)T}$ . Now we introduce  $U^{(2)} = \begin{bmatrix} U^{(1)} & V^{(1)} \end{bmatrix}$ , an orthogonal basis for a larger subspace of snapshots (observe that  $P^{(1)\perp}U^{(2)} = \begin{bmatrix} 0 & V^{(1)} \end{bmatrix}$ ,  $V^{(1)T}P^{(1)\perp} = UV^{(1)T}$ ). The easiest way to do this is to select a good number of snapshots, perform the SVD and then use a large threshold to choose the first set of basis functions. In the second pass one would lower the threshold and use the original and additional basis functions as  $U^{(2)}$  and so on.

With this new basis, we obtain a more accurate approximation by solving the reduced equation

$$b_u = U^{(2)T}[LU^{(2)}b + Bu(t)]. \quad (6)$$

Then, by solving this small dimensional problem, we obtain the corrected solution:  $w^{(2)} = U^{(2)}b$  and the approximation error  $\delta(t) = w - w^{(1)}$  for  $w^{(1)}$  can be estimated by  $\delta \approx w^{(2)} - w^{(1)}$ . It is important to remark that this is not a bound for the error, but it actually comes with a sign and, if the procedure has been performed appropriately, not only can be used to estimate the error with high precision but, just as in the deferred correction approach,  $w^{(2)}$  should be more accurate than  $w^{(1)}$  and the process can be iterated by adding more basis functions as required. If one starts with a lax threshold for the singular values and proceeds with this algorithm by refining the basis, then this is also a multi-resolution approach, in which one solves first for a low frequency band and keeps adding frequencies as needed. On the down side, the procedure will give a good estimate provided that  $w^{(2)}$  is a sufficiently accurate proxy for the HF solution.

If both solutions are inaccurate, this approach can lead to a significant under-estimation of the error.

## 6. Global Error Estimation-II

In this section we will pursue a different approach to solve the error estimation problem. In the first step we solve (4). We define  $\delta^{(1)}(t) = U^{(1)}e^{(1)}(t) + V^{(1)}d^{(1)}(t)$ , the approximation to  $\delta(t)$  that we wish to calculate. By subtracting (5) from (3) we get:

$\delta_u = Lw - P^{(1)}Lw^{(1)} + P^{(1)\perp}Bu(t)$ . This is the exact error equation, which shows the components in each subspace, reflecting the fact that since the HF solution  $w$  will not usually belong to the space spanned by  $U^{(1)}$ , there will certainly be an error component in the orthogonal complement of that space. That there is also a component of the error in the space of the columns of  $U^{(1)}$  comes from the fact that usually the projection of  $w$  into that space will not coincide with  $w^{(1)}$  [10]. A problem with this equation is that we do not know  $w$ . But if we replace it by  $w^{(1)} + \delta$ , then this can be solved approximately for  $\delta^{(1)}$  in the reduced space  $U^{(2)}$ :

$$\delta_u = L\delta + P^{(1)\perp} (Lw^{(1)} + Bu(t)),$$

or

$$U^{(1)}e_u^{(1)} + V^{(1)}d_u^{(1)} = L\delta + P^{(1)\perp} (Lw^{(1)} + Bu(t)).$$

Multiplying the previous equation successively by  $U^{(1)}$  and  $V^{(1)}$ , and calling  $res(w^{(1)}) = V^{(1)\top} (Lw^{(1)} + Bu(t))$ , we get the coupled system:

$$\begin{aligned} e_u^{(1)} &= U^{(1)\top} L\delta, \\ d_u^{(1)} &= V^{(1)\top} L\delta + res(w^{(1)}). \end{aligned}$$

Thus, we need now to pre-calculate:  $U^{(1)\top}LU^{(1)}$ ,  $ULV^{(1)}$ ,  $V^{(1)\top}LU^{(1)}$ , and  $V^{(1)\top}LV^{(1)}$ . We proceed in stages. First we solve the reduced system with the initial basis  $U^{(1)}$  producing  $a^{(1)}$ . Then we use this  $a^{(1)}$  to obtain  $w^{(1)}$ ,  $res(w^{(1)})$  and solve the error system to obtain  $e, d$  and from them  $\delta^{(1)} \approx \delta$ .

### 7. A Progressive Algorithm

The developments of the previous sections suggest the possibility of devising an incremental algorithm, in which a solution for a given basis is updated when new vectors are introduced in the basis.

