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On Canard Homoclinic of a Liénard Perturbation System

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Abstract

The classification on the orbits of some Liénard perturbation system with several parameters, which is relation to the example in [1] or [2], is discussed. The conditions for the parameters in order that the system has a unique limit cycle, homoclinic orbits, canards or the unique equilibrium point is globally asymptotic stable are given. The methods in our previous papers are used for the proofs.

Keywords: Liénard System, Canards, Limit Cycles, Homoclinic Orbits, Global Asymptotic Stability

1. Introduction

We shall consider the Liénard perturbation system

$$\begin{cases} \varepsilon \dot{x} = y - \lambda \left(\dfrac{x^3}{3} - \dfrac{x^2}{2} \right) \\ \dot{y} = -k \left(x^3 - a \right), \end{cases} \quad \text{(P1)}$$

where $a \in \mathbb{R}$, ε, λ and k are positive real numbers. System (P1) has a unique equilibrium point at $x = \sqrt[3]{3a}$ and the uniqueness of solutions of initial value problems for the system is guaranteed. In [1], it has been given that the unique equilibrium point of System (P1) for the case $a = 0$ and $\varepsilon = \lambda = 1$ is a global attractor but unstable. In [2], when $\lambda = 1$ and $a = 0$, the result that System (P1) has the special orbit called "a Canard Homoclinic" has been announced by the method of non standard. Our aim is to classify the orbits of System (P1) completely by the values of the parameters. Thus, we improve the results of the papers [1,2].

Our main results are the following

Theorem 1.1. *System* (P1) *has homoclinic orbits locally if and only if* ($a = 0$ *and* $\lambda > 8\varepsilon k$).

Then the system has no limit cycles.

Theorem 1.2. *System* (P1) *has a unique limit cycle if and only if* $0 < a < 1$.

Specially, if ε *and* $|a|$ *are sufficiently small, the orbit is called* "*a Canard Limit Cycle*".

Theorem 1.3. *The unique equilibrium point* (0, 0) *for System* (P1) *is globally asymptotic stable if and only if*

one of the following

$$a < 0 \quad or \quad \left(a = 0 \quad and \quad \lambda \le 8\varepsilon k \right) \quad or \quad a \ge 1$$

is satisfied.

In Section 2, we shall see that System (P1) is transformed to a usual Liénard system (see System (P3)) with the unique equilibrium point at the origin. In Section 3, the existence of the homoclinic orbit of the system will be discussed by using the method in [3]. In virtue of this result, the interesting fact that both the limit cycle and the homoclinic orbit of the system cannot coexist is given. If $a > 0$, $|a|$ and ε are sufficiently small, it has been well-known by E. Benoît ([4]) that System (P3) has the orbit called "a Canard". The orbit changes to the homoclinic orbit for the system as $a = 0$. So the orbit has been called "a Canard Homoclinic" ([2]). In Section 4, the fact that the system has at most one limit cycle will be proved by using the method of [5]. When $0 < a < 1$, $|a|$ and ε are sufficiently small, the orbit 'Canard' spirals to a unique limit cycle of the system. We call the orbit "a Canard Limit Cycle". In Section 5, it shall be seen from the facts of Section 3 and Section 4 that the unique equilibrium point for the system with the parameters in Theorem 1.3 is globally asymptotic stable. Finally, a phase portrait of System (P1) with respect to Theorem 1.1 will be presented in Section 6.

2. Transformation to a Liénard System

By using the transformation $t \to \varepsilon t$, $x \to -x$ and

$y \rightarrow -y$ for System (P1), the system is changed to the following

$$\begin{cases} \dot{x} = y - \lambda \left(\dfrac{x^3}{3} + \dfrac{x^2}{2} \right) \\ \dot{y} = -\varepsilon k \left(x^3 + a \right). \end{cases} \quad \text{(P2)}$$

Moreover, using the transformation $x \rightarrow x + \alpha$ and $y \rightarrow y + (\lambda/2)\alpha^2$ for $\alpha \in \mathbb{R}$ satisfying the equation $\alpha^3 + a = 0$, System (P2) is transformed to the system

$$\begin{cases} \dot{x} = y - \dfrac{\lambda}{6} x \left\{ 2x^2 + 3(2\alpha + 1)x + 6\alpha(\alpha + 1) \right\} \\ \dot{y} = -\varepsilon k x \left(x^2 + 3\alpha x + 3\alpha^2 \right). \end{cases} \quad \text{(P3)}$$

System (P3) has a unique equilibrium point (0, 0) and the uniqueness of solutions of initial value problems is also guaranteed.

