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Study of Stability and Vibration Reduction in Multi-Tool Ultrasonic Machining under Simultaneous Primary and Internal Resonance

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ABSTRACT

The main object of this paper is the mathematical study of the vibration behavior in ultrasonic machining (USM) described by non-linear differential equations. The ultrasonic machining (USM) consists of the tool holder and the absorbers representing the tools. This leads to four-degree-of-freedom system subject to multi-external excitation forces. The aim of this project is the reduction of the vibrations in the tool holder and have reasonable amplitudes for the tools represented by the multi-absorbers. Multiple scale perturbation method is applied to obtain the solution up to the second order approximation and to study the stability of the steady state solution near different simultaneous resonance cases. The resulting different resonance cases are reported and studied numerically. The stability of the steady state solution near the selected resonance cases is studied applying both frequency response equations and phase-plane technique. The effects of the different parameters of the system and the absorbers on the system behavior are studied numerically. Optimum working conditions for the tools were obtained. Comparison with the available published work is reported.

Keywords: Passive Vibration Control; Stability; Resonance; Ultrasonic Machining (USM)

1. Introduction

Ultrasonic machining (USM) is of particular interest for the machining of non-conductive, brittle materials such as engineering ceramics. Rupinder and Aspinwall [1,2] introduced a review for the fundamental principles of stationary ultrasonic machining, the material removal mechanisms involved and the effect of operating parameters on material removal rate, tool wear rate, and work piece surface finish of titanium and its alloys for application in manufacturing industry. The USM mechanism is dependent on vibration control of the machine head at resonance, while the tool represented by a dynamic absorber is doing the machining. Lim *et al.* [3] studied the behavior of the (USM) hypothesized theoretical model. The theoretical results showed that controlled variations in the softening stiffness can have a significant effect on the overall non-linear response of the system, by making the overall effect hardening, softening, or approximately linear. Experimentally, it has also been demonstrated that coupling of ultrasonic components with different non-linear characteristics can strongly influence the performance of the system. Amer [4] investigated the coupling of two non-linear oscillators

of the main system and absorber representing ultrasonic cutting process subjected to parametric excitation forces. A threshold value of main system linear damping has been obtained, where vibration can be reduced dramatically. This threshold value can be used effectively for passive vibration control, if it is economical. This will be more useful than usual passive control, and active control. It can be applicable for all excitation frequencies. Asfar, Eissa, El-Bassiouny and Shitikova [5-16] showed how effective is the passive vibration control reduction at resonance. Eissa, El-Bassiouny and Jaensch [17-23] showed how effective is the active control in vibration reduction at resonance at different modes of vibration. They demonstrated the advantages of active control over the passive one. Eissa *et al.* [24-26] investigated saturation phenomena in non-linear oscillating systems subject to multi-parametric and/or external excitations. The system represents the vibration of a single-degree-of-freedom cantilever or the wing of an aircraft. They reported the occurrence of saturation phenomena at different parameters values. They applied saturation values of different parameters as optimum working conditions for vibration suppression of the cantilever. El Ganaini *et al.* [27-29] studied USM model subject to multi-external or

both multi-external and multi-parametric and both multi-external and tuned excitation forces. The model consists of multi-degree-of-freedom system consisting of the tool holder and absorbers (tools) simulating ultrasonic machining process. The advantages of using multi-tools are to machine different materials and different shapes at the same time. This leads to time saving and higher machining efficiency. Besides, devoting all the available energy in the cutting process. The multiple time scale perturbation technique is applied throughout to get an approximate solution up to the second order approximation. The stability of the system is investigated applying both phase-plane and frequency response function methods. The effects of the different parameters of the absorbers on system behavior are studied numerically. The objective of this work is to study the model subject to multi-external excitation forces. The model is represented by a four-degree-of-freedom system consisting of the main system (machine head) and three absorbers (tools) simulateing ultrasonic machining process. The multiple time scale perturbation technique is applied throughout to get an approximate solution up to the second order approximation. The stability of the system is investigated applying both phase-plane and frequency response functions. The effects of the different parameters of the absorber on system behavior are studied numerically. Comparison with the available published work is reported.

