

# *L*-Fuzzy Vector Subspaces and Its Fuzzy Dimension

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## Abstract

In this paper, we introduce the definition of *L*-fuzzy vector subspace, define its dimension by an *L*-fuzzy natural number. For a finite-dimensional *L*-fuzzy vector subspace, we prove that the equality  $\dim(\tilde{E}_1 + \tilde{E}_2) + \dim(\tilde{E}_1 \cap \tilde{E}_2) = \dim \tilde{E}_1 + \dim \tilde{E}_2$  holds without any restricted conditions. At the same time, we deduce that the formula  $\dim(\widetilde{\text{im}} f) + \dim(\widetilde{\text{ker}} f) = \dim \tilde{E}$  holds.

## Keywords

*L*-Fuzzy Sets, *L*-Fuzzy Vector Subspace, *L*-Fuzzy Dimension

## 1. Introduction

Firstly, fuzzy vector subspace was introduced by Katsaras and Liu [1]. Then its properties and characters were investigated (see [2] [3] [4] [5], etc). The dimension of a fuzzy vector space was defined as a *n*-tuple by Lowen [6]. Subsequently, it was defined as a non-negative real number or infinity by Lubczonok [5], and proved that the formula

$$\dim(\tilde{E}_1 + \tilde{E}_2) + \dim(\tilde{E}_1 \cap \tilde{E}_2) = \dim \tilde{E}_1 + \dim \tilde{E}_2 \quad (1)$$

is valid under certain conditions, where  $\tilde{E}_1$  and  $\tilde{E}_2$  are fuzzy vector spaces. Recently, basis and dimension of a fuzzy vector space were redefined as a fuzzy set and a fuzzy natural number by Shi and Huang [7], respectively. Under the definitions, more properties of (crisp) vector spaces were correct in fuzzy vector spaces.

In this paper, we generalize the results in [7] to *L* lattice, and prove that some formulas still hold in the lattice *L*. In particular, we present the definition of *L*-fuzzy vector subspace and its -fuzzy dimension. The *L*-fuzzy dimension of a finite dimensional fuzzy vector subspace is a fuzzy natural number. We prove that (1) holds without any re-

stricted conditions and  $\dim(\widetilde{\ker f}) + \dim(\widetilde{\text{im} f}) = \dim \widetilde{E}$  holds.

## 2. Preliminaries

Given a set  $X$  and a completely distributive lattice  $L$ , we denote the power set of  $X$  and the set of all  $L$ -fuzzy sets on  $X$  (or  $L$ -sets for short) by  $2^X$  and  $L^X$ , respectively. For any  $A \subseteq X$ , we denote the cardinality of  $A$  by  $|A|$ .

An element  $a$  in  $L$  is called a prime element if  $a \geq b \wedge c$  implies  $a \geq b$  or  $a \geq c$ .  $a$  in  $L$  is called co-prime if  $a \leq b \vee c$  implies  $a \leq b$  or  $a \leq c$  [8]. The set of non-unit prime elements in  $L$  is denoted by  $P(L)$ . The set of non-zero co-prime elements in  $L$  is denoted by  $J(L)$ .

The binary relation  $<$  in  $L$  is defined as follows: for  $a, b \in L$ ,  $a < b$  if and only if for every subset  $D \subseteq L$ , the relation  $b \leq \sup D$  always implies the existence of  $d \in D$  with  $a \leq d$  [9].  $\{a \in L : a < b\}$  is called the greatest minimal family of  $b$  in the sense of [10], denoted by  $\beta(b)$ , and  $\beta^*(b) = \beta(b) \cap J(L)$ . Moreover, for  $b \in L$ , we define  $\alpha(b) = \{a \in L : a < {}^o b\}$  and  $\alpha^*(b) = \alpha(b) \cap P(L)$ . In a completely distributive lattice  $L$ , there exist  $\alpha(b)$  and  $\beta(b)$  for each  $b \in L$ , and  $b = \vee \beta(b) = \wedge \alpha(b)$  (see [10]).

In [10], Wang thought that  $\beta(0) = \{0\}$  and  $\alpha(1) = \{1\}$ . In fact, it should be that  $\beta(0) = \emptyset$  and  $\alpha(1) = \emptyset$ .

Throughout this paper,  $L$  denotes a completely distributive lattice, and  $E$  is a crisp vector space. We often do not distinguish a crisp subset  $A$  of  $E$  and its characteristic function  $\chi_A$ .

If  $A \in L^X$  and  $a \in L$ , we can define

$$A_{[a]} = \{x \in X : A(x) \geq a\}, \quad A_{(a)} = \{x \in X : a \in \beta(A(x))\},$$

$$A^{[a]} = \{x \in X : a \notin \alpha(A(x))\}, \quad A^{(a)} = \{x \in X : A(x) \not\leq a\}.$$

Some properties of these cut sets can be found in [11]-[16].

In [17] Shi introduced the concept of  $L$ -fuzzy natural numbers (denoted by  $\mathbb{N}(L)$ ), defined their operations and discussed the relation of  $\alpha$ -cut sets. We simply recall as follows: for any  $\lambda, \mu \in \mathbb{N}(L)$ ,  $a \in L$ ,

- (1)  $(\lambda + \mu)_{(a)} \subseteq \lambda_{(a)} + \mu_{(a)} \subseteq \lambda_{[a]} + \mu_{[a]} \subseteq (\lambda + \mu)_{[a]}$ ;
- (2)  $(\lambda + \mu)^{(a)} \subseteq \lambda^{(a)} + \mu^{(a)} \subseteq \lambda^{[a]} + \mu^{[a]} \subseteq (\lambda + \mu)^{[a]}$ ;
- (3) For any  $\lambda, \mu \in \mathbb{N}(L)$  and  $a \in P(L)$ , it follows that  $(\lambda + \mu)^{(a)} = \lambda^{(a)} + \mu^{(a)}$ .