We set

$$F(x) = \frac{\lambda}{6} x \left\{ 2x^2 + 3(2\alpha + 1)x + 6\alpha(\alpha + 1) \right\},$$

$$g(x) = \varepsilon k x \left(x^2 + 3\alpha x + 3\alpha^2 \right)$$

and

$$G(x) = \int_0^x g(\xi) \,\mathrm{d}\xi.$$

From the value of α, the graph of the characteristic curve $y = F(x)$ is divided into four cases;

$$\alpha \le -1 \, (i.e. \, a \ge 1), \quad -1 < \alpha < 0 \, (i.e. \, 0 < a < 1),$$

$$\alpha = 0 \, (i.e. \, a = 0) \quad \text{and} \quad \alpha > 0 \, (i.e. \, a < 0).$$

We will note that the situation of the graph is deeply concerned in the qualitative property of the orbits.

3. Proof of Theorem 1.1

In System (P3), a trajectory is said to be a homoclinic orbit if its α- and ω-limit sets are the origin. If System (P3) has a homoclinic orbit, $F(x) > 0$ (or $F(x) < 0$) in the neighborhood of the origin is necessary. Thus we have the assumption $\alpha = 0 \, (i.e. \, a = 0)$ in this section.

Consider a function $\varphi(x)$ with the condition

[C1] $\varphi \in C^1, \varphi(0) = 0$ and $xF(x)\varphi'(x) > 0$ for $x \ne 0$.

The following result has been given in [3].

Lemma 3.1. *System* (P3) *with* $\alpha = 0$ *has homoclinic orbits if and only if there exists a function* $\varphi(x)$ *with* [C1] *such that*

[C2] $F(x) - \varphi(x) > 0$ *and*

$$\varphi'(x) \left[F(x) - \varphi(x) > 0 \right] \ge g(x)$$

for $0 < |x| < \delta$.

As the supplement function in the above lemma, take $\varphi(x) = rx^2$ with $\lambda > 8\varepsilon k$ and $4\lambda r = \lambda \pm \sqrt{\lambda^2 - 8\varepsilon k \lambda}$. Then we have

$$F(x) - \varphi(x) = \frac{\lambda}{6} x^2 \left\{ 2x + 3(1 - 2r) \right\} > 0$$

and

$$\varphi'(x) \left[F(x) - \varphi(x) \right] - g(x)$$
$$= \frac{x^3}{3} \left\{ 2\lambda rx + 3\lambda r (1 - 2r) - 3\varepsilon k \right\} > 0$$

for $0 < |x| < \delta = 3(1 - 2r)/2$.
Thus, we see that the conditions [C1] and [C2] are satisfied. Hence System (P3) has (local) homoclinic orbits.

Moreover, the following is known by the Corollary 3 in [3].

Lemma 3.2. *If the conditions* [C1] *and* [C2] *hold for* $x_1 < x < +\infty$ *and* $x_1 < 0$, *then System* (P3) *with* $\alpha = 0$ *has homoclinic orbits locally, but no limit cycles.*

Taking $x_1 = -\delta$ in the proof of Lemma 3.1, we see that the above lemma holds. Therefore, the proof of Theorem 1.1 is completed now.

When $a > 0$, $|a|$ and ε are sufficiently small, it has been well-known from [4] that System (P3) has the orbit called "a Canard". The orbit "Canard" changes to the mentioned homoclinic orbit above as $a = 0$. So the orbit has been called "a Canard Homoclinic" by [2]. Thus, we see that there exists a canard homoclinic in System (P1).

4. Proof of Theorem 1.2

We shall assume the condition $-1 < \alpha < 0$ (*i.e.* $0 < a < 1$). Then we can easily check that System (P3) has at least one limit cycle. In facts, the unique equilibrium point (0, 0) is a unstable focus by $F'(0) < 0$ and the all orbits are uniformly ultimately bounded (for the details see [6]). Thus, by the well-known Poincaré-Bendixson theorem, the system has a limit cycle (for instance see [7]).

The following is a useful method ([5]) in order to guarantee that a Liénard system has at most one limit cycle.

