Applied Mathematics, 2011, 2, 1393-1396
doi:10.4236/am.2011.211197 Published Online November 2011 (http://www.SciRP.org/journal/am)

New Constructions of Edge Bimagic Graphs from Magic Graphs

Jayapal Baskar Babujee, Babitha Suresh
Department of Mathematics, Anna University, Chennai, India
E-mail: *baskarbabujee@yahoo.com*, *babi_mit@yahoo.co.in*
Received September 5, 2011; *revised October* 14, 2011; *accepted October* 21, 2011

Abstract

An edge magic total labeling of a graph $G(V,E)$ with p vertices and q edges is a bijection f from the set of vertices and edges to $1, 2, \cdots, p+q$ such that for every edge uv in E, $f(u) + f(uv) + f(v)$ is a constant k. If there exist two constants k_1 and k_2 such that the above sum is either k_1 or k_2, it is said to be an edge bimagic total labeling. A total edge magic (edge bimagic) graph is called a super edge magic (super edge bimagic) if $f(V(G)) = \{1, 2, \cdots, p\}$. In this paper we define super edge edge-magic labeling and exhibit some interesting constructions related to Edge bimagic total labeling.

Keywords: Graph, Labeling, Magic Labeling, Bimagic Labeling, Function

1. Introduction

A labeling of a graph G is an assignment f of labels to either the vertices or the edges or both subject to certain conditions. Labeled graphs are becoming an increasingly useful family of Mathematical Models from a broad range of applications. Graph labeling was first introduced in the late 1960's. A useful survey on graph labeling by J. A. Gallian (2010) can be found in [1]. All graphs considered here are finite, simple and undirected. We follow the notation and terminology of [2]. In most applications labels are positive (or nonnegative) integers, though in general real numbers could be used. A (p, q)-graph $G = (V, E)$ with p vertices and q edges is called total edge magic if there is a bijection $f : V \cup E \to \{1, 2, \cdots, p+q\}$ such that there exists a constant k for any edge uv in E, $f(u) + f(uv) + f(v) = k$. The original concept of total edge-magic graph is due to Kotzig and Rosa [3]. They called it magic graph. A total edge-magic graph is called a super edge-magic if $f(V(G))=\{1, 2, \cdots, p\}$. Wallis [4] called super edge-magic as strongly edge-magic. An Edge antimagic total labeling of a graph with p vertices and q edges is a bijection from the set of edges to $1, 2, \cdots, p+q$ such that the sums of the label of the edge and incident vertices are pairwise distinct.

It becomes interesting when we arrive with magic type labeling summing to exactly two distinct constants say k_1 or k_2. Edge bimagic totally labeling was introduced by J. Baskar Babujee [5] and studied in [6] as (1,1) edge bimagic labeling. A graph $G(p, q)$ with p vertices and q edges is called total edge bimagic if there exists a bijection $f : V \cup E \to \{1, 2, \cdots, p+q\}$. such that for any edge $uv \in E$, we have two constants k_1 and k_2 with $f(u) + f(v) + f(uv) = k_1$ or k_2. A total edge-bimagic graph is called *super edge-bimagic* if $f(V(G))=\{1, 2, \cdots, p\}$. Super edge-bimagic labeling for path, star- $K_{1,n}, K_{1,n,n}$ are proved in [7]. Super edge-bimagic labeling for cycles, Wheel graph, Fan graph, Gear graph, Maximal Planar class-Pl_n: $n \geq 5$,

$K_{1,m} \cup K_{1,n} (m, n \geq 1), P_n \cup P_{n+1} (n \geq 2), P_m \odot K_{1,n}$,

$C_3 \cup K_{1,n} (n \geq 1), P_n + N_2 (n \geq 3), P_2 \cup mK_1 + N_2)(m \geq 1)$,

(3, n)-kite graph $(n \geq 2)$, are proved in [8-10]. In this paper we define super edge edge-magic and exhibit some interesting constructions related to Edge bimagic total labeling. For our convenience, we state total edge-magic as edge-magic total labeling throughout the paper.

2. Main Results

On renaming Super edge-magic as Super vertex edge-magic it motivates us to define *super edge edge-magic* labeling in graphs.

Definition 2.1 A graph $G = (V, E)$ with p vertices and q edges is called total edge magic if there is a bijection function $f : V \cup E \to \{1, 2, \cdots, p+q\}$ such that for any edge uv in E we have a constant k with $f(u)+f(uv)+f(v)=k$. A total edge-magic graph is called *super edge edge-magic* if $f(E(G))=\{1,2,\cdots,q\}$.