## 3. $L$ -Fuzzy Vector Subspaces

**Definition 3.1.**  $L$ -fuzzy vector subspace is a pair  $\tilde{E} = (E, \mu)$  where  $E$  is a vector space on field  $F$ ,  $\mu : E \rightarrow L$  is a map with the property that for any  $x, y \in E, k, l \in F$ , we have  $\mu(kx + ly) \geq \mu(x) \wedge \mu(y)$ .

In this definition, when  $L = [0, 1]$ ,  $L$ -fuzzy vector subspace is exactly the fuzzy vector subspace defined in [1]. We denote the family of  $L$ -fuzzy vector subspaces by LFVS.

Let  $\tilde{E} = (E, \mu)$  be a member of LFVS, we denote

$$\begin{aligned} \tilde{E}_{[a]} = \mu_{[a]} &= \{x \in E : \mu(x) \geq a\}, & \tilde{E}_{(a)} = \mu_{(a)} &= \{x \in E : a \in \beta(\mu(x))\}. \\ \tilde{E}^{[a]} = \mu^{[a]} &= \{x \in E : a \notin \alpha(\mu(x))\}, & \tilde{E}^{(a)} = \mu^{(a)} &= \{x \in E : \mu(x) \not\leq a\}. \end{aligned}$$

We can obtain some properties of LFVS analogous to fuzzy vector subspaces as follows.

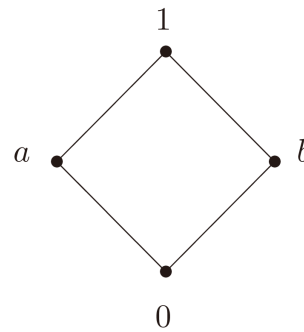
**Theorem 3.2.** Let  $\tilde{E} = (E, \mu)$  be a member of LFVS, then

- (1)  $\mu(0) = \sup_{x \in E} \mu(x)$ .
- (2) For any  $k \in F \setminus \{0\}$  and  $x \in E, \mu(kx) = \mu(x)$ .

The prove is trivial and omitted.

**Remark:** Since  $L$  is a completely distributive lattice, the property that if  $\mu(x) \neq \mu(y)$ , then  $\mu(x + y) = \mu(x) \wedge \mu(y)$  not holds for LFVS. This can be seen from the following example.

**Example 3.3.** Let  $L$  be a completely distributive lattice with four elements as follows.



Let  $\tilde{E} = (\mathbb{R}^2, \mu)$  be an  $L$ -fuzzy vector subspace on  $\mathbb{R}^2$  where  $\mu$  is defined by

$$\mu(x) = \begin{cases} 1, & x = (0, 0). \\ a, & x \in \{(y, 0) : y \in \mathbb{R} \setminus \{0\}\}. \\ b, & x \in \{(0, y) : y \in \mathbb{R} \setminus \{0\}\}. \\ 0, & \text{otherwise.} \end{cases}$$

We can easily check  $\tilde{E}$  is an  $L$ -fuzzy vector subspace on  $\mathbb{R}^2$ . Suppose that  $x = (3, 2)$  and  $y = (0, -2)$ , then  $\mu(x + y) = \mu(3, 0) = a > \mu(x) \wedge \mu(y) = 0 \wedge b = 0$ . This example illustrates for  $L$ -fuzzy vector subspace  $\mu(x) \neq \mu(y)$ ,  $\mu(x + y) > \mu(x) \wedge \mu(y)$ .

**Theorem 3.4.** Let  $E$  be a vector space,  $\mu \in L^E$  and  $\tilde{E} = (E, \mu)$ . Then the following statements are equivalent:

- (1)  $\tilde{E}$  is an  $L$ -fuzzy vector subspace.
- (2) (a) For all  $x \in E$  and  $k \in F, \mu(kx) \geq \mu(x)$ .  
(b) For any  $x, y \in E, \mu(x + y) \geq \mu(x) \wedge \mu(y)$ .
- (3) For any  $x_1, \dots, x_r \in E$  and  $k_1, \dots, k_r \in F$ , where  $r$  is a finite natural number, we have

$$\mu\left(\sum_{i=1}^r k_i x_i\right) \geq \bigwedge_{i=1}^r \mu(x_i).$$

The prove is trivial and omitted.

In the following paper, the vector spaces we discuss are finite-dimensional. For their  $L$ -fuzzy vector subspaces, the following observation will be useful.

**Remark:** Let  $\tilde{E} = (E, \mu)$  be a member of LFVS. Suppose that  $\mu(E) = \{\mu(x) : x \in E\}$ . Since  $E$  is finite-dimensional vector space, denotes  $\dim E = n$ , then  $\mu(E)$  is a finite subset of  $L$ .

