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Dividend Payments and Related Problems in a Markov-Dependent Insurance Risk Model under Absolute Ruin

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ABSTRACT

In this paper, we study the dividend payments prior to absolute ruin in a Markov-dependent risk process in which the claim occurrence and the claim amount are regulated by an external discrete time Markov chain. A system of integro-differential equations with boundary conditions satisfied by the moment-generating function, the nth moment of the discounted dividend payments prior to absolute ruin and the discounted penalty function, given the initial environment state, are derived. In the two-state risk model, explicit solutions to the integro-differential equations satisfied by the nth moment of the discounted dividend payments prior to absolute ruin are obtained when the claim size distribution is exponentially distributed. Finally, the matrix form of systems of integro-differential equations satisfied by the discounted penalty function are presented.

***Keywords*:** *Absolute Ruin, Markov-Dependent Insurance Risk Model, Debit Interest, Moment-Generating Function*

1. Introduction

Recently ruin theory under regime-switching model is becoming a popular topic. This model is proposed in Reinhard [1] and Asmussen [2]. Asmussen [2] calls it a Markov-modulated risk model in which both the frequency of the claim arrivals and the distribution of the claim amounts are influenced by an external environment process. This model is a generalization of the classical compound Poisson risk model and the primary motivation for this generalization is the enhanced flexibility that it permits for the modeling of the claim arrival process and the claim severity distribution assumed in the classical risk process. The model builds a Markov chain whose states represent different states of an economy into the insurance risk model. The regime-switching of the states of the economy can be attributed to the structural changes in the (macro-) economic conditions, the changes in political regimes, the impact of (macro-) economic news and business cycles, etc. There are many papers published on ruin probabilities and the related problems under the Markov-modulated (or Markov regime-switching) risk model. Ng and Yang [3] give closed form solutions for the joint distribution of the surplus before and after ruin when the initial surplus is zero or when the claim amount distributions are phase-type distributed. Li and Lu [4] study the moments of the present value of the dividend payments and the distribution of the total dividends prior to ruin for the Markov-modulated risk model modified by the introduction of a barrier dividend. Lu and Li [5] and Liu *et al*. [6] consider a regime-switching risk model with a threshold dividend strategy. Zhu and Yang [7] study a more general Markovian regime-switching risk model in which the premium, the claim intensity, the claim amount, the dividend payment rate and the dividend threshold level are influenced by an external Markovian environment process. Wei *et al*. [8] consider the Markov-modulated insurance risk model with tax.

Moreover, in recent years, semi-Markovian risk model has attracted attention in the literature. Albrecher and Boxma [9] study the expected discounted penalty function in a semi-Markovian dependent risk model in which at each instant of a claim, the underlying Markov chain jumps to a new state and the distribution of claim depends on this state. Liu *et al*. [10,11] consider the expected discounted penalty function under the constant dividend barrier and dividends payments under the threshold strategy in a Markov-dependent risk model, respectively. They consider the structure of a semi-Markovian dependence type as follows. Let W_i denote the time between the arrival of the $(i-1)$ th and the *i*th claims and $W_0 = X_0$ a.s., then

$$\begin{aligned}&\Pr\left[W_{n+1} \le x, X_{n+1} \le y, Z_{n+1} = j \middle| Z_n = i,\right.\\&\quad\left.(W_s, X_s, Z_s), 0 \le s \le n\right]\\&= \Pr\left(W_1 \le x, X_1 \le y, Z_1 = j \middle| Z_0 = i\right)\\&= \left(1-\mathrm{e}^{-\lambda_i x}\right) p_{ij} F_j(y)\end{aligned}$$

where $\{Z_n; n \ge 0\}$ is an irreducible discrete time Markov chain with state space $S = \{1,2,3,\cdots,m\}$ and transition matrix $\Lambda = \left(p_{i,j}\right)_{i,j=1}^{m}$, X_n is the amount of the *n*th claim. Thus at each instant of a claim, the Markov chain jumps to a state *j* and the distribution F_j of the claim depends on the new state *j*, and has a positive mean μ_j. Then, the next interarrival time is exponentially distributed with parameter λ_j. Note that given the states Z_{n-1} and Z_n, the quantities W_n and X_n are independent, but there is an autocorrelation among consecutive claim sizes and among consecutive interclaim times as well as cross-correlation between W_n and X_n.

Inspired by Albrecher and Boxma [9] and Liu *et al.* [10,11], in this paper we propose to generalize the semi-Markovian risk model to the absolute ruin risk model. In the new risk model, we assume that the insurer could borrow an amount of money equal to the deficit at a debit interest force β when the surplus is negative. Meanwhile, the insurer will repay the debts continuously from his/her premium income. When the negative surplus attains the level $-c/\beta$ or is below $-c/\beta$, the surplus is no longer able to be positive. Absolute ruin occurs at this moment. Moreover, when the surplus exceeds the constant barrier $b(\ge u)$, dividends are paid continuously so the surplus stays at the level *b* until a new claim occurs. Some recent references about absolute ruin risk model include Zhou and Zhang [12], Cai [13], Gerber and Yang [14], Yuen *et al.* [15], Yuan and Hu [16],Wang and Yin [17], Ming *et al.* [18], Wang *et al.* [19], Zhang *et al.* [20], Yu and Huang [21] and references therein.