Next we introduce a definition for vertex superimposing between two graphs.

Definition 2.2 If $G_1(p_1,q_1)$ and $G_2(p_2,q_2)$ are two connected graphs, $G_1\hat{o}G_2$ is obtained by superimposing any selected vertex of G_2 on any selected vertex of G_1. The resultant graph $G=G_1\hat{o}G_2$ consists of p_1+p_2-1 vertices and q_1+q_2 edges.

Theorems 2.3 *If G has super edge edge-magic total labeling then,* $G\hat{o}P_n$ *admits edge bimagic total labeling.*

***Proof*:** Let $G(p, q)$ be super edge edge-magic graph with the bijective function $f:V\cup E\to\{1,2,\cdots,p+q\}$. such that $f(u)+f(uv)+f(v)=k_1$. Let $w\in V$ be the vertex whose label $f(w)=p+q$ is the maximum value. Consider the path P_n with vertex set $\{x_i:1\le i\le n\}$ and edge set $\{x_ix_{i+1}:1\le i\le n-1\}$. We superimpose one of the pendent vertex of the path P_n say x_1 on the vertex $w\in V$ of G. Now we define the new graph called $G'=G\hat{o}P_n$ with vertex set $V'=V\cup\{x_i:2\le i\le n\}$ and $E'=E\cup\{wx_2,x_ix_{i+1}:2\le i\le n-1\}$. Consider the bijective function $g:V'\cup E'\to\{1,2,\cdots,p+q,\ p+q+1,\cdots,p+q+n,\cdots,\ p+q+2n-2\}$ defined by $g(v)=f(v)$ for all $v\in V$ and $g(uv)=f(uv)$ for all $uv\in E$.

From our construction of new graph G',

$$f(w)=g(x_1)=g(w)=p+q.$$

$$\text{For } 2\le i\le n,\ g(x_i)=\begin{cases} p+q+\left\lceil \dfrac{n}{2}\right\rceil+\dfrac{i-2}{2}, & i \text{ even} \\ p+q+\dfrac{i-1}{2}, & i \text{ odd} \end{cases}$$

Now

$$g(wx_2)=p+q+2(n-1);$$

$$g(x_ix_{i+1})=p+q+2n-i-1 \text{ for } 2\le i\le n-1.$$

Since the graph G is super edge edge-magic with common count k_1, implies that $g(u)+g(uv)+g(v)=k_1$ for all $uv\in E$. Now we have to prove that the remaining edges in the set $\{wx_2,x_ix_{i+1}:2\le i\le n-1\}$ have the common count k_2.

For the edge wx_2,

$$\begin{aligned} &g(w)+g(wx_2)+g(x_2) \\ &= p+q+p+q+2n-2+p+q+\left\lceil \frac{n}{2}\right\rceil \\ &=3p+3q+2n+\left\lceil \frac{n}{2}\right\rceil-2=k_2 \end{aligned}$$

For any edge x_ix_{i+1}, if i is even,

$$\begin{aligned} &g(x_i)+g(x_ix_{i+1})+g(x_{i+1}) \\ &=p+q+\left\lceil \frac{n}{2}\right\rceil+\frac{i-2}{2}+p+q+2n-i-1+p+q+\frac{i}{2} \\ &=3p+3q+2n+\left\lceil \frac{n}{2}\right\rceil-2=k_2 \end{aligned}$$

If i is odd,

$$\begin{aligned} &g(x_i)+g(x_ix_{i+1})+g(x_{i+1}) \\ &=p+q+\frac{i-1}{2}+p+q+2n-i-1+p+q+\left\lceil \frac{n}{2}\right\rceil+\frac{i-1}{2} \\ &=3p+3q+2n+\left\lceil \frac{n}{2}\right\rceil-2=k_2\cdots \end{aligned}$$

Thus $G'=G\hat{o}P_n$ has two common count k_1 and k_2. Hence $G\hat{o}P_n$ has edge bimagic total labeling.

Example 2.4 Taking $G=K_{1,6}$ which is super edge edge magic, by using the theorem 2.3, $G\hat{o}P_5=K_{1,6}\hat{o}P_5$ admits edge bimagic total labeling with two common count $k_1=26$ and $k_2=50$ is given in **Figure 1**.