2. Mathematical Modeling

The considered model is shown schematically in **Figure 1**, while **Figure 2** illustrates the principles of USM. It consists of the tool holder and absorbers (tools) simulating multi-tool ultrasonic machining process represented by a multi-degree-of-freedom non-linear system. The main system is exited by multi-external forces as shown in the following equations:

$$\ddot{X}_1 + 2\varepsilon\zeta_1\dot{X}_1 + 2\varepsilon\zeta_2\left(\dot{X}_1 - \dot{X}_2\right) + 2\varepsilon\zeta_3\left(\dot{X}_1 - \dot{X}_3\right)$$
$$+ 2\varepsilon\zeta_4\left(\dot{X}_1 - \dot{X}_4\right) + \varepsilon^2\zeta_5\dot{X}_1^2 + \omega_1^2 X_1 + \varepsilon\gamma_1\left(X_1 - X_2\right)$$
$$+ \varepsilon\gamma_2\left(X_1 - X_3\right) + \varepsilon\gamma_3\left(X_1 - X_4\right) + \varepsilon\eta_1 X_1^3$$
$$+ \varepsilon\eta_2\left(X_1 - X_2\right)^3 + \varepsilon\eta_3\left(X_1 - X_3\right)^3$$

$$\tag{1}$$

$$\ddot{X}_2 + 2\varepsilon\zeta_6\left(\dot{X}_2 - \dot{X}_1\right) + \omega_2^2\left(X_2 - X_1\right) + \varepsilon\eta_5\left(X_2 - X_1\right)^3 = 0$$

$$\tag{2}$$

$$\ddot{X}_3 + 2\varepsilon\zeta_7\left(\dot{X}_3 - \dot{X}_1\right) + \omega_3^2\left(X_3 - X_1\right) + \varepsilon\eta_6\left(X_3 - X_1\right)^3 = 0$$

$$\tag{3}$$

$$\ddot{X}_4 + 2\varepsilon\zeta_8\left(\dot{X}_4 - \dot{X}_1\right) + \omega_4^2\left(X_4 - X_1\right) + \varepsilon\eta_7\left(X_4 - X_1\right)^3 = 0$$

$$\tag{4}$$
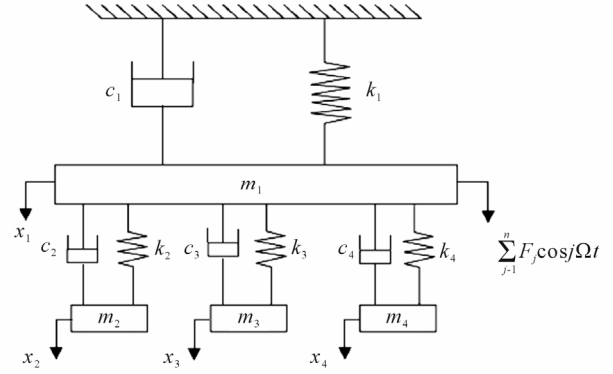
Figure 1. Schematic diagram of USM.
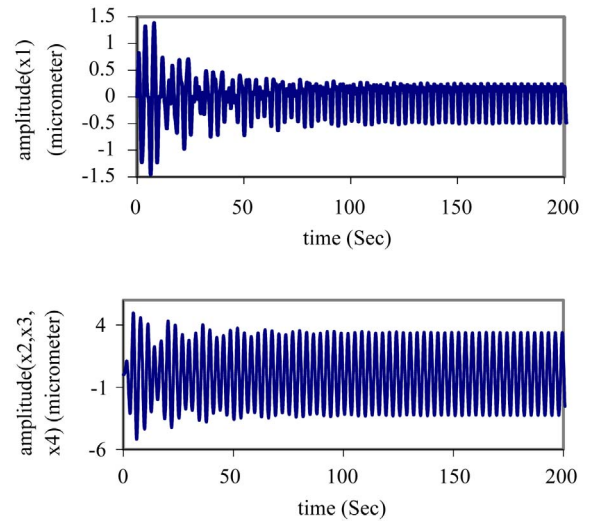
Figure 2. Response of the main system and absorbers at simultaneous primary and internal resonance case $\Omega \cong \omega_1 \cong \omega_2 \cong \omega_3 \cong \omega_4$.