Lemma 4.1. *If there exists a constant* $m \ge 0$ *such that* $F'(x)G(x) - mF(x)g(x) \ge 0$ *for* $x(\ne 0)$, *System* (P3) *has at most one limit cycle.*

We have

$$F'(x)G(x) - mF(x)g(x) = \frac{\lambda \varepsilon k}{12} x^2 \Phi(x; m, \alpha),$$

where

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$$\Phi(x; m, \alpha) = (3 - 4m)x^4 + 3\{6\alpha + 1 - 2m(4\alpha + 1)\}x^3$$
$$+ 15\alpha\{3\alpha + 1 - 2m(2\alpha + 1)\}x^2$$
$$+ 6\alpha^2\{8\alpha + 5 - 3m(4\alpha + 3)\}x$$
$$+ 18(1 - 2m)\alpha^3(\alpha + 1).$$

Let $m = 1/2$ and $\alpha = -1/4$ in $\Phi(x; m, \alpha)$. Then we have

$$\Phi\left(x; \frac{1}{2}, -\frac{1}{4}\right) = x^2\left\{\left(x - \frac{3}{4}\right)^2 + \frac{3}{8}\right\} > 0 \ (x \neq 0).$$

Thus, we see from Lemma 4.1 that System (P3) has at most one limit cycle. So we conclude that System (P3) has a unique limit cycle if $-1 < \alpha < 0$.

Conversely, suppose that System (P3) has a limit cycle. Then if System (P3) doesn't satisfy the condition $-1 < \alpha < 0$, this contradicts to the existence of the limit cycle by Theorem 1.1 and the proof of Theorem 1.3 (see Section 5).

Therefore, the proof of Theorem 1.2 is completed now.

In virtue of E. Benoît ([4]), if $a > 0$, $|a|$ and ε are sufficiently small, then System (P3) has the orbit called "a Canard". Then the orbit "Canard" spirals to a unique limit cycle of System (P3). So we call the special limit cycle "a Canard Limit Cycle".

5. Proof of Theorem 1.3

In this section, we shall assume the condition $\alpha \leq -1$ or $\alpha \geq 0$ (i.e. $a \leq 0$ or $a \geq 1$). The following is a powerful method (see [8]) to prove the non-existence of non-trivial closed orbits of a Liénard system.

Lemma 5.1. *If the curve $(F(x), G(x))$ has no intersecting points with itself, then System (P3) has no non-trivial closed orbits.*

From the situation of the graph $y = F(x)$, we shall prove the theorem by dividing into four cases;

i) $0 \leq \alpha < 1/2$,
ii) $1/2 \leq \alpha$,
iii) $-3/2 < \alpha \leq -1$,
iv) $\alpha \leq -3/2$.

First, we shall discuss the case i). The discussion is similar to the method shown in [9].

Let $p_i(\alpha)(i = 1, 2)$ be the solutions of the equation $F(x) = 0$ and $p_1(\alpha) < p_2(\alpha) < 0$. Now we consider the equation

$$F(x) = F(\xi) \ \text{for} \ p_1(\alpha) < \xi \leq -\alpha - 1.$$

This equation has two roots other than $x = \xi$. Let $u_1 = u_1(\xi)$ and $u_2 = u_2(\xi)$ denote these roots. Then we have $-\alpha - 1 < u_1 < p_2(\alpha)$ and $0 < u_2 < 1/2 - \alpha$.

From a property of the curve $(F(x), G(x))$, we shall

show that, if $F(u_1) = F(u_2)$ for $p_1(\alpha) < \xi \leq -\alpha - 1$, then $G(u_2) - G(u_1) < 0$.

From $F(u_1) = F(u_2)$ we have

$$2(u_1^2 + u_1 u_2 + u_2^2) + 3(2\alpha + 1)(u_1 + u_2) + 6\alpha(\alpha + 1) = 0.$$

Thus we get

$$G(u_2) - G(u_1) = \frac{\varepsilon k}{4}(u_2 - u_1)\{(u_1 + u_2)(u_1^2 + u_2^2)$$
$$+ 4\alpha(u_1^2 + u_1 u_2 + u_2^2) + 6\alpha^2(u_1 + u_2)\}.$$

Since u_1 and u_2 are solutions of the equation $F(x) = F(\xi)$, we have

$$u_1 + u_2 = -\frac{3(2\alpha + 1)}{2} \ \text{and} \ u_1 u_2 = 3\alpha(\alpha + 1).$$

Thus we get

$$u_1^2 + u_2^2 = 3\alpha^2 + 3\alpha + \frac{9}{4}.$$

By substituting $u_1 + u_2$ and $u_1^2 + u_2^2$ to $G(u_2) - G(u_1)$, we have

$$G(u_2) - G(u_1) = \frac{\varepsilon k}{4}(u_2 - u_1)L(\alpha),$$

where $L(\alpha) = -3\alpha^3 + (3/2)\alpha^2 - (9/4)\alpha - 27/8$. Then we have $L'(\alpha) < 0$ for all α and $L(0) = -27/8 < 0$. Thus we get $L(\alpha) < 0$ for $\alpha \geq 0$.

