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Rhotrix Linear Transformation

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ABSTRACT

This paper considers rank of a rhotrix and characterizes its properties, as an extension of ideas to the rhotrix theory rhomboidal arrays, introduced in 2003 as a new paradigm of matrix theory of rectangular arrays. Furthermore, we present the necessary and sufficient condition under which a linear map can be represented over rhotrix.

Keywords: Rhotrix; Rank; Rhotrix Rank; Linear Transformation; Rhotrix Linear Transformation

1. Introduction

By a rhotrix A of dimension *three*, we mean a rhomboidal array defined as

$$A=\left\langle \begin{array}{ccc} & a & \\ b & c & d \\ & e & \end{array} \right\rangle,$$

where, $a,b,c,d,e\in\Re$. The entry c in rhotrix A is called the heart of A and it is often denoted by $h(A)$. The concept of rhotrix was introduced by [1] as an extension of matrix-tertions and matrix noitrets suggested by [2]. Since the introduction of rhotrix in [1], many researchers have shown interest on development of concepts for Rhotrix theory that are analogous to concepts in Matrix theory (see [3-9]). Sani [7] proposed an alternative method of rhotrix multiplication, by extending the concept of row-column multiplication of two dimensional matrices to three dimensional rhotrices, recorded as follows:

$$A\circ B=\left\langle \begin{array}{ccc} & a & \\ b & h(A) & d \\ & e & \end{array} \right\rangle\circ\left\langle \begin{array}{ccc} & f & \\ g & h(B) & i \\ & j & \end{array} \right\rangle=\left\langle \begin{array}{ccc} & af+dg & \\ bf+eg & h(A)h(B) & ai+dj \\ & bi+ej & \end{array} \right\rangle,$$

where, A and B belong to set of all three dimensional rhotrices, $R_3(\Re)$.

The definition of rhotrix was later generalized by [6] to include any finite dimension $n\in 2Z^{+}+1$. Thus; by a rhotrix A of dimension $n\in 2Z^{+}+1$, we mean a rhomboidal array of cardinality $\frac{1}{2}(n^2+1)$. Implying a rhotrix R of dimension n can be written as

$$R_n=\left\langle \begin{array}{ccccccc} & & & a_{11} & & & \\ & & a_{21} & c_{11} & a_{12} & & \\ & \cdots & \cdots & \cdots & \cdots & \cdots & \\ a_{t1} & \cdots & \cdots & \cdots & \cdots & \cdots & a_{tt} \\ & \cdots & \cdots & \cdots & \cdots & \cdots & \\ & & a_{tt-1} & c_{t-1t-1} & a_{t-1t} & & \\ & & & a_{tt} & & & \end{array} \right\rangle$$

The element $a_{ij}\ (i,j=1,2,\cdots,t)$ and $c_{kl}\ (k,l=1,2,\cdots,t-1)$ are called the major and minor entries of R respectively. A generalization of row-column multiplication method for n-dimensional rhotrices was given by [8]. That is, given any n-dimensional rhotrices $R_n=\langle a_{ij},c_{kl}\rangle$ and $Q_n=\langle b_{ij},d_{kl}\rangle$, the multiplication of R_n and Q_n is as follows:

$$R_n\circ Q_n=\left\langle \sum_{i,j=1}^{t}\left(a_{ij}b_{ij}\right),\sum_{k,l=1}^{t-1}\left(c_{kl}d_{kl}\right)\right\rangle, t=\frac{(n+1)}{2}.$$

The method of converting a rhotrix to a special matrix called "coupled matrix" was suggested by [9]. This idea was used to solve systems of $n\times n$ and $(n-1)\times(n-1)$ matrix problems simultaneously. The concept of vectors and rhotrix vector spaces and their properties were introduced by [3] and [4] respectively. To the best of our knowledge, the concept of rank and linear transformation of rhotrix has not been studied. In this paper, we consider the rank of a rhotrix and characterize its properties. We also extend the idea to suggest the necessary and sufficient condition for representing rhotrix linear transformation.

2. Preliminaries

The following definitions will help in our discussion of a useful result in this section and other subsequent ones.

2.1. Definition

Let $R_n = \langle a_{ij}, c_{kl} \rangle$ be an *n*-dimensional rhotrix. Then, a_{ij} is the (i, j)-entries called the major entries of R_n and c_{kl} is the (k,l)-entries called the minor entries of R_n.