Theorem 2.5 $G\hat{o}K_{1,n}$ *is total edge bimagic for any arbitrary super edge edge-magic Graph G.*

***Proof*:** Let $G(p,q)$ be super edge edge-magic graph with the bijective function $f:V\cup E\to\{1,2,\cdots,p+q\}$ such that $f(u)+f(uv)+f(v)=k_1$. Let $w\in V$ be the vertex whose label $f(w)=p+q$ is the maximum value. Consider the star $K_{1,n}$ with vertex set $\{x_0,x_i:1\le i\le n\}$ and edge set $\{x_0x_i:1\le i\le n\}$. We superimpose the vertex x_0 of the star $K_{1,n}$ graph on the vertex $w\in V$ of G. Now we define the new graph called $G'=G\hat{o}K_{1,n}$ with vertex set $V'=V\cup\{x_i:1\le i\le n\}$ and $E'=E\cup\{wx_i:1\le i\le n\}$. Consider the bijective function $g:V'\cup E'\to\{1,2,\cdots,p+q,\ p+q+1,\cdots,p+q+n,\cdots,p+q+2n\}$ defined by $g(v)=f(v)$ for all $v\in V$ and $g(uv)=f(uv)$ for all $uv\in E$.

From our construction of new graph G',

$$f(w)=g(x_0)=g(w)=p+q.$$

$$g(x_i)=p+q+2n-(i-1),\ \text{for } 1\le i\le n.$$

and $$g(wx_i)=p+q+i;\ \text{ for } 1\le i\le n.$$

Since the graph G is super edge edge-magic with common count k_1, implies that $g(u)+g(uv)+g(v)=k_1$ for

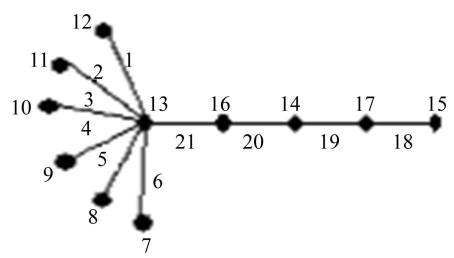

Figure1. Edge bimagic total labeling of $K_{1,6}\hat{o}P_5$.

all $uv \in E$. Now we have to prove that the remaining edges joining w and $x_i : 1 \le i \le n$ have the common count k_2.

For any edge wx_i,

$$g(w) + g(wx_i) + g(x_i)$$
$$= p + q + p + q + i + p + q + 2n - (i-1)$$
$$= 3p + 3q + 2n + 1 = k_2.$$

Thus we have $G' = G\hat{o}K_{1,n}$ has two common count k_1 and k_2. Hence $G\hat{o}K_{1,n}$ has edge bimagic total labeling.

Theorem 2.6 *If G has super edge edge-magic total labeling then,* $G\hat{o}F_{1,n}$ *admits edge bimagic total labeling.*

***Proof*:** Let $G(p,q)$ be super edge edge-magic graph with the bijective function $f : V \cup E \to \{1, 2, \cdots, p+q\}$ such that $f(u) + f(uv) + f(v) = k_1$. Let $w \in V$ be the vertex whose label $f(w) = p + q$ is the maximum value. Consider the Fan $F_{1,n}$ with vertex set $\{x_0, x_i : 1 \le i \le n\}$ and edge set $\{x_0 x_i : 1 \le i \le n\} \cup \{x_i x_{i+1} : 1 \le i \le n-1\}$. We superimpose the vertex x_0 of the Fan $F_{1,n}$ graph on the vertex $w \in V$ of G. Now we define the new graph called $G' = G\hat{o}F_{1,n}$ with vertex set $V' = V \cup \{x_i : 1 \le i \le n\}$ and $E' = E \cup \{wx_i : 1 \le i \le n\} \cup \{x_i x_{i+1} : 1 \le i \le n-1\}$. Consider the bijective function $g : V' \cup E' \to \{1, 2, \cdots, p+q, p+q+1, \cdots, p+q+n, \cdots, p+q+2n-1, \cdots, p+q+3n-1\}$ defined by $g(v) = f(v)$ for all $v \in V$ and $g(uv) = f(uv)$ for all $uv \in E$.

From our construction of new graph G',

$$f(w) = g(x_0) = g(w) = p + q.$$

Now

$$g(x_i) = p + q - 1 + \frac{1 - 5(-1)^i + 6i}{4}, \text{for } 1 \le i \le n.$$

and $g(x_i x_{i+1}) = p + q + 3n - 3i$, for $1 \le i \le n-1$;

$$g(wx_i) = p + q - 1 + \frac{12n + 7 + 5(-1)^i - 6i}{4}, \text{for } 1 \le i \le n.$$

Since the graph G is super edge edge-magic with common count k_1, implies that $g(u) + g(uv) + g(v) = k_1$ for all $uv \in E$. Now we have to prove that the remaining edges in the set $\{wx_i : 1 \le i \le n\} \cup \{x_i x_{i+1} : 1 \le i \le n-1\}$ have the common count k_2.