2.1. Perturbation Analysis

Multiple scale perturbation method is conducted to obtain an approximate solution for Equations (1)-(4). Assuming the solution in the form:

$$x_n\left(t;\varepsilon\right) = x_{n0}\left(T_0, T_1\right) + \varepsilon x_{n1}\left(T_0, T_1\right) \quad n = \left(1, 2, 3, 4\right) \tag{5}$$

and the time derivatives became

$$\frac{\mathrm{d}}{\mathrm{d}t} = D_0 + \varepsilon D_1, \quad \frac{\mathrm{d}^2}{\mathrm{d}t^2} = D_0^2 + 2\varepsilon D_0 D_1 \tag{6}$$

where $T_n = \varepsilon^n t$. ($n = 0, 1$) are the fast and slow time scales respectively.

Substituting Equations (5) and (6) into Equations (1)-(4), and equating the coefficients of the same power of ε in both sides, we obtain

$$\left(D_0^2 + \omega_1^2\right)x_{10} = 0 \tag{7}$$

$$\left(D_0^2 + \omega_m^2\right)x_{m0} = \omega_m^2 x_{10}, \quad m = \left(2, 3, 4\right) \tag{8}$$

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$$\left(D_0^2 + \omega_1^2\right)x_{11} = \sum_{s=1}^{n} F_s \cos s\Omega T_0 - 2\left(D_1 + \zeta_1 + \zeta_2 + \zeta_3 + \zeta_4\right)\left(D_0 x_{10}\right) + 2\zeta_2 D_0 x_{20} + 2\zeta_3 D_0 x_{30} + 2\zeta_4 D_0 x_{40}$$
$$- \left(\gamma_1 + \gamma_2 + \gamma_3\right)x_{10} + \gamma_1 x_{20} + \gamma_2 x_{30} + \gamma_3 x_{40} - \eta_1 x_{10}^3 - \eta_2\left(x_{10} - x_{20}\right)^3 - \eta_3\left(x_{10} - x_{30}\right)^3 - \eta_4\left(x_{10} - x_{40}\right)^3 \tag{9}$$

$$\left(D_0^2 + \omega_2^2\right)x_{21} = -2D_0 D_1 x_{20} + 2\zeta_6 D_0 x_{10} - 2\zeta_6 D_0 x_{20} + \omega_2^2 x_{11} + \eta_5\left(x_{10} - x_{20}\right)^3 \tag{10}$$

$$\left(D_0^2 + \omega_3^2\right)x_{31} = -2D_0 D_1 x_{30} + 2\zeta_7 D_0 x_{10} - 2\zeta_7 D_0 x_{30} + \omega_3^2 x_{11} + \eta_6\left(x_{10} - x_{30}\right)^3 \tag{11}$$

$$\left(D_0^2 + \omega_4^2\right)x_{41} = -2D_0 D_1 x_{40} + 2\zeta_8 D_0 x_{10} - 2\zeta_8 D_0 x_{40} + \omega_4^2 x_{11} + \eta_7\left(x_{10} - x_{40}\right)^3 \tag{12}$$

The solution of Equation (7) can be expressed in the form

$$x_{10} = A_1 \exp\left(i\omega_1 T\right)_0 + cc \tag{13}$$

Using Equation (13) into Equation (8) yields

$$x_{m0} = A_m \exp\left(i\omega_m T_0\right) + E_{m-1} \exp\left(i\omega_1 T_0\right) + cc, \quad m = (2,3,4), \tag{14}$$

where $E_{m-1} = \dfrac{\omega_m^2}{\left(\omega_m^2 - \omega_1^2\right)}$, A_m are complex functions in T_1, which can be determined from eliminating the secular terms at the next approximation, and cc, stands for the conjugate of the preceding terms. Substituting Equations (13) and (14) into Equation (9), eliminating the secular terms, then the first order approximation is given by:

$$x_{11} = \sum_{S=1}^{n} Q_{S1} \exp(is\Omega T_0) + E_4 \exp\left(i\omega_2 T_0\right) + E_5 \exp\left(i\omega_3 T_0\right) + E_6 \exp\left(i\omega_4 T_0\right) + E_7 \exp\left(3i\omega_1 T_0\right) + E_8 \exp\left(3i\omega_2 T_0\right)$$
$$+ E_9 \exp\left(3i\omega_3 T_0\right) + E_{10} \exp\left(3i\omega_4 T_0\right) + E_{11} \exp\left(i\left(\omega_1 + 2\omega_2\right)T_0\right) + E_{12} \exp\left(i\left(\omega_1 - 2\omega_2\right)T_0\right) + E_{13} \exp(i(2\omega_1 + \omega_2)T_0)$$
$$+ E_{14} \exp\left(i\left(2\omega_1 - \omega_2\right)T_0\right) + E_{15} \exp\left(i\left(\omega_1 + 2\omega_3\right)T_0\right) + E_{16} \exp\left(i\left(\omega_1 - 2\omega_3\right)T_0\right) + E_{17} \exp\left(i\left(2\omega_1 + \omega_3\right)T_0\right)$$
$$+ E_{18} \exp\left(i\left(2\omega_1 - \omega_3\right)T_0\right) + E_{19} \exp\left(i\left(\omega_1 + 2\omega_4\right)T_0\right) + E_{20} \exp\left(i\left(\omega_1 - 2\omega_4\right)T_0\right) + E_{21} \exp\left(i\left(2\omega_1 + \omega_4\right)T_0\right)$$
$$+ E_{22} \exp\left(i\left(2\omega_1 - \omega_4\right)T_0\right) + cc \tag{15}$$

where Q_{S1} and $E_s \left(s = 4,5,\cdots,22\right)$ are complex functions in T_1. From Equations (13)-(15) into Equations (10)-(12) and eliminating the secular terms to obtain the solutions are given by:

$$x_{21} = \sum_{S=1}^{n} Q_{S2} \exp\left(is\Omega T_0\right) + E_{23} \exp\left(i\omega_1 T_0\right) + E_{24} \exp\left(i\omega_3 T_0\right) + E_{25} \exp\left(i\omega_4 T_0\right) + E_{26} \exp\left(3i\omega_1 T_0\right) + E_{27} \exp\left(3i\omega_2 T_0\right)$$
$$+ E_{28} \exp\left(3i\omega_3 T_0\right) + E_{29} \exp\left(3i\omega_4 T_0\right) + E_{30} \exp\left(i\left(\omega_1 + 2\omega_2\right)T_0\right) + E_{31} \exp\left(i\left(\omega_1 - 2\omega_2\right)T_0\right) + E_{32} \exp\left(i\left(2\omega_1 + \omega_2\right)T_0\right)$$
$$+ E_{33} \exp\left(i\left(2\omega_1 - \omega_2\right)T_0\right) + E_{34} \exp\left(i\left(\omega_1 + 2\omega_3\right)T_0\right) + E_{35} \exp\left(i\left(\omega_1 - 2\omega_3\right)T_0\right) + E_{36} \exp\left(i\left(2\omega_1 + \omega_3\right)T_0\right)$$
$$+ E_{37} \exp\left(i\left(2\omega_1 - \omega_3\right)T_0\right) + E_{38} \exp\left(i\left(\omega_1 + 2\omega_4\right)T_0\right) + E_{39} \exp\left(i\left(\omega_1 - 2\omega_4\right)T_0\right) + E_{40} \exp\left(i\left(2\omega_1 + \omega_4\right)T_0\right)$$
$$+ E_{41} \exp\left(i\left(2\omega_1 - \omega_4\right)T_0\right) + cc \tag{16}$$