The surplus process $\{U_b(t); t \ge 0\}$ under the Markov-dependent risk model is given by

$$dU_b(t) = \left[c + \beta U(t) I\left(U(t) < 0\right)\right]dt - d\left(\sum_{n=1}^{N(t)} X_n\right)$$

where $U(0) = u$ is the initial surplus, *c* the premium rate, β the debit interest, $\{N(t); t \ge 0\}$ the number of claims up to time *t*, and $I(B)$ means the indicator function of an event *B*. Furthermore, we assume the net profit condition holds, that is

$$\sum_{i=1}^{m} \pi_i \mu_i < c \sum_{i=1}^{m} \pi_i \frac{1}{\lambda_i}$$

where $\pi = (\pi_1, \pi_2, \cdots, \pi_m)$ is the stationary distribution of process $\{Z_n, n \ge 0\}$.

Let $D(t)$ be the cumulative amount of dividends paid out up to time *t* and $\delta > 0$ the force of interest, then

$$D_{u,b} = \int_0^{T_b} \mathrm{e}^{-\delta t} \mathrm{d}D(t)$$

is the present value of all dividends until time of ruin T_b, where T_b denoted by $T_b = \inf\{t \ge 0 : U_b(t) \le -c/\beta\}$ is the time of absolute ruin.

In the sequel we will be interested in the moment-generating function

$$M_i(u, y; b) = E\left[\mathrm{e}^{yD_{u,b}} \middle| Z_0 = i\right], \quad i \in S$$

and the n th moment function

$$V_{n,i}(u; b) = E\left[D_{u,b}^n \middle| Z_0 = i\right], \quad n \in N, \quad i \in S$$

with $V_{0,i}(u; b) = 1$, and the expected discounted penalty function, for $i \in S$

$$\begin{aligned}\Phi_i(u, b) = E\Big[&\mathrm{e}^{-\delta T_b} \omega\left(U_b(T_b-), \left|U_b(T_b)\right|\right)\\&\times I(T_b < \infty) \Big| U_b(0) = u, Z_0 = i\Big]\end{aligned}$$

where, $U_b(T_b-)$ is the surplus prior to absolute ruin and $\left|U_b(T_b)\right|$ is the deficit at absolute ruin. The penalty function $\omega(x_1, x_2)$ is an arbitrary nonnegative measurable function defined on $(-c/\beta, +\infty) \times (c/\beta, +\infty)$. Throughout this paper we assume that $M_i(u, y; b)$, $V_{n,i}(u; b)$ and $\Phi_i(u, b)$ are sufficiently smooth functions in *u* and *y*, respectively.

Then, fix $n \in N$, the expected present value of the total dividend payments until ruin in the stationary case is given by

$$V_n(u, b) = \sum_{i=1}^{m} \pi_i V_{n,i}(u, b)$$

The rest of the paper is organized as follows. In Sections 2, we get integro-differential equations for the moment-generating function and boundary conditions in a Markov-dependent risk model. In section 3, the integro-differential equations satisfied by higher moment of the dividend payments and boundary conditions are derived. Examples for a two-state risk model are illustrated in section 4 when the claim size distribution is exponentially distributed. In the last section, we obtain the systems of integro-differential equations for the discounted penalty function and its matrix form.

2. Moment-Generating Function of $D_{u,b}$

In this section, we discuss the integro-differential equations satisfied by the moment-generating function at absolute ruin. We point out that $M_i(u,y;b)$ has different paths for $0 \le u < b$ and $-c/\beta < u \le 0$. For $i \in S$, we define

$$M_i(u,y;b) = \begin{cases} M_{1,i}(u,y;b), & 0 \le u < b \\ M_{2,i}(u,y;b), & -c/\beta < u \le 0 \end{cases}$$