For any edge wx_i,

$$g(w) + g(wx_i) + g(x_i) = p + q + p + q - 1$$
$$+ \frac{12n + 7 + 5(-1)^i - 6i}{4} + p + q - 1 + \frac{1 - 5(-1)^i + 6i}{4}$$
$$= 3(p + q + n) = k_2.$$

And for the edge $x_i x_{i+1}$,

$$g(x_i) + g(x_i x_{i+1}) + g(x_{i+1}) = p + q - 1 + \frac{1 - 5(-1)^i + 6i}{4}$$
$$+ p + q + 3n - 3i + p + q - 1 + \frac{1 - 5(-1)^{i+1} + 6(i+1)}{4}$$
$$= 3(p + q + n) = k_2.$$

Thus we have $G' = G\hat{o}F_{1,n}$ has two common count k_1 and k_2. Hence $G\hat{o}F_{1,n}$ has edge bimagic total labeling.

Example 2.7 illustrates the labeling technique used in the above theorem 2.6.

Example 2.7 Let k_1 be the constant edge count of an arbitrary graph $G(p, q)$ which is super edge edge-magic with maximum label $p + q = 15$ for one of its vertex. The bimagic labeling for $G\hat{o}F_{1,5}$ with $k_2 = 3(p + q + n) = 60$ can be verified from the given **Figure 2**.

Theorem 2.8 *If* G_1 *has super edge edge-magic labeling and* G_2 *has super vertex edge-magic labeling then,* $G_1\hat{o}G_2$ *admits edge bimagic total labeling.*

***Proof*:** Let G_1 be the super edge edge-magic then there exist the bijective function $f : V_1 \cup E_1 \to \{1, 2, \cdots, p_1 + q_1\}$ such that $f(u) + f(uv) + f(v) = k_1$ for all $u, v \in V_1$, and Let G_2 be the super vertex edge-magic then there exist the bijective function $g : V_2 \cup E_2 \to \{1, 2, \cdots, p_2 + q_2\}$ such that $g(u) + g(uv) + g(v) = k_2$, for all $u, v \in V_2$. Let $w \in V_1$ be the vertex whose label is the maximum value $p_1 + q_1$ and $x_1 \in V_2$ be the vertex with label 1. We superimpose the vertex x_1 of G_2 graph on the vertex $w \in V_1$. Now we define the new graph called $G' = G_1\hat{o}G_2$ with vertex set $V' = V_1 \cup \{V_2 - x_1\}$ and $E' = E_1 \cup E_2$. Consider the bijective function $h : V' \cup E' \to \{1, 2, \cdots, p_1 + q_1 - 1, p_1 + q_1, p_1 + q_1 + 1, \cdots, p_1 + q_1 + p_2 + q_2\}$ defined by

$$h(u) = f(u) \text{ for all } u \in V'(G_1) - w;$$

$$h(uv) = f(uv) \text{ for all } uv \in E'(G_1);$$

$$h(w) = h(x_1) = p_1 + q_1;$$

$$h(v) = p_1 + q_1 - 1 + g(v) \text{ for all } v \in V'(G_2) - x_1;$$

$$h(uv) = p_1 + q_1 - 1 + g(uv) \text{ for all } uv \in E'(G_2).$$

For the edges in G_1, we have

$$h(u) + h(uv) + h(v) = f(u) + f(uv) + f(v) = k_1.$$

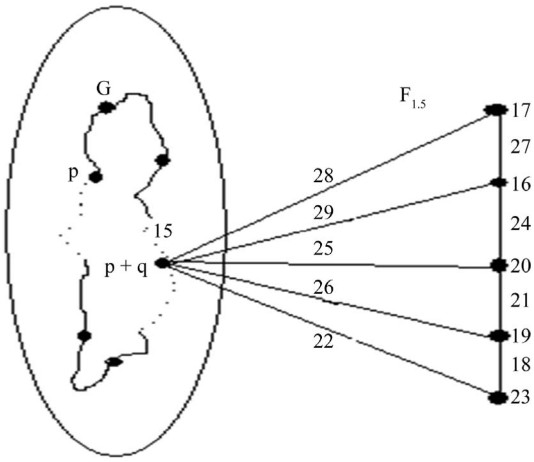

Figure 2. Construction of bimagic for $G\hat{o}F_{1,5}$.