$$x_{31} = \sum_{S=1}^{n} Q_{S3} \exp\left(is\Omega T_0\right) + E_{42} \exp\left(i\omega_1 T_0\right) + E_{43} \exp\left(i\omega_2 T_0\right) + E_{44} \exp\left(i\omega_4 T_0\right) + E_{45} \exp\left(3i\omega_1 T_0\right) + E_{46} \exp\left(3i\omega_2 T_0\right)$$
$$+ E_{47} \exp\left(3i\omega_3 T_0\right) + E_{48} \exp\left(3i\omega_4 T_0\right) + E_{49} \exp\left(i\left(\omega_1 + 2\omega_2\right)T_0\right) + E_{50} \exp\left(i\left(\omega_1 - 2\omega_2\right)T_0\right) + E_{51} \exp\left(i\left(2\omega_1 + \omega_2\right)T_0\right)$$
$$+ E_{52} \exp\left(i\left(2\omega_1 - \omega_2\right)T_0\right) + E_{53} \exp\left(i\left(\omega_1 + 2\omega_3\right)T_0\right) + E_{54} \exp\left(i\left(\omega_1 - 2\omega_3\right)T_0\right) + E_{55} \exp\left(i\left(2\omega_1 + \omega_3\right)T_0\right)$$
$$+ E_{56} \exp\left(i\left(2\omega_1 - \omega_3\right)T_0\right) + E_{57} \exp\left(i\left(\omega_1 + 2\omega_4\right)T_0\right) + E_{58} \exp\left(i\left(\omega_1 - 2\omega_4\right)T_0\right) + E_{59} \exp\left(i\left(2\omega_1 + \omega_4\right)T_0\right)$$
$$+ E_{60} \exp\left(i\left(2\omega_1 - \omega_4\right)T_0\right) + cc \tag{17}$$

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$$x_{41} = \sum_{S=1}^{n} Q_{S4} \exp\left(is\,\Omega T_0\right) + E_{61} \exp\left(i\,\omega_1 T_0\right) + E_{62} \exp\left(i\,\omega_2 T_0\right) + E_{63} \exp\left(i\,\omega_3 T_0\right) + E_{64} \exp\left(3i\omega_1 T_0\right) + E_{65} \exp\left(3i\omega_2 T_0\right)$$

$$+ E_{66} \exp\left(3i\omega_3 T_0\right) + E_{67} \exp\left(3i\omega_4 T_0\right) + E_{68} \exp\left(i\left(\omega_1 + 2\omega_2\right)T_0\right) + E_{69} \exp\left(i\left(\omega_1 - 2\omega_2\right)T_0\right) + E_{70} \exp\left(i\left(2\omega_1 + \omega_2\right)T_0\right)$$

$$+ E_{71} \exp\left(i\left(2\omega_1 - \omega_2\right)T_0\right) + E_{72} \exp\left(i\left(\omega_1 + 2\omega_3\right)T_0\right) + E_{73} \exp\left(i\left(\omega_1 - 2\omega_3\right)T_0\right) + E_{74} \exp\left(i\left(2\omega_1 + \omega_3\right)T_0\right)$$

$$+ E_{75} \exp\left(i\left(2\omega_1 - \omega_3\right)T_0\right) + E_{76} \exp\left(i\left(\omega_1 + 2\omega_4\right)T_0\right) + E_{77} \exp\left(i\left(\omega_1 - 2\omega_4\right)T_0\right) + E_{78} \exp\left(i\left(2\omega_1 + \omega_4\right)T_0\right)$$

$$+ E_{79} \exp\left(i\left(2\omega_1 - \omega_4\right)T_0\right) + cc$$

$$(18)$$

where Q_{S2}, Q_{S3}, Q_{S4} and $E_s\left(s = 23, \cdots, 79\right)$ are complex functions in T_1.

The reported resonance cases at this approximation order are:

1) Trivial resonance: $\Omega \cong \pm\omega_1 \cong \pm\omega_2 \cong \pm\omega_3 \cong \pm\omega_4 = 0$. Primary resonance:

$$\Omega \cong \pm\omega_1,\ \Omega \cong \pm\omega_2,\ \Omega \cong \pm\omega_3,\ \Omega \cong \pm\omega_4$$

2) Sub-harmonic resonance:

$$\Omega \cong \pm3\omega_1,\ \Omega \cong \pm3\omega_2,\ \Omega \cong \pm3\omega_3,\ \Omega \cong \pm3\omega_4$$

3) Super-harmonic resonance:

$$\Omega \cong \pm\omega_1/2,\ \Omega \cong \pm\omega_1/3,\ \Omega \cong \pm\omega_2/2,\ \Omega \cong \pm\omega_2/3$$

$$\Omega \cong \pm\omega_3/2,\ \Omega \cong \pm\omega_3/3,\ \Omega \cong \pm\omega_4/2,\ \Omega \cong \pm\omega_4/3$$

$$\Omega \cong \pm3\omega_1/2,\ \Omega \cong \pm3\omega_2/2,\ \Omega \cong \pm3\omega_3/2,\ \Omega \cong \pm3\omega_4/2$$