2.2. Definition 2.2 [7]

A rhotrix $R_n = \langle a_{ij}, c_{kl} \rangle$ of *n*-dimension is a coupled of two matrices (a_{ij}) and (c_{kl}) consisting of its major and minor matrices respectively. Therefore, (a_{ij}) and (c_{kl}) are the major and minor matrices of R_n.

2.3. Definition

Let $R_n = \langle a_{ij}, c_{kl} \rangle$ be an *n*-dimensional rhotrix. Then, rows and columns of (a_{ij}) ((c_{kl})) will be called the major (minor) rows and columns of R_n respectively.

2.4. Definition

For any odd integer *n*, an $n \times n$ matrix (a_{ij}) is called a filled coupled matrix if $a_{ij} = 0$ for all i, j whose sum $i + j$ is odd. We shall refer to these entries as the *null* entries of the filled coupled matrix.

2.5. Theorem

There is one-one correspondence between the set of all *n*-dimensional rhotrices over F and the set of all $n \times n$ filled coupled matrices over F.

3. Rank of a Rhotrix

Let $R_n = \langle a_{ij}, c_{kl} \rangle$, the entries $a_{rr}\ (1 \le r \le t)$ and $c_{ss}\ (1 \le s \le t-1)$ in the main diagonal of the major and minor matrices of R respectively, formed the main diagonal of *R*. If all the entries to the left (right) of the main diagonal in R are zeros, R is called a right (left) triangular rhotrix. The following lemma follows trivially.

3.1. Lemma

Let $R_n = \langle a_{ij}, c_{kl} \rangle$, is a left (right) triangular rhotrix if and only if (a_{ij}) and (c_{kl}) are lower (upper) triangular matrices.

Proof

This follows when the rhotrix R_n is being rotated through 45˚ in anticlockwise direction.

In the light of this lemma, any *n*-dimensional rhotrix R can be reduce to a right triangular rhotrix by reducing its major and minor matrix to echelon form using elementary row operations. Recall that, the rank of a matrix A denoted by $\text{rank}(A)$ is the number of non-zero row(s) in its reduced row echelon form. If $R_n = \langle a_{ij}, c_{kl} \rangle$, we define rank of R denoted by $\text{rank}(R)$ as:

$$\text{rank}(R) = \text{rank}(a_{ij}) + \text{rank}(c_{kl}). \tag{3}$$

It follows from Equation (3) that many properties of rank of matrix can be extended to the rank of rhotrix. In particular, we have the following:

3.2. Theorem

Let $R_n = \langle a_{ij}, c_{kl} \rangle$, and $Q_n = \langle b_{ij}, d_{kl} \rangle$, be any two *n*-dimensional rhotrices, where $n \in 2Z^{+} + 1$. Then

1) $\text{rank}(R) \le n$;
2) $\text{rank}(R+S) \le \text{rank}(R) + \text{rank}(S)$;
3) $\text{rank}(R) + \text{rank}(S) - n \le \text{rank}(R \circ S)$;
4) $\text{rank}(R \circ S) \le \min\{\text{rank}(R), \text{rank}(S)\}$.

Proof

The first two statements follow directly from the definition. To prove the third statement, we apply the corresponding inequality for matrices, that is, $\text{rank}(AB) \ge \text{rank}(A) + \text{rank}(B) - n$, where A is $m \times n$ and B is $n \times p$. Thus,

$$\begin{aligned}\text{rank}(RS) &= \text{rank}\left[(a_{ij})(b_{ij})\right] + \text{rank}\left[(c_{kl})(d_{kl})\right] \\ &\ge \left[\text{rank}(a_{ij}) + \text{rank}(b_{ij}) - \left(\frac{n+1}{2}\right)\right] \\ &\quad + \left[\text{rank}(c_{kl}) + \text{rank}(d_{kl}) - \left(\frac{n+1}{2}\right) + 1\right] \\ &= \text{rank}(R) + \text{rank}(S) - n.\end{aligned}$$