In the fact, let  $B$  be a basis of  $E$ , then  $|B| = n$ . Suppose that  $\mu(E)$  is infinite, then for all  $a \in L$ , the total number of  $\tilde{E}_{[a]}$  is infinite. Since  $B \cap \tilde{E}_{[a]}$  is a basis of  $\tilde{E}_{[a]}$ , we have  $\tilde{E}_{[a]} = \langle B \cap \tilde{E}_{[a]} \rangle$ . Again since  $B$  is finite, the total number of  $\tilde{E}_{[a]}$  is also finite. It contradicts with the hypothesis. Therefore  $\mu(E)$  is a finite subset of  $L$  with at most  $2^n + 1$  values;  $2^n$  values which can be attained at the vectors of  $E \setminus \{0\}$  and the maximum which is attained at 0.

**Theorem 3.5.** Let  $E$  be a vector space,  $\mu \in L^E$  and  $\tilde{E} = (E, \mu)$ . Then the following statements equivalent:

- (1)  $\tilde{E}$  is an  $L$ -fuzzy vector subspace.
- (2) For all  $a \in L$ ,  $\tilde{E}_{[a]}$  is a vector space.
- (3) For all  $a \in J(L)$ ,  $\tilde{E}_{[a]}$  is a vector space.
- (4) For all  $a \in L$ ,  $\tilde{E}^{[a]}$  is a vector space.
- (5) For all  $a \in P(L)$ ,  $\tilde{E}^{[a]}$  is a vector space.
- (6) For all  $a \in P(L)$ ,  $\tilde{E}^{(a)}$  is a vector space.

*Proof.* We prove (1)  $\Leftrightarrow$  (4) and (1)  $\Leftrightarrow$  (6), the others can be proved analogously.

(1)  $\Rightarrow$  (4) We show that  $\tilde{E}^{[a]}$  is a vector space as follows. Suppose that  $x, y \in \tilde{E}^{[a]}$ , then  $a \notin \alpha(\mu(x))$  and  $a \notin \alpha(\mu(y))$ , i.e.  $a \notin \alpha(\mu(x)) \cup \alpha(\mu(y)) = \alpha(\mu(x) \wedge \mu(y))$ . Since  $\tilde{E} = (E, \mu)$  be an  $L$ -fuzzy vector subspace, then  $\alpha(\mu(x) \wedge \mu(y)) \supseteq \alpha(\mu(kx + ly))$ , we have  $a \notin \alpha(\mu(kx + ly))$ , this means  $kx + ly \in \tilde{E}^{[a]}$ . Therefore  $\tilde{E}^{[a]}$  is a vector space.

(4)  $\Rightarrow$  (1) Suppose that for all  $a \in L$ ,  $\tilde{E}^{[a]}$  is a vector space. Let  $x, y \in E$  and  $k, l \in F$ . Since  $\tilde{E}^{[a]}$  is a vector space, then  $kx + ly \in \tilde{E}^{[a]}$  if and only if  $x \in \tilde{E}^{[a]}$  and  $y \in \tilde{E}^{[a]}$ . We have

$$\begin{aligned} \mu(kx + ly) &= \bigwedge_{a \in L} (a \wedge \tilde{E}^{[a]})(kx + ly) \\ &= \bigwedge_{a \in L} (a \vee (\tilde{E}^{[a]}(x) \wedge \tilde{E}^{[a]}(y))) \\ &= \left( \bigwedge_{a \in L} (a \vee \tilde{E}^{[a]}(x)) \right) \wedge \left( \bigwedge_{a \in L} (a \vee \tilde{E}^{[a]}(y)) \right) \\ &= \mu(x) \wedge \mu(y). \end{aligned}$$

Therefore  $\tilde{E}$  is an  $L$ -fuzzy vector subspace.

(1)  $\Rightarrow$  (6) Suppose that  $x, y \in E^{(a)}$ , then  $\mu(x) \not\leq a$  and  $\mu(y) \not\leq a$ . Since  $a \in P(L)$ , then  $\mu(x) \wedge \mu(y) \not\leq a$ . Because  $\tilde{E} = (E, \mu)$  is an  $L$ -fuzzy vector subspace, we can have  $\mu(kx + ly) \not\leq a$ , this implies  $kx + ly \in E^{(a)}$ . Thus  $E^{(a)}$  is a vector space.

(6)  $\Rightarrow$  (1) Let  $x, y \in E$  and  $k, l \in F$ . Since  $\tilde{E}^{(a)}$  is a vector space, then

$kx + ly \in \tilde{E}^{(a)}$  if and only if  $x \in \tilde{E}^{(a)}$  and  $y \in \tilde{E}^{(a)}$ . We have the following implications.

$$\begin{aligned} \mu(kx + ly) &= \bigwedge_{a \in P(L)} (a \vee \tilde{E}^{(a)})(kx + ly) \\ &= \bigwedge_{a \in P(L)} (a \vee (\tilde{E}^{(a)}(x) \wedge \tilde{E}^{(a)}(y))) \\ &= \left( \bigwedge_{a \in P(L)} (a \vee \tilde{E}^{(a)}(x)) \right) \wedge \left( \bigwedge_{a \in P(L)} (a \vee \tilde{E}^{(a)}(y)) \right) \\ &= \mu(x) \wedge \mu(y). \end{aligned}$$

Therefore  $\tilde{E}$  is an  $L$ -fuzzy vector subspace.

**Theorem 3.6.** Let  $E$  be a vector space,  $\mu : E \rightarrow L$  be a map,  $\tilde{E} = (E, \mu)$ , and for all  $a, b \in L, \beta(a \wedge b) = \beta(a) \cap \beta(b)$ . Then the following statements equivalent:

(1)  $\tilde{E}$  is an  $L$ -fuzzy vector subspace.