4) Internal resonance: $\omega_1 \cong \pm\omega_2 \cong \pm\omega_3 \cong \pm\omega_4$,

$$\omega_1 \cong \pm3\omega_n,\ \omega_1 \cong \pm4\omega_n,\ \omega_1 \cong \pm5\omega_n\ ,$$

$$3\omega_1 \cong 5\omega_n,\ 5\omega_1 \cong 3\omega_n,\ n = \left(2,3,4\right).$$

$$\omega_2 \cong \pm3\omega_s,\ \omega_2 \cong \pm4\omega_s,\ \omega_2 \cong \pm5\omega_s,$$

$$3\omega_2 \cong 5\omega_s,\ 5\omega_2 \cong 3\omega_s,\ s = \left(1,3,4\right).$$

$$\omega_3 \cong \pm3\omega_m,\ \omega_3 \cong \pm4\omega_m,\ \omega_3 \cong \pm5\omega_m,$$

$$3\omega_3 \cong 5\omega_m,\ 5\omega_3 \cong 3\omega_m,\ n = \left(1,2,4\right).$$

$$\omega_4 \cong \pm3\omega_L,\ \omega_4 \cong \pm4\omega_L,\ \omega_4 \cong \pm5\omega_L,$$

$$3\omega_4 \cong 5\omega_L,\ 5\omega_4 \cong 3\omega_L,\ L = \left(1,2,3\right).$$

$$\omega_n \cong \pm\left(\omega_1 \pm 2\omega_2\right),\ \omega_n \cong \pm\left(\omega_1 \pm 2\omega_3\right),\ \omega_n \cong \pm\left(\omega_1 \pm 2\omega_4\right)$$

$$\omega_n \cong \pm\left(2\omega_1 \pm \omega_2\right),\ \omega_n \cong \pm\left(2\omega_1 \pm \omega_3\right),\ \omega_n \cong \pm\left(2\omega_1 \pm \omega_4\right),$$

$$n = \left(1,2,3,4\right).$$

5) Combined resonance:

$$\Omega \cong \left(\omega_1 + \omega_2\right),\ \Omega \cong \pm\left(\omega_1 - \omega_2\right),\ \Omega \cong \left(\omega_1 + 2\omega_2\right),$$

$$\Omega \cong \pm\left(\omega_1 - 2\omega_2\right),\ \Omega \cong \left(2\omega_1 + \omega_2\right),\ \Omega \cong \pm\left(2\omega_1 - \omega_2\right)$$

6) Simultaneous or incident resonance: Any combination of the above resonance cases is considered as simultaneous resonance.

2.2. Numerical Results

Table 1 illustrates the selected values of the equations parameters used in resonance case calculations and its units.

Table 2 summarizes some of different resonance cases and the effectiveness of the absorbers.

Figure 2 illustrates the response for the system with absorber at the simultaneous primary resonance $\Omega \cong \omega_1 \cong \omega_2 \cong \omega_3 \cong \omega_4$. The effectiveness of the absorber E_a (the steady state amplitude of the main system without absorber/the steady state amplitude of main system with absorber) is about 7, which means that the maximum amplitude is reduced to about 14% of its original value.

3. Results and Discussion

One of the effective resonance cases where the tool holder has low amplitude and at the same time, the absorbers have high amplitudes is studied in the next section.

3.1. Stability of the System

Introducing the detuning parameters σ_1, σ_2, σ_3 and σ_4 in the primary and internal resonance to convert the small-divisor terms into the secular terms, according to:

$$\Omega_1 \cong \omega_1 + \varepsilon\sigma_1,\ \omega_2 \cong \omega_1 + \varepsilon\sigma_2$$

$$\omega_3 \cong \omega_1 + \varepsilon\sigma_3,\ \omega_4 \cong \omega_1 + \varepsilon\sigma_4$$

$$(19)$$

Table 1. The values of the equations parameters.