For the last statement, consider

$$\begin{aligned}&\text{rank}(RS) \\ &= \text{rank}\left[(a_{ij})(b_{ij})\right] + \text{rank}\left[(c_{kl})(d_{kl})\right] \\ &\le \min\left\{(a_{ij}), \text{rank}(b_{ij})\right\} + \min\left\{(c_{kl}), \text{rank}(d_{kl})\right\} \\ &\le \min\left\{(a_{ij}) + \text{rank}(c_{kl}), (b_{ij}) + \text{rank}(d_{kl})\right\} \\ &= \min\left\{\text{rank}(R) + \text{rank}(S)\right\}.\end{aligned}$$

3.3. Example

Let

$$A = \left\langle \begin{array}{ccccc} & & 1 & & \\ & 0 & 2 & -2 & \\ 1 & -1 & 3 & 1 & 2 \\ & -2 & 1 & 1 & \\ & & 2 & & \end{array} \right\rangle.$$

Then, the filled coupled matrix of A is given by

$$m(A)=\begin{pmatrix}1&0&-2&0&2\\0&2&0&1&0\\0&0&3&0&1\\0&-1&0&1&0\\1&0&-2&0&2\end{pmatrix}.$$

Now reducing $m(A)$ to reduce row echelon form $(rref)$, we obtain

$$rref(m(A))=\begin{pmatrix}1&0&-2&0&2\\0&2&0&1&0\\0&0&3&0&1\\0&0&0&3&0\\0&0&0&0&0\end{pmatrix},$$

which is a coupled of (2×2) and (3×3) matrices, *i.e.* $A(\text{say})=\begin{pmatrix}2&0\\0&3\end{pmatrix}$ and $B(\text{say})=\begin{pmatrix}1&-2&2\\0&3&1\\0&0&0\end{pmatrix}$ respectively.

Notice that,

$$\begin{aligned}&\text{rank}(A)+\text{rank}(B)\\&=2+2=4=\text{rank}\big(rref(m(A))\big).\end{aligned}$$

Hence, $=\text{rank}(A)=4$.

4. Rhotrix Linear Transformation

One of the most important concepts in linear algebra is the concept of representation of linear mappings as matrices. If V and W are vector spaces of dimension n and m respectively, then any linear mapping T from V to W can be represented by a matrix. The matrix representation of T is called the matrix of T denoted by $m(T)$. Recall that, if F is a field, then any vector space V of finite dimension n over F is isomorphic to F^n. Therefore, any $n\times n$ matrix over F can be considered as a linear operator on the vector space F^n in the fixed standard basis. Following this ideas, we study in this section, a rhotrix as a linear operator on the vector space F^n. Since the dimension of a rhotrix is always odd, it follow that, in representing a linear map T on a vector space V by a rhotrix, the dimension of V is necessarily odd. Therefore, throughout what follows, we shall consider only odd dimensional vector spaces. For any $n\in 2\mathrm{Z}^{+}+1$ and F be an arbitrary field, we find the coupled F^t, F^{t-1} of F^t $F^t=\{(\alpha_1,\alpha_2,\cdots,\alpha_t)\,|\,\alpha_1,\cdots,\alpha_t\in F\}$ and $F^{t-1}=\{(\beta_1,\beta_2,\cdots,\beta_t)\,|\,\beta_1,\beta_2,\cdots,\beta_{t-1}\in F^{t-1}\}$ by

$$\begin{aligned}(F^t,F^{t-1})=\{&(\alpha_1,\alpha_2,\cdots,\alpha_t,\beta_1,\beta_2,\cdots,\beta_{t-1}):\\&\alpha_1,\alpha_2,\cdots,\alpha_t,\beta_1,\beta_2,\cdots,\beta_{t-1}\in F^t\}.\end{aligned}$$

It is clear that (F^t,F^{t-1}) coincides with F^n and so, if $n\in 2\mathrm{Z}^{+}+1$, any n-dimensional vector spaces V_1 and V_2 is of dimensions $\frac{n+1}{2}$ and $\frac{n+1}{2}-1$ respectively. Less obviously, it can be seen that not every linear map T of F^n can be represented by a rhotrix in the standard basis. For instance, the map

$$T:F^3\to F^3$$

defined by

$$T(x,y,z)=(x-y,x+z,y+z)$$

is a linear mapping on F^3 which cannot be represented by a rhotrix in the standard basis. The following theorem characterizes when a linear map T on F^n can be represented by a rhotrix.