Using the notation above we consider not the error equation but rather the solution of the reduced equation with an enlarged basis:  $w^{(2)} = U^{(1)}a + V^{(1)}b$ . Replacing this *Ansatz* in the wave equation we get:

$$U^{(1)}a_u + V^{(1)}b_u = LU^{(1)}a + Bu + LV^{(1)}b.$$

Replacing  $U^{(1)}a \approx w^{(1)}$ , we get:

$$w_u^{(1)} + V^{(1)}b_u = P(Lw^{(1)} + Bu) + P^\perp(Lw^{(1)} + Bu) + LV^{(1)}b.$$

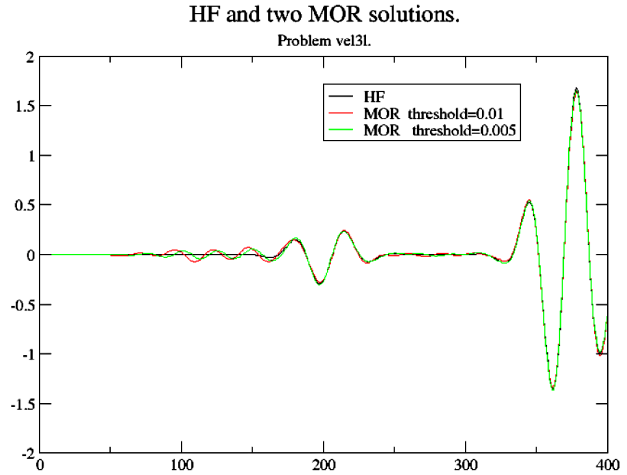
The first terms on both sides cancel out and we are left with an equation on the subspace spanned by the columns of  $V^{(1)}$ :

$$b_u = V^{(1)\top}LV^{(1)}b + res(w^{(1)}). \tag{7}$$

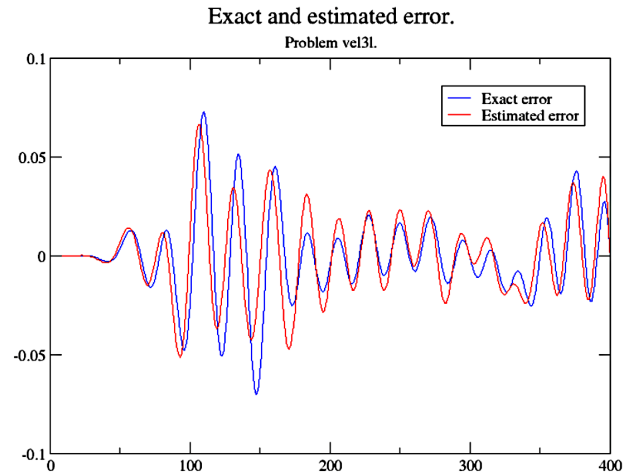
Observe that this is the same as the second equation for the error if we set the component  $e = 0$ . Thus, we first solve for  $w^{(1)}$  in the the space spanned by the columns of  $U^{(1)}$  and then we can add a correction to that solution by solving Equation (7) for  $b^{(1)}$ :  $w^{(2)} = U^{(1)}a^{(1)} + V^{(1)}b^{(1)}$ .

### 8. Numerical Test

We test first the direct procedure with the problem for model vel31 described in Section 3, Example 1. We run the HF code and MOR with SV thresholds 0.01, 0.005. This leads to basis with 57 and 64 snapshots, respectively. The results are shown in **Figures 11** and **12**, with the



**Figure 11.** HF vs two MOR runs with different thresholds for first shot.



**Figure 12.** Exact and estimated errors for first shot.

estimated error tracking closely the exact one (including its sign!).

Now we test the second procedure on the same problem, selecting for MOR a threshold of 0.05 that results in 45 snapshots been selected. For the extended space we choose the same threshold as above. Results can be seen in **Figures 13-16**. We see that both procedures give very good results.

### 9. Adaptive Snapshot Selection

An interesting and novel approach to a dynamic selection of snapshots can be achieved by using a progressive QR decomposition [19]. This will also eliminate the need for the SVD of the snapshot’s matrix. The strategy is as follows:

- Within the HF integration process, as a snapshot is produced, calculate one step of the QR reduction via a Householder transformation. If the diagonal term in R is below a given threshold, reject it, otherwise accept