Theorem 2.1 For $i \in S$, $0 \le u \le b$, we have

$$c\frac{\partial M_{1,i}(u,y;b)}{\partial u} = \delta y \frac{\partial M_{1,i}(u,y;b)}{\partial y} + \lambda_i M_{1,i}(u,y;b) + \lambda_i \sum_{j=1}^{m} p_{i,j}\left[\int_0^u M_{1,j}(u-x,y;b)\mathrm{d}F_j(x) + \int_u^{u+\frac{c}{\beta}} M_{2,j}(u-x,y;b)\mathrm{d}F_j(x) + \bar{F}_j(u+c/\beta)\right] \tag{2.1}$$

and, for $-c/\beta < u \le 0$,

$$(\beta u + c)\frac{\partial M_{2,i}(u,y;b)}{\partial u} = \delta y \frac{\partial M_{2,i}(u,y;b)}{\partial y} + \lambda_i M_{2,i}(u,y;b) - \lambda_i \sum_{j=1}^{m} p_{i,j}\left[\int_0^{u+\frac{c}{\beta}} M_{2,j}(u-x,y;b)\mathrm{d}F_j(x) + \bar{F}_j(u+c/\beta)\right] \tag{2.2}$$

with boundary conditions, for $i \in S$,

$$\left.\frac{\partial M_{1,i}(u,y;b)}{\partial u}\right|_{u=b} = yM_{1,i}(b,y;b) \tag{2.3}$$

$$M_{2,i}(-c/\beta,y;b) = 1 \tag{2.4}$$

where, $\bar{F}_j(x) = 1 - F_j(x)$.

Proof. Fix $i \in S$, and $0 \le u \le b$. Considering a small time interval $[0,t]$, such that $u + ct < b$. In view of the strong Markov property of the surplus process $\{U_b(t), t \ge 0\}$, we have

$$M_i(u,y;b) = E\left[M_i\left(U_b(t), ye^{-\alpha t};b\right)\right] \tag{2.5}$$

Conditioning on the event occurring in the interval $[0,t]$, we obtain

$$M_{1,i}(u,y;b) = (1-\lambda_i t)M_{1,i}\left(u+ct, ye^{-\delta t};b\right) + \lambda_i t \sum_{j=1}^{m} p_{i,j}\left[\int_0^{u+ct} M_{1,j}\left(u+ct-x, ye^{-\delta t};b\right)dF_j(x) + \int_{u+ct}^{u+ct+\frac{c}{\beta}} M_{2,j}\left(u+ct-x, ye^{-\alpha t};b\right)dF_j(x) + \bar{F}_j(u+ct+c/\beta)\right] + o(t) \tag{2.6}$$

Taylor's expansion gives

$$M_{1,i}\left(u+ct, ye^{-\delta t};b\right) = M_{1,i}(u,y;b) + ct\frac{\partial M_{1,i}(u,y;b)}{\partial u} - \delta yt\frac{\partial M_{1,i}(u,y;b)}{\partial y} + o(t) \tag{2.7}$$

Substituting (2.7) into (2.6), dividing both sides by t, and letting $t \to 0$, we obtain (2.1).

Similarly, when $-c/\beta < u \le 0$, we still consider a small time interval $[0,t]$, with $t(t>0)$ being sufficiently small so that the surplus will not reach 0 in the time interval. Let t_0 be the solution to

$$h_\beta(u,t) = ue^{\beta t} + c\left(\mathrm{e}^{\beta t} - 1\right)/\beta = 0$$

Then $h_\beta(u,t)$ is the surplus at time $t \le t_0$ if no claim occurs prior to time t_0. We assume $t \le t_0$. So conditioning on the time and the amount of the first claim, we have

$$M_{2,i}(u,y;b) = (1-\lambda_i t)M_{2,i}\left(h_\beta(u,t), y\mathrm{e}^{-\delta t};b\right) + \lambda_i t \times \sum_{j=1}^{m} p_{i,j}\left[\int_0^{h_\beta(u,t)+\frac{c}{\beta}} M_{2,j}\left(h_\beta(u,t)-x, y\mathrm{e}^{-\delta t};b\right)\mathrm{d}F_j(x) + \bar{F}_j\left(h_\beta(u,t)+c/\beta\right)\right] + o(t) \tag{2.8}$$

By Taylor's expansion

$$M_{2,i}\left(h_\beta(u,t), y\mathrm{e}^{-\delta t};b\right) = M_{2,i}(u,y;b) + (\beta u + c)t\frac{\partial M_{2,i}(u,y;b)}{\partial u} - \delta yt\frac{\partial M_{2,i}(u,y;b)}{\partial y} + o(t) \tag{2.9}$$

Substituting (2.9) into (2.8), dividing both sides by *t*, and letting $t \to 0$, we obtain (2.2).

When the initial surplus is *b*, we obtain

$$\begin{aligned} M_{1,i}(b,y;b) &= (1-\lambda_i t)\mathrm{e}^{yct} M_{1,i}\left(b, y\mathrm{e}^{-\delta t}; b\right) \\ &+ \lambda_i t\mathrm{e}^{yct}\sum_{j=1}^{m} p_{i,j}\left[\int_0^b M_{1,j}\left(b-x, y\mathrm{e}^{-\delta t}; b\right)\mathrm{d}F_j(x) + \int_b^{b+\frac{c}{\beta}} M_{2,j}\left(b-x, y\mathrm{e}^{-\alpha t}; b\right)\mathrm{d}F_j(x) + \bar{F}_j(b+c/\beta)\right] + o(t) \end{aligned} \tag{2.10}$$