4.1. Theorem

Let $n\in 2\mathrm{Z}^{+}+1$ and F be a field. Then, a linear map $T:F^n\to F^n$ can be represented by a rhotrix with respect to the standard basis if and only if T is defined as

$$\begin{aligned}&T(x_1,y_1,x_2,y_2,\cdots,y_{t-1},x_t)\\&=\big(\alpha_1(x_1,x_2,\cdots,x_t),\beta_1(y_1,y_2,\cdots,y_{t-1}),\\&\quad\alpha_2(x_1,x_2,\cdots,x_t),\beta_2(y_1,y_2,\cdots,y_{t-1}),\cdots,\\&\quad\beta_{t-1}(y_1,y_2,\cdots,y_{t-1}),\alpha_t(x_1,x_2,\cdots,x_t)\big),\end{aligned}$$

where $t=\frac{n+1}{2},\alpha_1,\cdots,\alpha_t$ and $\beta_1,\cdots,\beta_{t-1}$ are any linear map on F^t and F^{t-1} respectively.

Proof:

Suppose $T:F^n\to F^n$ is defined by

$$\begin{aligned}&T(x_1,y_1,x_2,y_2,\cdots,y_{t-1},x_t)\\&=\big(\alpha_1(x_1,x_2,\cdots,x_t),\beta_1(y_1,y_2,\cdots,y_{t-1}),\\&\quad\alpha_2(x_1,x_2,\cdots,x_t),\beta_2(y_1,y_2,\cdots,y_{t-1}),\cdots,\\&\quad\beta_{t-1}(y_1,y_2,\cdots,y_{t-1}),\alpha_t(x_1,x_2,\cdots,x_t)\big),\end{aligned}$$

where, $t=\frac{n+1}{2},\alpha_1,\cdots,\alpha_t$ and $\beta_1,\cdots,\beta_{t-1}$ are any linear map on F^t and F^{t-1} respectively, and consider the standard basis $\{(1,0,\cdots,0),(0,1,0,\cdots,0),\cdots,(0,0,\cdots,1)\}$. Note that, for $1\le i\le t$ and $1\le j\le t-1$. Since α_i,β_j are linear maps, $\alpha_i(0,\cdots,0)=\beta_j(0,\cdots,0)=0$. Thus,

$$\left.\begin{array}{l} T(1,0,\cdots,0)=\left[\alpha_1(1,0,\cdots,0),0,\cdots,\alpha_t(1,0,\cdots,0)\right] \\ T(1,0,\cdots,0)=\left[0,\beta_1(1,0,\cdots,0),\cdots,\beta_{t-1}(1,0,\cdots,0)\right] \\ \qquad\vdots \\ T(0,\cdots,0,1)=\left[0,\beta_1(0,\cdots,0,1),\cdots,\beta_{t-1}(0,\cdots,0),1\right] \\ T(0,\cdots,0,1)=\left[\alpha_1(0,\cdots,0,1),0,\cdots,\alpha_t(0,0,\cdots,0,1)\right] \end{array}\right\} \quad (5)$$

Let $\alpha_{ij}=\alpha_j\left(0,\cdots,\underset{i^{\text{th}}-\text{position}}{1},\cdots,0\right)$ for $(1\le i,j\le t)$ and $\beta_{kl}=\beta_l\left(0,\cdots,\underset{j^{\text{th}}-\text{position}}{1},\cdots,0\right)$ for $(1\le k,l\le t-1)$. Then from (5), we have the matrix of T is

$$\begin{pmatrix} \alpha_{11} & 0 & \alpha_{12} & \ldots & \alpha_{1t-1} & 0 & \alpha_{1t} \\ 0 & \beta_{11} & 0 & \ldots & 0 & \beta_{1t-1} & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & \beta_{t-1t} & 0 & \ldots & 0 & \beta_{t-1t-1} & 0 \\ \alpha_{t1} & 0 & \alpha_{t2} & \ldots & \alpha_{tt-1} & 0 & \alpha_{tt} \end{pmatrix}. \quad (6)$$

This is a filled coupled matrix from which we obtain the rhotrix representation of T as $\langle\alpha_{ij},\beta_{kl}\rangle$.