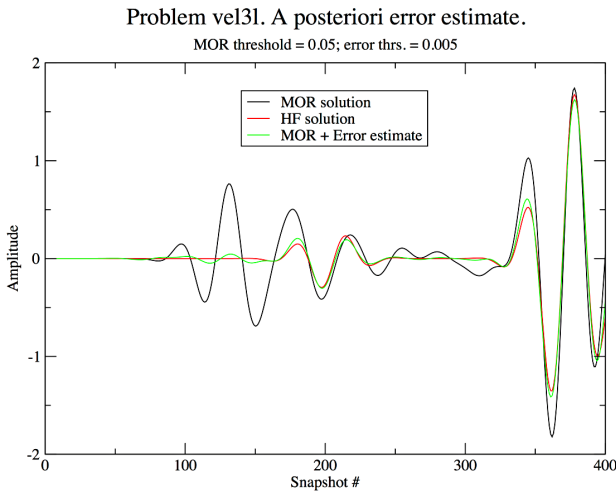


Figure 13. HF, MOR and corrected MOR for first shot.

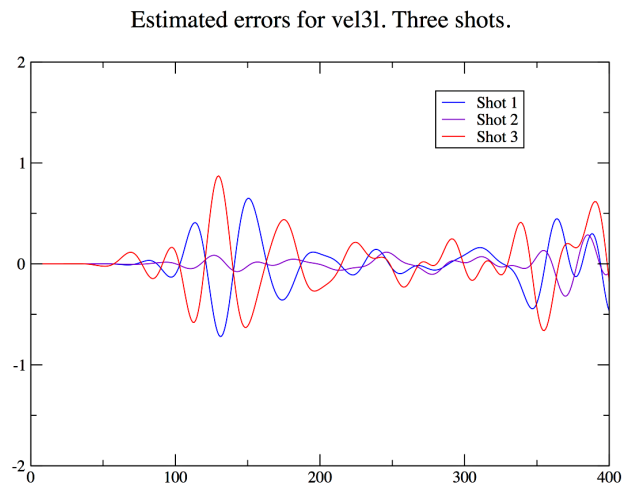


Figure 16. Errors for the three shots.

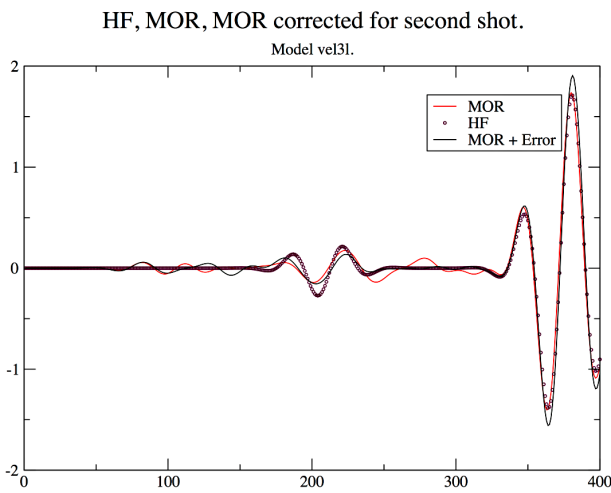


Figure 14. HF, MOR and corrected MOR for second shot.

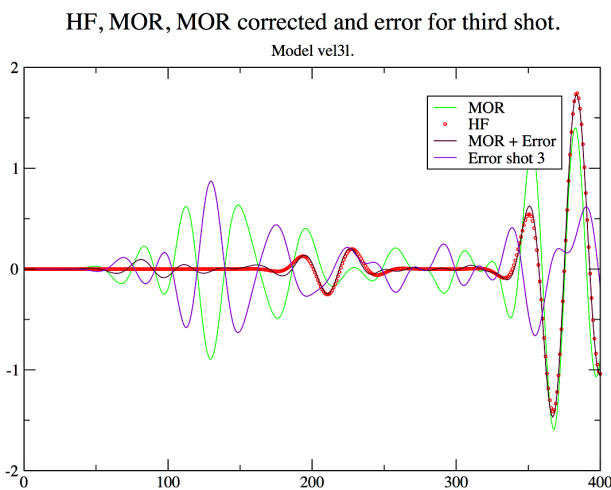


Figure 15. HF, MOR and corrected MOR for third shot.

it into the basis.

- After that, each time a new snapshot is calculated we

add it tentatively at the end of the already orthogonalized basis and update the QR decomposition. Again, if the new diagonal term is below a given threshold the snapshot is rejected, otherwise the new Householder transformation is saved.

- At the end of this process we would have selected a well conditioned set of snapshots and moreover an orthogonal basis (Q) would have been produced, thus eliminating the need for an SVD.