Since magic labeling is preserved in a graph if all the vertices and edges are increased by any constants, for the edges in G_2, we have

$$h(u)+h(uv)+h(v)$$
$$= p_1+q_1-1+g(u)+p_1+q_1-1$$
$$+g(uv)+p_1+q_1-1+g(v)$$
$$=3(p_1+q_1-1)+k_2=k_3.$$

So $G'=G_1\hat{o}G_2$ has two common count k_1 and k_3. Hence $G_1\hat{o}G_2$ admits edge bimagic total labeling.

Theorem 2.9 *If G has super edge edge-magic total labeling then, $G+K_1$ admits edge bimagic total labeling.*

Proof: Let $G(p,q)$ be super edge edge-magic. Then there exist a bijective function function $f:V\cup E\to\{1,2,\cdots,p+q\}$ such that $f(u)+f(uv)+f(v)=k_1$. Now we define the new graph called $G'=G+K_1$ with vertex set $V'=V\cup\{x\}$ and $E'=E\cup\{xv_i:1\le i\le p\}$. Consider the bijective function

$g:V'\cup E'\to\{1,2,\cdots,p+q,p+q+1,\cdots,2p+q+1\}$

defined as follows,

Since there are p vertices in the graph G,

$$g(v_i)=p+q-i+1 \text{ for } 1\le i\le p \text{ and}$$

$$g(uv)=f(uv) \text{ for all } uv\in E.$$

$$g(x)=2p+q+1 \text{ and } g(xv_i)=p+q+i; \text{ for } 1\le i\le p.$$

Since the graph G is super edge edge-magic with common count k_1, implies that $g(u)+g(uv)+g(v)=k_1$. Now we have to prove that the remaining p edges joining V and x have the common count k_2.

For any edge xv_i,

$$g(x)+g(xv_i)+g(v_i)$$
$$=2p+q+1+p+q+i+p+q-i+1$$
$$=4p+3q+2 = k_2.$$

Thus we have $G'=G+K_1$ has two common count k_1 and k_2. Hence $G+K_1$ has edge bimagic total labeling.

3. Concluding Remarks

Theorem 2.8 shows that $G_1\hat{o}G_2$ admit edge bimagic total labeling if G_1 has super edge edge-magic labeling and G_2 has super vertex edge-magic labeling. Further investigation can be done to obtain the conditions at which $G_1\hat{o}G_2$ admits edge bimagic total labeling for any two arbitrary total magic graphs.

4. Acknowledgements

The referee is gratefully acknowledged for their suggestions that improved the manuscript.

5. References

[1] J. A. Gallian, "A Dynamic Survey of Graph Labeling," *Electronic Journal of Combinatorics*, Vol. 17, No. 1, 2010, pp. 1-246.

[2] N. Hartsfield and G. Ringel, "Pearls in Graph Theory," Academic Press, Cambridge, 1990.

[3] A. Kotzig and A. Rosa, "Magic Valuations of Finite Graphs," *Canadian Mathematical Bulletin*, Vol. 13, 1970, pp. 451-461. doi:10.4153/CMB-1970-084-1

[4] W. D. Wallis, "Magic Graphs," Birkhauser, Basel, 2001. doi:10.1007/978-1-4612-0123-6

[5] J. B. Babujee, "Bimagic Labeling in Path Graphs," *The Mathematics Education*, Vol. 38, No. 1, 2004, pp. 12-16.

[6] J. B. Babujee, "On Edge Bimagic Labeling," *Journal of Combinatorics Information & System Sciences*, Vol. 28, No. 1-4, 2004, pp. 239-244.

[7] J. B. Babujee and R. Jagadesh, "Super Edge Bimagic Labeling for Trees" *International Journal of Analyzing methods of Components and Combinatorial Biology in Mathematics*, Vol. 1 No. 2, 2008, pp. 107-116.

[8] J. B. Babujee and R. Jagadesh, "Super Edge Bimagiclabeling for Graph with Cycles," *Pacific-Asian Journal of Mathematics*, Vol. 2, No. 1-2, 2008, pp. 113-122.

[9] J. B. Babujee and R. Jagadesh, "Super Edge Bimagic Labeling for Disconnected Graphs," *International Journal of Applied Mathematics & Engineering Sciences*, Vol. 2, No. 2, 2008, pp. 171-175.

[10] J. B. Babujee and R. Jagadesh, "Super Edge Bimagic Labeling for Some Class of Connected Graphs Derived from Fundamental Graphs," *International Journal of Combinatorial Graph Theory and Applications*, Vol. 1, No. 2, 2008, pp. 85-92.