(2) For all  $a \in L, \tilde{E}_{(a)}$  is a vector space.

*Proof.* (1)  $\Rightarrow$  (2) Suppose that  $x, y \in \tilde{E}_{(a)}$ , then  $a \in \beta(\mu(x))$  and  $a \in \beta(\mu(y))$ , i.e.  $a \in \beta(\mu(x) \cap \mu(y))$ . Since for all  $a, b \in L, \beta(a \wedge b) = \beta(a) \cap \beta(b)$  and  $\tilde{E}$  is an  $L$ -fuzzy vector subspace, we can know  $a \in \beta(\mu(x) \wedge \mu(y)) \subseteq \beta(\mu(ax + by))$ , this implies  $ax + by \in \tilde{E}_{(a)}$ . Therefore  $\tilde{E}_{(a)}$  is a vector space.

(2)  $\Rightarrow$  (1) Suppose that for all  $a \in L, \tilde{E}_{(a)}$  is a vector space. Let  $x, y \in E$  and  $k, l \in F$ . Since  $\tilde{E}_{(a)}$  is a vector space, then  $kx + ly \in \tilde{E}_{(a)}$  if and only if  $x \in \tilde{E}_{(a)}$  and  $y \in \tilde{E}_{(a)}$ . We have

$$\begin{aligned} \mu(kx + ly) &= \bigvee_{a \in L} (a \wedge \tilde{E}_{(a)})(kx + ly) \\ &= \bigvee_{a \in L} (a \wedge (\tilde{E}_{(a)}(x) \wedge \tilde{E}_{(a)}(y))) \\ &= \left( \bigvee_{a \in L} (a \wedge \tilde{E}_{(a)}(x)) \right) \wedge \left( \bigvee_{a \in L} (a \wedge \tilde{E}_{(a)}(y)) \right) \\ &= \mu(x) \wedge \mu(y). \end{aligned}$$

Therefore  $\tilde{E}$  is an  $L$ -fuzzy vector subspace.

We can define the operations between two  $L$ -fuzzy vector subspaces analogous to fuzzy vector subspaces.

**Definition 3.7.** Let  $\tilde{E}_1 = (E, \mu_1)$  and  $\tilde{E}_2 = (E, \mu_2)$  be two  $L$ -fuzzy vector subspaces on  $E$ . Define the intersection of  $\tilde{E}_1$  and  $\tilde{E}_2$  to be  $\tilde{E}_1 \cap \tilde{E}_2 = (E, \mu_1 \wedge \mu_2)$ . Define the sum of  $\tilde{E}_1$  and  $\tilde{E}_2$  to be  $\tilde{E}_1 + \tilde{E}_2 = (E, \mu_1 + \mu_2)$  where  $\mu_1 + \mu_2$  is defined by for all  $x \in E$

$$\begin{aligned} (\mu_1 + \mu_2)(x) &= \bigvee_{x=x_1+x_2} (\mu_1(x_1) \wedge \mu_2(x_2)) \\ &= \bigvee_{x_1 \in E} (\mu_1(x_1) \wedge \mu_2(x - x_1)). \end{aligned}$$

**Definition 3.8.** Let  $\tilde{E}_1 = (E_1, \mu_1)$  and  $\tilde{E}_2 = (E_2, \mu_2)$  be two members of LFVS and  $E = E_1 \oplus E_2$ . We define the direct sum of  $\tilde{E}_1$  and  $\tilde{E}_2$  to be  $\tilde{E}_1 \oplus \tilde{E}_2 = (E, \mu_1 \oplus \mu_2)$  where  $\mu_1 \oplus \mu_2$  is defined by for all  $x \in E, x = x_1 \oplus x_2, x_i \in E_i, i = 1, 2$

$$(\mu_1 \oplus \mu_2)(x) = (\mu_1 \oplus \mu_2)(x_1 \oplus x_2) = \mu_1(x_1) \wedge \mu_2(x_2).$$

**Theorem 3.9.** Let  $\tilde{E}_1 = (E, \mu_1)$  and  $\tilde{E}_2 = (E, \mu_2)$  be two members of LFVS on  $E$ . We have

- (1)  $\tilde{E}_1 \cap \tilde{E}_2$  is a member of LFVS on  $E$ .
- (2)  $\tilde{E}_1 + \tilde{E}_2$  is a member of LFVS on  $E$ .

The proof of the theorem is trivial and it is omitted.

**Theorem 3.10.** Let  $\tilde{E}_1 = (E, \mu_1)$  and  $\tilde{E}_2 = (E, \mu_2)$  be the members of LFVS. We have

- (1) For all  $a \in L$ ,  $(\tilde{E}_1 \cap \tilde{E}_2)_{[a]} = (\tilde{E}_1)_{[a]} \cap (\tilde{E}_2)_{[a]}$ .
- (2) For all  $a \in L$ ,  $(\tilde{E}_1 \cap \tilde{E}_2)^{[a]} = (\tilde{E}_1)^{[a]} \cap (\tilde{E}_2)^{[a]}$ .
- (3) For any  $a \in P(L)$ ,  $(\tilde{E}_1 \cap \tilde{E}_2)^{(a)} = (\tilde{E}_1)^{(a)} \cap (\tilde{E}_2)^{(a)}$ .
- (4) For any  $a \in P(L)$ ,  $(\tilde{E}_1 + \tilde{E}_2)^{(a)} = (\tilde{E}_1)^{(a)} + (\tilde{E}_2)^{(a)}$ .