In detail. Let  $v$  be a candidate snapshot and let  $H = I - 2uu^T / \langle u, u \rangle$  be the Householder (orthogonal) transformation that reduces  $v$  to  $Hv = \alpha e_1$ , where  $e_1 = (1, 0, 0, 0, \dots)^T$ . The  $u$  that does this is:

$$u = v - \text{sign}(v(1)) \|v\|_2 e_1 \text{ and the resulting}$$

$\alpha = -\text{sign}(v(1)) \|v\|_2$  [20]. At the  $k$ th step, first we apply the previously calculated Householder transformation to the candidate snapshot  $v$ :  $\hat{v} = H_{k-1} H_{k-2} \dots H_1 v$  and then we calculate and apply a tentative  $H_k$  of size  $(n - k + 1)$  to the vector  $(\hat{v}_k, \hat{v}_{k+1}, \dots, \hat{v}_n)^T$ . If  $|\alpha| / \|v\| < \epsilon$  then the snapshot is rejected, otherwise it is incorporated into the basis, together with the new  $H_k$ . In this case,

$$|\alpha| = \sqrt{\sum_{i=k}^n \hat{v}_i^2}.$$

Observe that during the process we would only need to save the defining vectors of the Householder transformations, since

$Hv = v - 2u \langle u, v \rangle / \langle u, u \rangle$ . In this way, at the end we would have selected a basis of snapshots  $U$  and simultaneously calculated its  $U = QR$  decomposition, where  $Q = H_1 H_2 \dots H_k$  and  $R$  (not used) is upper triangular.

If desired, the actual  $n \times k$  orthogonal matrix  $Q$  can be created by applying successively the Householder transformations to the identity matrix. This also opens up an opportunity for further adaptivity by modifying the snapshot time interval  $\Delta t$ . If we see that there are no rejected snapshots for a number of steps this may be an indication that  $\Delta t$  is too large and that we should make it smaller. This is an important indicator that without this

vital information could only be detected before by experimentation. Systematic skipping more than one snapshot indicates that the step is too small and that can be enlarged, making the process more economical. These step modifications should be done with caution to avoid too many fluctuations, although they do not change much the integration aspects that are actually controlled by RKF45. Output points do not really increase or alter the flow of the integration. The only loose end is how to choose  $\epsilon$  appropriately. Observe that  $|\alpha|$  is the norm of the component of  $v$  orthogonal to the current basis, or in other words  $|\alpha|/\|v\|_2 = \sin\phi$ , where  $\phi$  is the angle that  $v$  forms with the current basis. The larger  $\phi$  is the more independent  $v$  will be from the existing basis. For instance, choosing  $\epsilon = 0.174$  implies that no  $v$  that forms an angle  $\phi < 10^\circ$  with the existing basis subspace would be accepted.

### 10. Numerical Examples for Progressive QR

We test the new strategy from the previous section on model vel31. This time we consider 22 shots separated by 10 m and collect snapshots from the two ends and center shots. The statistics are shown in **Tables 9** and **10**. The line MOR + HF accounts for the pro-rated cost of three HF solves (320/22). The ratio between HF and this more realistic cost is 3.1495. MOR uses 175 basis functions selected and orthogonalized in HF.

The accuracy at the center of the two intervals (shots 5 and 15) is good. The threshold used for the angle between a new snapshot and the existing basis was  $2.87^\circ$  ( $\sin(2.87) = 0.05$ ). Now we consider the larger model BP1 with 21 shots separated by 10ms and using three of those to extract snapshots (1, 11, 21). MOR used 120 basis functions in this calculation. Again, the accuracy in the middle shots is good. The compression factor is 5.6.

### 11. Conclusion

We have shown that a POD reduced-order modeling can be successfully employed to reduce the cost of repeated solutions of the acoustic wave equation in two-dimen-

sions, while still preserving enough accuracy. This has been exemplified using some complex earth models from oil exploration, an important area of application. A posteriori error estimation was discussed and two procedures were described and exemplified. The POD procedure was implemented via the traditional SVD approach and through a more economical and flexible progressive QR algorithm, which also lead to an adaptive selection of snapshots and a well-conditioned basis. For additional comparisons, including a realistic full seismic survey see [21].

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**Table 9. Errors for vel31 with gradients in the layer**

Method	Preproc.	Integration	Total
HF	9	1242	1251
MOR	198	19.8	217.8
MOR + HF (3 shots)	377.4	19.8	397.2

**Table 10. Model BP1. 21 shots.**

Method	Preproc.	Integration	Total
HF	300	23,262	23,562
MOR	504	33.6	538
MOR + HF (3 shots)	4170	33.6	4201

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