Using Taylor's expansion, we have,

$$\begin{aligned} \delta y\frac{\partial M_{1,i}(b,y;b)}{\partial y} + (\lambda_i - cy)M_{1,i}(b,y;b) &= \lambda_i \sum_{j=1}^{m} p_{i,j}\left[\int_0^b M_{1,j}\left(b-x, ye^{-\delta t}; b\right)dF_j(x)\right. \\ &= \lambda_i \sum_{j=1}^{m} p_{i,j}\left[\int_0^b M_{1,j}(b-x, ye^{-\delta t}; b)dF_j(x) + \bar{F}_j(b+c/\beta)\right] + o(t) \end{aligned} \tag{2.11}$$

Letting $u \uparrow b$ in (2.1) and comparing it with (2.11), we obtain (2.3). Where "↑" denoting increasing approach.

When $u = -c/\beta$, absolute ruin is immediate. Thus, no dividend is paid. So we obtain (2.4). Theorem 2.1 is proved.

Theorem 2.2 For $i \in S$,

$$M_{1,i}(0+, y; b) = M_{2,i}(0-, y; b) \tag{2.12}$$

Proof. For $-c/\beta < u \le 0$, letting τ_0 be the time that the surplus reach 0 for the first time from $u < 0$ and using the Markov property of the surplus process, we obtain

$$\begin{aligned} M_{2,i}(u,y;b) &= E_i^u\left[I(\tau_0 < T_b)\mathrm{e}^{yD_{u,b}}\right] + E_i^u\left[I(\tau_0 \ge T_b)\mathrm{e}^{yD_{u,b}}\right] \\ &= E_i^u\left[I(\tau_0 < T_b)\exp\left\{y\int_0^{T_b-\tau_0}\mathrm{e}^{-\delta t}\mathrm{d}D(t+\tau_0)\right\}\right] + P(\tau_0 \ge T_b) \\ &= E_i^u\left[I(\tau_0 < T_b)\exp\left\{y\mathrm{e}^{-\delta\tau_0}\int_{\tau_0}^{T_b}\mathrm{e}^{-\delta t}\mathrm{d}D(t)\right\}\right] + P(\tau_0 \ge T_b) \\ &\le M_{1,i}(0,y;b) + P(\tau_0 \ge T_b) \end{aligned} \tag{2.13}$$

Similarly, we obtain

$$\begin{aligned} M_{2,i}(u,y;b) &\ge E_i^u\left[I(\tau_0 < T_b, \tau_0 = t_0)\mathrm{e}^{yD_{u,b}}\right] + E_i^u\left[I(\tau_0 \ge T_b)\mathrm{e}^{yD_{u,b}}\right] \\ &= M_{1,i}(0,y;b)E_i^u\left[\mathrm{e}^{-\delta\tau_0}I(\tau_0 < T_b, \tau_0 = t_0)\right] + P(\tau_0 \ge T_b) \\ &= M_{1,i}(0,y;b)\mathrm{e}^{-\delta\tau_0}P(T_1 > t_0) + P(\tau_0 \ge T_b) \\ &= \mathrm{e}^{-(\lambda_i+\delta)t_0}M_{1,i}(0,y;b) + P(\tau_0 \ge T_b) \end{aligned} \tag{2.14}$$

where T_1 is the time of the first claim.

When $u \uparrow 0$, we notice that τ_0 and t_0 both go into zero. Letting $u \uparrow 0$ in (2.13) and (2.14) and in view of

$$\lim_{u\uparrow 0} P(\tau_0 \ge T_b) = 0$$

we obtain (2.12). Theorem 2.2 is proved.

3. Higher Moment of the Dividend Payments

We now derive a system of integro-differential equations satisfied by $V(u,b)$. By the definitions of $M(u,y;b)$ and $V(u,b)$, we obtain, for $i \in S$,

$$M_{1,i}(u,y;b) = 1 + \sum_{n=1}^{\infty}\frac{y^n}{n!}V_{n,1,i}(u,b) \tag{3.1}$$

$$M_{2,i}(u,y;b) = 1 + \sum_{n=1}^{\infty}\frac{y^n}{n!}V_{n,2,i}(u,b) \tag{3.2}$$

where, $V_{n,i}(u;b)$ is defined by

$$V_{n,i}(u;b) = \begin{cases} V_{n,1,i}(u;b), & 0 \le u < b \\ V_{n,2,i}(u;b), & -c/\beta < u \le 0 \end{cases}$$