*Proof.* The proofs of (1) and (2) are easy by the definition of  $\tilde{E}_1 \cap \tilde{E}_2$  and the properties of  $L$ -fuzzy sets.

- (3) For any  $a \in P(L)$ , we have

$$\begin{aligned} x \in (\tilde{E}_1 \cap \tilde{E}_2)^{(a)} &\Leftrightarrow \mu_1(x) \wedge \mu_2(x) \not\leq a \\ &\Leftrightarrow \mu_1(x) \not\leq a \text{ and } \mu_2(x) \not\leq a \\ &\Leftrightarrow x \in (\tilde{E}_1)^{(a)} \cap (\tilde{E}_2)^{(a)} \end{aligned}$$

- (4) By the definition of the sum of  $L$ -fuzzy vector subspaces, for any  $a \in P(L)$  we have

$$\begin{aligned} x \in (\tilde{E}_1 + \tilde{E}_2)^{(a)} &\Leftrightarrow \bigvee_{x=x_1+x_2} (\mu_1(x_1) \wedge \mu_2(x_2)) \not\leq a \\ &\Leftrightarrow \exists x_1, x_2 \in E \text{ and } x = x_1 + x_2, \text{ such that } \mu_1(x_1) \wedge \mu_2(x_2) \not\leq a. \\ &\Leftrightarrow \exists x_1, x_2 \in E, \mu_1(x_1) \not\leq a \text{ and } \mu_2(x_2) \not\leq a. \\ &\Leftrightarrow x = x_1 + x_2 \in \tilde{E}_1^{(a)} + \tilde{E}_2^{(a)}. \end{aligned}$$

**Theorem 3.11.** Let  $\tilde{E}_1 = (E, \mu_1)$  and  $\tilde{E}_2 = (E, \mu_2)$  be two members of LFVS. Suppose that for any  $a, b \in L$ , we have  $\beta(a \wedge b) = \beta(a) \cap \beta(b)$ . Then

- (1)  $(\tilde{E}_1 \cap \tilde{E}_2)_{(a)} = (\tilde{E}_1)_{(a)} \cap (\tilde{E}_2)_{(a)}$ ,
- (2)  $(\tilde{E}_1 + \tilde{E}_2)_{(a)} = (\tilde{E}_1)_{(a)} + (\tilde{E}_2)_{(a)}$ .

The prove is trivial and omitted.

### 4. Fuzzy Dimension of $L$ -Fuzzy Vector Subspaces

**Definition 4.1.** Let  $\mathbb{N}(L)$  be the family of  $L$ -fuzzy natural number. The map  $\dim : \text{LFVS} \rightarrow \mathbb{N}(L)$  is defined by

$$\dim \tilde{E}(n) = \bigvee_{a \in L} (a \wedge \dim \tilde{E}_{[a]})(n)$$

is called the  $L$ -fuzzy dimensional function of the  $L$ -fuzzy vector subspace  $\tilde{E}$ , and  $\dim \tilde{E}$  is called the  $L$ -fuzzy dimension of  $\tilde{E}$ , it is an  $L$ -fuzzy natural number. We

usually use another form of  $\dim \tilde{E}$  as follows.

$$\dim \tilde{E}(n) = \vee \{a \in L : \dim \tilde{E}_{[a]} \geq n\}.$$

**Theorem 4.2.** For each  $\tilde{E} \in \text{LFVS}$  and  $n \in \mathbb{N}$ , we have

$$\dim \tilde{E}(n) = \vee_{a \in L} (a \wedge \dim \tilde{E}_{(a)})(n) = \vee \{a \in L : \dim \tilde{E}_{(a)} \geq n\}$$

*Proof.* For any  $n \in \mathbb{N}$ , let  $\lambda = \vee_{a \in L} (a \wedge \dim \tilde{E}_{(a)})(n)$ . Obviously  $\lambda \leq \dim \tilde{E}(n)$ . Next we show that  $\lambda \geq \dim \tilde{E}(n)$ . Suppose that  $b \in L$  and  $b \in \beta(\dim \tilde{E}(n))$ , then there exists  $a \in L$  and  $\dim \tilde{E}_{[a]} \geq n$  such that  $b \in \beta(a)$ . In this case,  $n \leq \dim \tilde{E}_{[a]} \leq \dim \tilde{E}_{(b)} \leq \dim \tilde{E}_{[b]}$  which implies  $\lambda = \vee \{a \in L : \dim \tilde{E}_{(a)} \geq n\} \geq b$ . Thus we have

$$\lambda \geq \vee \{b \in \beta(\dim \tilde{E}(n))\} = \dim \tilde{E}(n).$$

This completes the proof.

**Theorem 4.3.** Let the pair  $\tilde{E} = (E, \mu)$  be a member of LFVS. Then for any  $a \in L$ ,

$$(\dim \tilde{E})_{(a)} \leq \dim \tilde{E}_{[a]} \leq (\dim \tilde{E})_{[a]}.$$

If  $\beta(a \wedge b) = \beta(a) \cap \beta(b)$  for all  $a, b \in L$ , then

$$(\dim \tilde{E})_{(a)} \leq \dim \tilde{E}_{(a)} \leq \dim \tilde{E}_{[a]} \leq (\dim \tilde{E})_{[a]}.$$

In particular,  $(\dim \tilde{E})_{[a]} = \dim \tilde{E}_{[a]}$  for any  $a \in J(L)$ .