From these facts and $u_2 - u_1 > 0$, if $\alpha \geq 0$, we conclude that $G(u_2) < G(u_1) < G(\xi)$ for $p_1(\alpha) < \xi \leq -\alpha - 1$. Namely, the curve $(F(x), G(x))$ has no intersecting points with itself. This means that System (P3) with $0 \leq \alpha < 1/2$ has no non-trivial closed orbits.

Similarly we can check that $L(\alpha) < 0$ for $\alpha \geq 1/2$ and $L(\alpha) > 0$ for $\alpha \leq -1$. Thus we have $G(u_2) - G(u_1) < 0$ for $\alpha \geq 1/2$ or $\alpha \leq -1$. Therefore, we conclude that System (P3) also has no non-trivial closed orbits for the another cases ii), iii) and iv).

We say that the equilibrium point $E(0, 0)$ is globally asymptotically stable if E is stable and every orbit of System (P3) tends to E. We will see the global asymptotic stability of E from the following conditions:

[i] all orbits of System (P3) are bounded in the future,
[ii] System (P3) has no non-trivial closed orbits,
[iii] System (P3) has no homoclinic orbits,
[iv] E is asymptotic stable.

The condition [i], [ii] or [iii] has been checked in Section 3, Section 4 or the mentioned fact above. So we shall check the condition [iv].

Suppose that E is not stable. Then, by checking the direction of the vector $(y - F(x), -g(x))$ for System (P3), we have that every positive semi-trajectories of System (P3) starting in the neighborhood of E keep on rotating around E and go away from E. Hence, by the
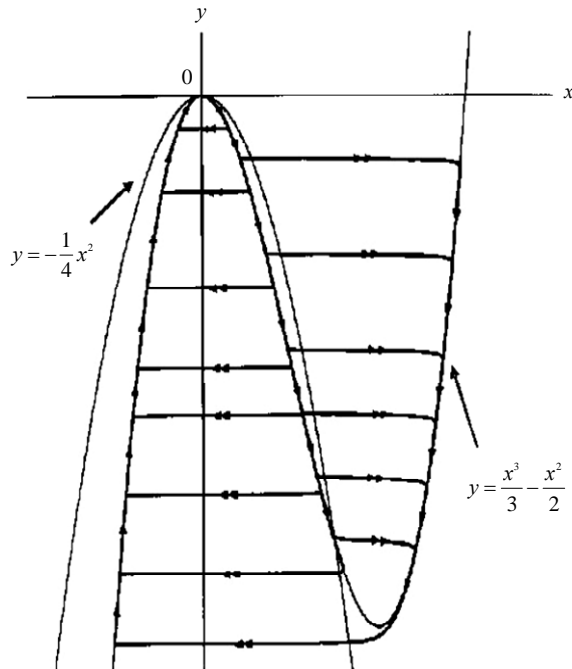
$y = -\dfrac{1}{4}x^2$

$y = \dfrac{x^3}{3} - \dfrac{x^2}{2}$

Figure 1. $a = 0$, $\lambda = 1$ and $\varepsilon = k = 1/100$.

fact [i] and the Poincaré-Bendixson theorem, the system has a closed orbit. This contradicts to the fact [ii]. It follows from the direction of the vector field that E is asymptotically stable.

Conversely, suppose that E is globally asymptotic stable. Then we see from Theorem 1.1 and 1.2 that System (P3) must satisfy the condition in Theorem 1.3.

Therefore, the proof of Theorem 1.3 is completed now.

Remark. In the case of $(\alpha = 0$ and $\lambda - 8\varepsilon k \le 0)$ or $\alpha = -1$, E is a non-hyperbolic equilibrium point. We see from the fact $G(u_2) - G(u_1) < 0$ and [10] that E is a stable focus. Thus, a unique equilibrium point of System (P1) cannot be "Center".

6. A Numerical Example

We shall present a phase portrait of System (P1). We

consider the example of the case $a = 0$, $\lambda = 1$ and $\varepsilon = k = 1/100$. Then we have $\lambda > 8\varepsilon k$. Thus we shall see that the system has a homoclinic orbit, but no limit cycles as is shown in the **Figure 1**.

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