Substituting (3.1) and (3.2) into (2.1) and (2.2), respectively, and comparing the coefficients of y^n yield the following integro-differential equations:

$$cV'_{n,1,i}(u,b)=(\lambda_i+n\delta)V_{n,1,i}(u,b)-\lambda_i\sum_{j=1}^{m}p_{i,j}\left[\int_0^u V_{n,1,j}(u-x,b)\mathrm{d}F_j(x)+\int_u^{u+\frac{c}{\beta}}V_{n,2,j}(u-x,b)\mathrm{d}F_j(x)\right] \quad (3.3)$$

for $0\le u<b$, and for $-c/\beta<u\le 0$,

$$(\beta u+c)V'_{n,2,i}(u,b)=(\lambda_i+n\delta)V_{n,2,i}(u,b)-\lambda_i\sum_{j=1}^{m}p_{i,j}\int_0^{u+\frac{c}{\beta}}V_{n,2,j}(u-x,b)\mathrm{d}F_j(x) \quad (3.4)$$

Substituting (3.1) into (2.3), similarly, we obtain

$$V'_{n,1,i}(u,b)\big|_{u=b}=nV_{n-1,1,i}(b,b) \quad (3.5)$$

Thus, $V_{1,1,i}(b,b)=1$ is an obvious result since $V_{0,1,i}(b,b)=1$.

Substituting (3.1) and (3.2) into (2.4) and (2.12), we obtain, for $n\in N^+$

$$V_{n,2,i}(-c/\beta,b)=0 \quad (3.6)$$

$$V_{n,1,i}(0+,b)=V_{n,2,i}(0-,b) \quad (3.7)$$

Letting $u\downarrow 0$ in (3.3) and $u\uparrow 0$ in (3.4) and using (3.7), we obtain, for $n\in N^+$.

$$V'_{n,1,i}(0+,b)=V'_{n,2,i}(0-,b) \quad (3.8)$$

where, "$\downarrow$" denoting decreasing approach.

4. Explicit Expressions for Exponential Claims to $V_n(u,b)$ for a Two-State Model

In this section, we consider a two-state Markov-dependent risk model. Then $\{Z_n;n\ge 0\}$ is a two-state Markov chain, which reflects the random environmental effects due to "normal" vs. "abnormal", or "high season" vs. "low season" conditions. We derive the explicit formulae for $V_n(u,b)$ when the claim size is exponentially dis tributed $F_i(x)=1-\mathrm{e}^{-x/\mu_i}$, $i=1,2$. Set $p_{11}=p_{22}=0$, $p_{12}=p_{21}=1$. In view of Equation (3.3) and the expo nential density function, Equation (3.3) are reduced to, for $0\le u\le b$

$$cV'_{n,1,1}(u,b)=(\lambda_1+n\delta)V_{n,1,1}(u,b)-\frac{\lambda_1}{\mu_2}\mathrm{e}^{-u/\mu_2}\int_0^u V_{n,1,2}(x,b)\mathrm{e}^{x/\mu_2}\mathrm{d}x-\frac{\lambda_1}{\mu_2}\mathrm{e}^{-u/\mu_2}\int_{-\frac{c}{\beta}}^0 V_{n,2,2}(x,b)\mathrm{e}^{x/\mu_2}\mathrm{d}x \quad (4.1)$$

$$cV'_{n,1,2}(u,b)=(\lambda_2+n\delta)V_{n,1,2}(u,b)-\frac{\lambda_2}{\mu_1}\mathrm{e}^{-u/\mu_1}\int_0^u V_{n,1,1}(x,b)\mathrm{e}^{x/\mu_1}\mathrm{d}x-\frac{\lambda_2}{\mu_1}\mathrm{e}^{-u/\mu_1}\int_{-\frac{c}{\beta}}^0 V_{n,2,1}(x,b)\mathrm{e}^{x/\mu_1}\mathrm{d}x \quad (4.2)$$

Applying the operator $\left(\frac{d}{du}+\frac{1}{\mu_2}\right)$ and $\left(\frac{d}{du}+\frac{1}{\mu_1}\right)$ on (4.1) and (4.2), respectively, and rearranging them, we obtain, for $0\le u\le b$

$$V''_{n,1,1}(u,b)+\left(\frac{1}{\mu_2}-\frac{\lambda_1+n\delta}{c}\right)V'_{n,1,1}(u,b)-\frac{\lambda_1+n\delta}{\mu_2 c}V_{n,1,1}(u,b)=-\frac{\lambda_1}{\mu_2 c}V_{n,1,2}(u,b) \quad (4.3)$$

$$V''_{n,1,2}(u,b)+\left(\frac{1}{\mu_1}-\frac{\lambda_2+n\delta}{c}\right)V'_{n,1,2}(u,b)-\frac{\lambda_2+n\delta}{\mu_1 c}V_{n,1,2}(u,b)=-\frac{\lambda_2}{\mu_1 c}V_{n,1,1}(u,b) \quad (4.4)$$

They are second-order linear non-homogeneous differential equations with constant coefficients. For convenient writing, let

$$q_1=\frac{1}{\mu_2}-\frac{\lambda_1+n\delta}{c},\quad q_2=-\frac{\lambda_1+n\delta}{\mu_2 c}$$

$$\eta_1=\frac{1}{\mu_1}-\frac{\lambda_2+n\delta}{c},\quad \eta_2=-\frac{\lambda_2+n\delta}{\mu_1 c}$$

$$g_1(u)=-\frac{\lambda_1 V_{n,1,2}(u,b)}{\mu_2 c},\quad g_2(u)=-\frac{\lambda_2 V_{n,1,1}(u,b)}{\mu_1 c}$$