*Proof.* In order to prove  $(\dim \tilde{E})_{(a)} \leq \dim \tilde{E}_{(a)}$ . Suppose that  $n \leq (\dim \tilde{E})_{(a)}$ , then  $a \in \beta(\dim \tilde{E}(n))$ . Since  $\beta$  is a preserve-union map, there is  $b \in L$  and  $n \leq \dim \tilde{E}_{[b]}$ , such that  $a \in \beta(b)$ . Because  $\tilde{E}_{[b]} \subseteq \tilde{E}_{(a)} \subseteq \tilde{E}_{[a]}$ , thus  $n \leq \dim \tilde{E}_{(a)}$ . Therefore  $(\dim \tilde{E})_{(a)} \leq \dim \tilde{E}_{(a)}$ .

$\dim \tilde{E}_{(a)} \leq \dim \tilde{E}_{[a]}$  is obvious. Moreover, we can obtain that  $\dim \tilde{E}_{[a]} \leq (\dim \tilde{E})_{[a]}$  from the definition of  $\dim(\tilde{E})$ .

In order to prove for any  $a \in J(L)$ ,  $(\dim \tilde{E})_{[a]} = \dim \tilde{E}_{[a]}$ , we only need to show  $(\dim \tilde{E})_{[a]} \subseteq \dim \tilde{E}_{[a]}$ . Since the set  $\mu(E)$  is finite, for any  $a \in J(L)$  we have

$$\begin{aligned} n \leq (\dim \tilde{E})_{[a]} &\Rightarrow \dim \tilde{E}(n) \geq a \\ &\Rightarrow \vee \{b \in L : \dim \tilde{E}_{[b]} \geq n\} \geq a \\ &\Rightarrow \exists a \leq b, \text{ such that } n \leq \dim \tilde{E}_{[b]} \\ &\Rightarrow n \leq \dim \tilde{E}_{[a]} \end{aligned}$$

Therefore  $(\dim \tilde{E})_{[a]} = \dim \tilde{E}_{[a]}$ .

**Theorem 4.4.** Let  $\tilde{E} = (E, \mu)$  be a member of LFVS. Then

$$(\dim \tilde{E})^{(a)} \leq \dim \tilde{E}^{(a)} \leq \dim \tilde{E}^{[a]} \leq (\dim \tilde{E})^{[a]}.$$

In particular,  $(\dim \tilde{E})^{(a)} = \dim \tilde{E}^{(a)}$  for any  $a \in P(L)$ .

*Proof.*  $(\dim \tilde{E})^{(a)} \leq \dim \tilde{E}^{(a)}$  can be proved from the following implications.

$$\begin{aligned} n \leq (\dim \tilde{E})^{(a)} &\Leftrightarrow \dim \tilde{E}(n) \not\leq a \\ &\Leftrightarrow \vee \{b \in L : \dim \tilde{E}_{[b]} \geq n\} \not\leq a \\ &\Leftrightarrow \exists b \not\leq a, \text{ such that } n \leq \dim \tilde{E}_{[b]} \\ &\Rightarrow \dim \tilde{E}^{(a)} = \dim \left( \bigcup_{b \not\leq a} \tilde{E}_{[b]} \right) \geq n. \end{aligned}$$

Let  $a \in P(L)$ . In order to show  $\dim \tilde{E}^{(a)} \leq (\dim \tilde{E})^{(a)}$ , we need to show that

$$\dim \left( \bigcup_{b \not\leq a} \tilde{E}_{[b]} \right) \leq \vee_{b \not\leq a} \dim \tilde{E}_{[b]}.$$

Suppose that  $n \leq \dim \left( \bigcup_{b \not\leq a} \tilde{E}_{[b]} \right)$ . Since the number of  $\tilde{E}_{[a]}$

is finite, then when  $b \not\leq a$ , the number of  $\tilde{E}_{[b]}$  is finite, denotes  $\tilde{E}_{[a_1]}, \tilde{E}_{[a_2]}, \dots, \tilde{E}_{[a_r]}$ ,

where  $a_i \not\leq a$  for any  $i \in \{1, 2, \dots, r\}$ . Thus  $\bigcup_{b \not\leq a} \tilde{E}_{[b]} = \bigcup_{i=1}^r \tilde{E}_{[a_i]}$ . Since  $a \in P(L)$ , then

we have  $c = a_1 \wedge a_2 \wedge \dots \wedge a_r \not\leq a$ . Further we have  $\bigcup_{i=1}^r \tilde{E}_{[a_i]} \subseteq \tilde{E}_{[c]}$ . Thus for any

$$n \leq \dim \left( \bigcup_{b \not\leq a} \tilde{E}_{[b]} \right) = \dim \left( \bigcup_{i=1}^r \tilde{E}_{[a_i]} \right) \leq \dim \tilde{E}_{[c]} \leq \vee_{b \not\leq a} \dim \tilde{E}_{[b]} \leq \vee_{b \not\leq a} (\dim \tilde{E})_{[b]} = (\dim \tilde{E})^{(a)}.$$

Therefore for any  $a \in P(L)$ ,  $(\dim \tilde{E})^{(a)} = \dim \tilde{E}^{(a)}$ .