Conversely:

Suppose $T:F^n\to F^n$ has a rhotrix representation $\langle\alpha_{ij},\beta_{kl}\rangle$ in the standard basis. Then, the corresponding matrix representation of T is the filled coupled given in (6) above. Thus, we obtain the system

$$\left.\begin{array}{l} T(1,0,\cdots,0)=(\alpha_{11},0,\alpha_{12},\cdots,\alpha_{1t-1},0,\alpha_{1t}) \\ T(1,0,\cdots,0)=(0,\beta_{1t-1},0,\cdots,\beta_{1t-1},0) \\ \qquad\vdots \\ T(0,\cdots,0,1)=(0,\beta_{t-1t},0,\cdots,\beta_{t-1t-1},0) \\ T(0,\cdots,0,1)=(\alpha_{t1},0,\alpha_{t2},\cdots,\alpha_{tt-1},0,\alpha_{tt}) \end{array}\right\} \quad (7)$$

From this system, it follows that for each $(x_1,y_1,x_2,y_2,\cdots,y_{t-1},x_t)\in F^n$ we have the linear transformation T defined by

$$\begin{aligned} &T(x_1,y_1,x_2,y_2,\cdots,y_{t-1},x_t) \\ &=\big(\alpha_1(x_1,x_2,\cdots,x_t),\beta_1(y_1,y_2,\cdots,y_{t-1}), \\ &\quad \alpha_2(x_1,x_2,\cdots,x_t),\beta_2(y_1,y_2,\cdots,y_{t-1}),\cdots, \\ &\quad \beta_{t-1}(y_1,y_2,\cdots,y_{t-1}),\alpha_t(x_1,x_2,\cdots,x_t)\big), \end{aligned}$$

where, $t=\frac{n+1}{2},\alpha_1,\cdots,\alpha_t$ and $\beta_1,\cdots,\beta_{t-1}$ are any linear map on F^t with $\alpha_j\left(0,\cdots,\underset{i^{\text{th}}-\text{position}}{1},\cdots,0\right)=\alpha_{ij}$ for $(1\le i,j\le t)$ and $\beta_l\left(0,\cdots,\underset{j^{\text{th}}-\text{position}}{1},\cdots,0\right)=\beta_{kl}$ for $(1\le k,l\le t-1)$.

4.2. Example

Consider the linear mappings $T:\Re\to\Re$ define by $T(x,y,z)=(2x-z,4y,x-3z)$. To find the rhotrix of T relative to the standard basis. We proceed by finding the matrices of T. Thus,

$$\begin{aligned} T(1,0,0)&=(2,0,1) \\ T(0,1,0)&=(0,4,0) \\ T(0,0,1)&=(-1,0,-3) \end{aligned}$$

Therefore, by definition of matrix of T with respect to the standard basis, we have

$$m(T)=\begin{pmatrix} 2 & 0 & 1 \\ 0 & 4 & 0 \\ -1 & 0 & -3 \end{pmatrix},$$

which is a filled coupled matrix from which we obtain the rhotrix of T in R_3, $r(T)=\left\langle\begin{array}{ccc} & 2 & \\ -1 & 4 & 1 \\ & -3 & \end{array}\right\rangle$.

Now starting with the rhotrix $r(T)=\left\langle\begin{array}{ccc} & 2 & \\ -1 & 4 & 1 \\ & -3 & \end{array}\right\rangle$ the filled coupled matrix of $r(T)$ is $\begin{pmatrix} 2 & 0 & 1 \\ 0 & 4 & 0 \\ -1 & 0 & -3 \end{pmatrix}$.

And so, defining $T:R_3\to R_3$

$$\begin{aligned} T(1,0,0)&=2(1,0,0)+0(0,1,0)+1(0,0,1) \\ T(0,1,0)&=0(1,0,0)+4(0,1,0)+0(0,0,1) \\ T(0,0,1)&=-1(1,0,0)+0(0,1,0)-3(0,0,1) \end{aligned}$$

Thus, if $(x,y,z)=x(1,0,0)+y(0,1,0)+z(0,0,1)$.
Therefore,

$$\begin{aligned} T(x,y,z)&=xT(1,0,0)+yT(0,1,0)+zT(0,0,1) \\ &=x(2,0,1)+y(0,4,0)+z(-1,0,-3) \\ &=(2x-z,4y,x-3z) \end{aligned}$$

5. Conclusion

We have considered the rank of a rhotrix and characterize its properties as an extension of ideas to the rhotrix theory rhomboidal arrays. Furthermore, a necessary and sufficient condition under which a linear map can be represented over rhotrix had been presented.

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