Then Equation (4.3) and Equation (4.4) can be rewrite-ten as

$$V''_{n,1,1}(u,b)+q_1V'_{n,1,1}(u,b)+q_2V_{n,1,1}(u,b)=g_1(u) \quad (4.5)$$

$$V''_{n,1,2}(u,b)+\eta_1V'_{n,1,2}(u,b)+\eta_2V_{n,1,2}(u,b)=g_2(u) \quad (4.6)$$

The corresponding homogeneous differential Equations of (4.5) and (4.6) are

$$V''_{n,1,1}(u,b)+q_1V'_{n,1,1}(u,b)+q_2V_{n,1,1}(u,b)=0 \quad (4.7)$$

$$V''_{n,1,2}(u,b)+\eta_1V'_{n,1,2}(u,b)+\eta_2V_{n,1,2}(u,b)=0 \quad (4.8)$$

The general solutions of Equations (4.7) and (4.8) are

$$V_{n,1,1}(u,b)=c_1\mathrm{e}^{r_1u}+c_2\mathrm{e}^{r_2u} \quad (4.9)$$

$$V_{n,1,2}(u,b)=c_3\mathrm{e}^{\eta_1u}+c_4\mathrm{e}^{\eta_2u} \quad (4.10)$$

where c_1, c_2, c_3, c_4 are arbitrary constants,

$$r_1=\frac{-q_1+\sqrt{q_1^2-4q_2}}{2}$$

$$r_2 = \frac{-q_1 - \sqrt{q_1^2 - 4q_2}}{2}$$

$$\eta_1 = \frac{-q_3 + \sqrt{q_3^2 - 4q_4}}{2}$$

$$\eta_2 = \frac{-q_3 - \sqrt{q_3^2 - 4q_4}}{2}$$

According to the variation of constants method, we assume $c_1(u)e^{r_1 u} + c_2(u)e^{r_2 u}$ and $c_3(u)e^{\eta_1 u} + c_4(u)e^{\eta_2 u}$ are special solutions of Equations (4.5) and (4.6), respectively. Then we have

$$\begin{cases} c_1'(u)e^{r_1 u} + c_2'(u)e^{r_2 u} = 0 \\ c_1'(u)r_1 e^{r_1 u} + c_2'(u)r_2 e^{r_2 u} = g_1(u) \end{cases} \tag{4.11}$$

$$\begin{cases} c_3'(u)e^{\eta_1 u} + c_4'(u)e^{\eta_2 u} = 0 \\ c_3'(u)\eta_1 e^{\eta_1 u} + c_4'(u)\eta_2 e^{\eta_2 u} = g_2(u) \end{cases} \tag{4.12}$$

Solving the above two equations, we obtain

$$c_1'(u) = g_1(u)(r_1 - r_2)e^{-r_1 u}$$

$$c_2'(u) = g_1(u)(r_2 - r_1)e^{-r_2 u}$$

$$c_3'(u) = g_2(u)(\eta_1 - \eta_2)e^{-\eta_1 u}$$

$$c_4'(u) = g_2(u)(\eta_2 - \eta_1)e^{-\eta_2 u}$$

then we have, for $0 \le u \le b$

$$c_1(u) = (r_1 - r_2)\int_0^u g_1(x)e^{-r_1 x}dx$$

$$c_2(u) = (r_2 - r_1)\int_0^u g_1(x)e^{-r_2 x}dx$$

$$c_3(u) = (\eta_1 - \eta_2)\int_0^u g_2(x)e^{-\eta_1 x}dx$$

$$c_4(u) = (\eta_2 - \eta_1)\int_0^u g_2(x)e^{-\eta_2 x}dx$$

So the general solutions of Equations (4.5) and (4.6) are, for $0 \le u \le b$,

$$V_{n,1,1}(u,b) = (r_1 - r_2)e^{r_1 u}\int_0^u g_1(x)e^{-r_1 x}dx + (r_2 - r_1)e^{r_2 u}\int_0^u g_1(x)e^{-r_2 x}dx + c_1 e^{r_1 u} + c_2 e^{r_2 u} \tag{4.13}$$

$$V_{n,1,2}(u,b) = (\eta_1 - \eta_2)e^{\eta_1 u}\int_0^u g_2(x)e^{-\eta_1 x}dx + (\eta_2 - \eta_1)e^{\eta_2 u}\int_0^u g_2(x)e^{-\eta_2 x}dx + c_3 e^{\eta_1 u} + c_4 e^{\eta_2 u} \tag{4.14}$$