$\dim \tilde{E}^{(a)} \leq \dim \tilde{E}^{[a]}$  is obvious. We show that  $\dim \tilde{E}^{[a]} \leq (\dim \tilde{E})^{[a]}$  in the following implications.

$$\begin{aligned} \dim \tilde{E}^{[a]} &= \dim \bigcap_{\substack{a \in \alpha(b) \\ b \in P(L)}} \tilde{E}^{(b)} \leq \bigwedge_{\substack{a \in \alpha(b) \\ b \in P(L)}} \dim \tilde{E}^{(b)} \\ &= \bigwedge_{\substack{a \in \alpha(b) \\ b \in P(L)}} (\dim \tilde{E})^{(b)} = (\dim \tilde{E})^{[a]}. \end{aligned}$$

**Theorem 4.5.** Let  $\tilde{E}_1 = (E, \mu_1)$  and  $\tilde{E}_2 = (E, \mu_2)$  be two L-fuzzy vector subspaces. Then the following equality holds

$$\dim(\tilde{E}_1 + \tilde{E}_2) + \dim(\tilde{E}_1 \cap \tilde{E}_2) = \dim \tilde{E}_1 + \dim \tilde{E}_2.$$

*Proof.* We denote the sum of  $\tilde{E}_1$  and  $\tilde{E}_2$  by  $\tilde{E}_1 + \tilde{E}_2 = (E, \mu)$ . From Theorem 11, we know that  $\tilde{E}_1 + \tilde{E}_2$  is a L-fuzzy vector subspace. By the properties of L-fuzzy natural numbers, Theorem 12 and the dimensional formulation of vector spaces, we know for any  $a \in P(L)$ ,



$$\begin{aligned}
 & \left( \dim(\tilde{E}_1 + \tilde{E}_2) + \dim(\tilde{E}_1 \cap \tilde{E}_2) \right)^{(a)} \\
 &= \left( \dim(\tilde{E}_1 + \tilde{E}_2) \right)^{(a)} + \left( \dim(\tilde{E}_1 \cap \tilde{E}_2) \right)^{(a)} \\
 &= \dim(\tilde{E}_1 + \tilde{E}_2)^{(a)} + \dim(\tilde{E}_1 \cap \tilde{E}_2)^{(a)} \\
 &= \dim(\tilde{E}_1^{(a)} + \tilde{E}_2^{(a)}) + \dim(\tilde{E}_1^{(a)} \cap \tilde{E}_2^{(a)}) \\
 &= \dim \tilde{E}_1^{(a)} + \dim \tilde{E}_2^{(a)} - \dim(\tilde{E}_1^{(a)} \cap \tilde{E}_2^{(a)}) + \left( \dim(\tilde{E}_1^{(a)} \cap \tilde{E}_2^{(a)}) \right) \\
 &= \dim \tilde{E}_1^{(a)} + \dim \tilde{E}_2^{(a)}
 \end{aligned}$$

Therefore  $\dim(\tilde{E}_1 + \tilde{E}_2) + \dim(\tilde{E}_1 \cap \tilde{E}_2) = \dim \tilde{E}_1 + \dim \tilde{E}_2$ .

**Definition 4.6.** Suppose that  $\tilde{E} = (E, \mu)$  is an  $L$ -fuzzy vector subspace. A map  $f : E \rightarrow E$  is called an  $L$ -fuzzy linear transformation, if it satisfies the following conditions:

- (1)  $f$  is a linear map on  $E$ .
- (2) For all  $x \in E$ ,  $\mu(f(x)) \geq \mu(x)$ .

**Theorem 4.7.** Suppose that  $\tilde{E} = (E, \mu)$  is an  $L$ -fuzzy vector subspace,  $f$  is an  $L$ -fuzzy linear transformation on  $E$ , then  $\widetilde{\ker f} = (\ker f, \mu|_{\ker f})$  and  $\widetilde{\text{im} f} = (\text{im} f, \mu|_{\text{im} f})$  are  $L$ -fuzzy vector subspaces.

The prove is trivial and omitted.

**Theorem 4.8.** Suppose that  $\tilde{E} = (E, \mu)$  is an  $L$ -fuzzy vector subspace,  $f : E \rightarrow E$  is an  $L$ -fuzzy linear transformation, then

$$\dim(\widetilde{\ker f}) + \dim(\widetilde{\text{im} f}) = \dim \tilde{E}$$

*Proof.* Suppose that  $\varphi$  is a linear transformation on (crisp) vector spaces  $V$ , then the equality  $\dim(\text{im} \varphi) + \dim(\ker \varphi) = \dim V$  holds. Hence, for all  $a \in P(L)$ , we have

$$\begin{aligned}
 \left( \dim(\widetilde{\text{im} f}) + \dim(\widetilde{\ker f}) \right)^{(a)} &= \left( \dim(\widetilde{\text{im} f}) \right)^{(a)} + \left( \dim(\widetilde{\ker f}) \right)^{(a)} \\
 &= \dim(\widetilde{\text{im} f})^{(a)} + \dim(\widetilde{\ker f})^{(a)} \\
 &= \dim(\tilde{E}^{(a)} \cap \text{im} f) + \dim(\tilde{E}^{(a)} \cap \ker f)
 \end{aligned}$$

Since  $f|_{\tilde{E}^{(a)}}$  is a linear transformation on  $\tilde{E}^{(a)}$ , we have

$$\begin{aligned}
 \left( \dim(\widetilde{\text{im} f}) + \dim(\widetilde{\ker f}) \right)^{(a)} &= \dim(\text{im} f|_{\tilde{E}^{(a)}}) + \dim(\ker f|_{\tilde{E}^{(a)}}) \\
 &= \dim \tilde{E}^{(a)} = \left( \dim \tilde{E} \right)^{(a)}.
 \end{aligned}$$

Therefore  $\dim(\widetilde{\ker f}) + \dim(\widetilde{\text{im} f}) = \dim \tilde{E}$ .

### 5. Conclusion

In this paper,  $L$ -fuzzy vector subspace is defined and showed that its dimension is an  $L$ -fuzzy natural number. Based on the definitions, some good properties of crisp vector spaces are hold in a finite-dimensional  $L$ -fuzzy vector subspace. In particular, the equality  $\dim(\tilde{E}_1 + \tilde{E}_2) + \dim(\tilde{E}_1 \cap \tilde{E}_2) = \dim \tilde{E}_1 + \dim \tilde{E}_2$  holds without any restricted

conditions. At the same time,  $\dim(\widetilde{\text{im}} f) + \dim(\widetilde{\text{ker}} f) = \dim \widetilde{E}$  holds.

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## References

- [1] Katsaras, A.K. and Liu, D.B. (1977) Fuzzy Vector Spaces and Fuzzy Topological Vector Spaces. *Journal of Mathematical Analysis and Applications*, **58**, 135-146. [https://doi.org/10.1016/0022-247X\(77\)90233-5](https://doi.org/10.1016/0022-247X(77)90233-5)
- [2] Lubczonok, G. and Murali, V. (2002) On Flags and Fuzzy Subspaces of Vector Spaces. *Fuzzy Sets and Systems*, **125**, 201-207. [https://doi.org/10.1016/S0165-0114\(00\)00129-9](https://doi.org/10.1016/S0165-0114(00)00129-9)
- [3] Abdukhalikov, K.S., Tulenbaev, M.S. and Umirbarov, U.U. (1994) On Fuzzy Bases of Vector Spaces. *Fuzzy Sets and Systems*, **63**, 201-206. [https://doi.org/10.1016/0165-0114\(94\)90350-6](https://doi.org/10.1016/0165-0114(94)90350-6)
- [4] Abdukhalikov, K.S. (1996) The Dual of a Fuzzy Subspace. *Fuzzy Sets and Systems*, **82**, 375-381. [https://doi.org/10.1016/0165-0114\(96\)84977-3](https://doi.org/10.1016/0165-0114(96)84977-3)
- [5] Lubczonok, P. (1990) Fuzzy Vector Spaces. *Fuzzy Sets and Systems*, **38**, 329-343. [https://doi.org/10.1016/0165-0114\(90\)90206-L](https://doi.org/10.1016/0165-0114(90)90206-L)
- [6] Lowen, R. (1980) Convex Fuzzy Sets. *Fuzzy Sets and Systems*, **3**, 291-310. [https://doi.org/10.1016/0165-0114\(80\)90025-1](https://doi.org/10.1016/0165-0114(80)90025-1)
- [7] Shi, F.G. and Huang, C.E. (2010) Fuzzy Bases and the Fuzzy Dimension of Fuzzy Vector Spaces. *Mathematical Communications*, **15**, 303-310.
- [8] Gierz, G., *et al.* (2003) Continuous Lattices and Domains. Encyclopedia of Mathematics and its Applications, Cambridge University Press, Cambridge. <https://doi.org/10.1017/CBO9780511542725>
- [9] Dwinger, P. (1982) Characterizations of the Complete Homomorphic Images of a Completely Distributive Complete Lattice I. *Indagationes Mathematicae (Proceedings)*, **85**, 403-414. [https://doi.org/10.1016/1385-7258\(82\)90034-8](https://doi.org/10.1016/1385-7258(82)90034-8)
- [10] Wang, G.-J. (1992) Theory of Topological Molecular Lattices. *Fuzzy Sets and Systems*, **47**, 351-376. [https://doi.org/10.1016/0165-0114\(92\)90301-J](https://doi.org/10.1016/0165-0114(92)90301-J)
- [11] Huang, H.-L. and Shi, F.-G. (2008)  $L$ -Fuzzy Numbers and Their Properties. *Information Sciences*, **178**, 1141-1151. <https://doi.org/10.1016/j.ins.2007.10.001>
- [12] Shi, F.-G. (2000)  $L$ -Fuzzy Relation and  $L$ -Fuzzy Subgroup. *Journal of Fuzzy Mathematics*, **8**, 491-499.
- [13] Negoita, C.V. and Ralescu, D.A. (1975) Applications of Fuzzy Sets to Systems Analysis, Interdisciplinary Systems Research Series 11, Birkhaeuser, Basel. <https://doi.org/10.1007/978-3-0348-5921-9>
- [14] Shi, F.-G. (1995) Theory of  $L_\beta$ -Nested Sets and  $L_\alpha$ -Nested Sets and Its Applications. *Fuzzy*

*Systems and Mathematics*, **4**, 65-72. (In Chinese)

- [15] Shi, F.-G. (1996) *L-Fuzzy Sets and Prime Element Nested Sets*. *Journal of Mathematical Research and Exposition*, **16**, 398-402. (In Chinese)
- [16] Shi, F.-G. (1996) *Theory of Molecular Nested Sets and Its Applications*. *Journal of Yantai Teachers University*, **1**, 33-36. (In Chinese)
- [17] Shi, F.-G. (2009) *A New Approach to the Fuzzification of Matroids*. *Fuzzy Sets and Systems*, **160**, 696-705. <https://doi.org/10.1016/j.fss.2008.05.007>



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