5. The Discounted Penalty Function

In this section, we derive integro-differential equations for the discounted penalty functions. For $i \in S$, define

$$\Phi_i(u,b) = \begin{cases} \Phi_{1,i}(u,b), & 0 \le u < b \\ \Phi_{2,i}(u,b), & -c/\beta < u \le 0 \end{cases}$$

Note that in the stationary case, we have

$$\Phi(u,b) = \sum_{i=1}^m \pi_i \Phi_i(u,b)$$

Theorem 5.1 For $i \in S$, $0 \le u \le b$,

$$c\Phi_{1,i}'(u,b) = (\lambda_i + \delta)\Phi_{1,i}(u,b) - \lambda_i \sum_{j=1}^m p_{i,j}\left[\int_0^u \Phi_{1,j}(u-x,b)dF_j(x) + \int_u^{u+\frac{c}{\beta}} \Phi_{2,j}(u-x,b)dF_j(x) + A_j(u)\right] \tag{5.1}$$

and, for $-c/\beta < u \le 0$,

$$(\beta u + c)\Phi_{2,i}'(u,b) = (\lambda_i + \delta)\Phi_{2,i}(u,b) - \lambda_i \sum_{j=1}^m p_{i,j}\int_0^{u+\frac{c}{\beta}} \Phi_{2,j}(u-x,b)dF_j(x) + A_j(u) \tag{5.2}$$

with boundary conditions

$$\Phi_{1,i}'(b,b) = 0 \tag{5.3}$$

$$\Phi_{1,i}(0+,b) = \Phi_{2,i}(0-,b) \tag{5.4}$$

$$\Phi_{1,i}'(0+,b) = \Phi_{2,i}'(0-,b) \tag{5.5}$$

where

$$A_i(u) = \int_{u+\frac{c}{\beta}}^{\infty} \omega(u, x-u)dF_i(x)$$

Proof. For $i \in S$ and $0 \le u \le b$. Similar to argument as in Section 2, we condition on the events that can occur in the small time interval $[0,t]$.

$$\begin{aligned} \Phi_{1,i}(u,b) = {} & (1-\lambda_i t)e^{-\delta t}\Phi_{1,i}(u+ct,b) + \lambda_i t e^{-\delta t}\sum_{j=1}^m p_{i,j}\left[\int_0^{u+ct} \Phi_{1,j}(u+ct-x,b)dF_j(x)\right. \\ & \left. + \int_{u+ct}^{u+ct+\frac{c}{\beta}} \Phi_{2,j}(u+ct-x,b)F_j(x) + \int_{u+\frac{c}{\beta}}^{\infty} \omega(u,x-u)dF_j(x)\right] + o(t) \end{aligned} \tag{5.6}$$

Since

$$\mathrm{e}^{-\delta t} = 1-\delta h+o(h)$$

we then get

$$\begin{aligned}\Phi_{1,i}(u,b) &= \left[1-(\lambda_i+\delta)t\right]\Phi_{1,i}(u+ct,b)+\lambda_i t\sum_{j=1}^{m}p_{i,j}\ \int_0^{u+ct}\Phi_{1,j}(u+ct-x,b)dF_j(x)\\ &\quad +\int_{u+ct}^{u+ct+\frac{c}{\beta}}\Phi_{2,j}(u+ct-x,b)dF_j(x)+\int_{u+\frac{c}{\beta}}^{\infty}\omega(u,x-u)dF_j(x)\Bigg]+o(t)\end{aligned} \tag{5.7}$$

Equation (5.7) can be rewritten as

$$\begin{aligned}\frac{\Phi_{1,i}(u+ct,b)-\Phi_{1,i}(u,b)}{t} &= (\lambda_i+\delta)\Phi_{1,i}(u+ct,b)-\lambda_i\sum_{j=1}^{m}p_{i,j}\left[\int_0^{u+ct}\Phi_{1,j}(u+ct-x,b)\mathrm{d}F_j(x)\right.\\ &\quad \left.+\int_{u+ct}^{u+ct+\frac{c}{\beta}}\Phi_{2,j}(u+ct-x,b)\mathrm{d}F_j(x)+\int_{u+\frac{c}{\beta}}^{\infty}\omega(u,x-u)\mathrm{d}F_j(x)\right]+\frac{o(t)}{t}\end{aligned} \tag{5.8}$$

Letting $t\to 0$ in (5.8), we obtain (5.1).

For $i\in S$ and $-c/\beta<u\le 0$, we have

$$\begin{aligned}\Phi_{2,i}(u,b) &= (1-\lambda_i t)\mathrm{e}^{-\delta t}\Phi_{2,i}\left(h_\beta(u,t),b\right)+\lambda_i t\mathrm{e}^{-\delta t}\\ &\quad\times\sum_{j=1}^{m}p_{i,j}\left[\int_0^{h_\beta(u,t)+\frac{c}{\beta}}\Phi_{2,j}\left(h_\beta(u,t)-x,b\right)\mathrm{d}F_j(x)+\int_{h_\beta(u,t)+\frac{c}{\beta}}^{\infty}\omega\left(h_\beta(u,t),x-h_\beta(u,t)\right)\mathrm{d}F_j(x)\right]+o(t)\end{aligned} \tag{5.9}$$

By Taylor's expansion

$$\begin{aligned}\Phi_{2,i}\left(h_\beta(u,t),b\right) &= \Phi_{2,i}(u,b)\\ &\quad+(\beta u+c)t\ \Phi'_{2,i}(u,b)+o(t)\end{aligned} \tag{5.10}$$

Substituting (5.10) into (5.9), dividing both sides by t, and letting $t\to 0$, we obtain (5.2). Theorem 5.1 is proved.

Integro-differential Equations (5.1) and (5.2) can eas ily be rewritten in matrix form.

Let

$$\mathbf{\Phi}_j(u,b)=\left(\Phi_{j,1}(u,b),\cdots,\Phi_{j,m}(u,b)\right)^{\mathrm{T}},\quad j=1,2.$$

"T" denoting transpose. Rewritten (5.1) and (5.2) in ma- trix, then we have

Theorem 5.2 The integro-differential equation in ma trix form for $\mathbf{\Phi}_1(u,b)$ and $\mathbf{\Phi}_2(u,b)$, for $0\le u\le b$,

$$\begin{aligned}\mathbf{\Phi}'_1(u,b) &= \mathbf{H}_1\mathbf{\Phi}_1(u,b)+\int_0^u \boldsymbol{G}_1(x)\mathbf{\Phi}_1(u-x,b)\mathrm{d}x\\ &\quad+\int_u^{u+\frac{c}{\beta}}\boldsymbol{G}_1(x)\mathbf{\Phi}_2(u-x,b)\mathrm{d}x+\mathbf{A}_1(u)\end{aligned} \tag{5.11}$$

and, for $-c/\beta<u\le 0$,

$$\begin{aligned}\mathbf{\Phi}'_2(u,b) &= \boldsymbol{H}_2(u)\mathbf{\Phi}_2(u,b)\\ &\quad+\int_0^{u+\frac{c}{\beta}}\boldsymbol{G}_2(x)\mathbf{\Phi}_2(u-x,b)\mathrm{d}x+\boldsymbol{A}_2(u)\end{aligned} \tag{5.12}$$

where

$$\boldsymbol{H}_1=\begin{pmatrix}\frac{\lambda_1+\delta}{c} & & & \\ & \frac{\lambda_2+\delta}{c} & & \\ & & \ddots & \\ & & & \frac{\lambda_m+\delta}{c}\end{pmatrix}$$

$$\boldsymbol{H}_2(u)=\begin{pmatrix}\frac{\lambda_1+\delta}{\beta u+c} & & & \\ & \frac{\lambda_2+\delta}{\beta u+c} & & \\ & & \ddots & \\ & & & \frac{\lambda_m+\delta}{\beta u+c}\end{pmatrix}$$

$$\boldsymbol{G}_1(x)=-\begin{pmatrix}\frac{\lambda_1}{c} & & \\ & \ddots & \\ & & \frac{\lambda_m}{c}\end{pmatrix}\Lambda\begin{pmatrix}f_1(x) & & \\ & \ddots & \\ & & f_m(x)\end{pmatrix}$$

$$\boldsymbol{G}_2(x)=-\begin{pmatrix}\frac{\lambda_1}{\beta u+c} & & \\ & \ddots & \\ & & \frac{\lambda_m}{\beta u+c}\end{pmatrix}\times\Lambda\begin{pmatrix}f_1(x) & & \\ & \ddots & \\ & & f_m(x)\end{pmatrix}$$

are all $m \times m$ matrices, and $\boldsymbol{A}_1(u)$ and $\boldsymbol{A}_2(u)$ defined by

$$\boldsymbol{A}_1(u) = \int_{u+\frac{c}{\beta}}^{\infty} \omega(u, x-u)\boldsymbol{G}_1(x)\boldsymbol{I}\mathrm{d}x$$

$$\boldsymbol{A}_2(u) = \int_{u+\frac{c}{\beta}}^{\infty} \omega(u, x-u)\boldsymbol{G}_2(x)\boldsymbol{I}\mathrm{d}x$$

are all m-dimensional vector, in which $\boldsymbol{I} = (1,1,\cdots,1)^{\mathrm{T}}$ is an $m \times 1$ column vector. The continuity condition and derivative condition for $\boldsymbol{\Phi}_1(u,b)$ and $\boldsymbol{\Phi}_2(u,b)$ is

$$\boldsymbol{\Phi}_1(0+,b) = \boldsymbol{\Phi}_2(0-,b),$$

$$\boldsymbol{\Phi}_1'(0+,b) = \boldsymbol{\Phi}_2'(0-,b).$$

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