Damping coefficients values (Newton sec/micrometer)	$\zeta_1 = 0.01$ $\zeta_2 = 0.001$ $\zeta_3 = 0.001$ $\zeta_4 = 0.001$ $\zeta_5 = 0.01$ $\zeta_6 = 0.01$ $\zeta_7 = 0.01$ $\zeta_8 = 0.01$
Non-linear parameters values (Newton/micrometer)	$\eta_1 = 0.01$ $\eta_2 = 0.005$ $\eta_3 = 0.005$ $\eta_4 = 0.005$ $\eta_5 = 0.05$ $\eta_6 = 0.05$ $\eta_7 = 0.05$ $\gamma_1 = 0.4$ $\gamma_2 = 0.4$ $\gamma_3 = 0.4$
Natural and Excitation frequencies values (Hertz)	$\dfrac{\Omega}{\omega_1} = 1$, $\omega_1 = \omega_2 = \omega_3 = \omega_4$
Excitation amplitudes values (micrometer/sec²)	$F_1 = 4$ $F_2 = 2$ $F_3 = 1$

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Table 2. Summarizes the resonance cases for the tool holder and absorber.

Cases	Conditions	x_1/F_1	x_2/F_1	x_3/F_1	x_4/F_1	Remarks*
$\Omega \cong \omega_1$	$\omega_1 \cong \omega_2 \cong \omega_3 \cong \omega_4$	12.5%	85%	85%	85%	Limit cycle
	$\omega_1 \cong \omega_2/2 \cong \omega_3 \cong \omega_4$	15.5%	50%	118%	118%	Limit cycle
	$\omega_1 \cong \omega_2/3 \cong \omega_3 \cong \omega_4$	18.75%	30%	118%	118%	Limit cycle
	$\omega_1 \cong \omega_2/5 \cong \omega_3 \cong \omega_4$	20%	22%	118 %	118%	Limit cycle
	$\omega_1 \cong \omega_2 \cong \omega_3/2 \cong \omega_4/3$	34%	165%	74%	43%	Limit cycle
	$\omega_1 \cong \omega_2 \cong \omega_3/3 \cong \omega_4/5$	39%	170%	50%	40%	Limit cycle
$\Omega \cong 3\omega_1$	$\omega_1 \cong \omega_2 \cong \omega_3 \cong \omega_4$	3.5 %	0.4%	0.4%	0.4%	Limit cycle
	$\omega_1 \cong \omega_2/2 \cong \omega_3 \cong \omega_4$	3.75%	27%	0.45%	0.45%	Limit cycle
$\Omega \cong \omega_1/2$	$\omega_1 \cong \omega_2 \cong \omega_3 \cong \omega_4$	39%	100%	100 %	100%	Limit cycle
	$\omega_1 \cong \omega_2/2 \cong \omega_3 \cong \omega_4$	36%	40%	113%	113%	Limit cycle
$\Omega \cong \omega_1/3$	$\omega_1 \cong \omega_2 \cong \omega_3 \cong \omega_4$	65%	85%	85%	85%	Limit cycle
	$\omega_1 \cong \omega_2/2 \cong \omega_3 \cong \omega_4$	63%	67%	85%	85%	Limit cycle
$\Omega \cong 3\omega_1/2$	$\omega_1 \cong \omega_2 \cong \omega_3 \cong \omega_4$	37%	31%	31%	31%	Limit cycle
	$\omega_1 \cong \omega_2/2 \cong \omega_3 \cong \omega_4$	36%	25%	31%	31%	Limit cycle

This case represent the system best case and at the same time absorber high amplitude. Substituting Equation (19) into Equations (9)-(12) and eliminating the secular terms, leads to the solvability conditions for the first order approximation noting that A_1, A_2, A_3 and A_4 are functions in T_1 we get

$$2i\omega_1 \left[D_1 + \zeta_1 + \zeta_2 + \zeta_3 + \zeta_4 \right] A_1 + \left(\gamma_1 + \gamma_2 + \gamma_3 \right) A_1 + 3\eta_1 A_1^2 \bar{A}_1 + \eta_2 \left(3A_1^2 \bar{A}_1 + 6A_1 A_2 \bar{A}_2 \right)$$
$$+ \eta_3 \left(3A_1^2 \bar{A}_1 + 6A_1 A_3 \bar{A}_3 \right) + \eta_4 \left(3A_1^2 \bar{A}_1 + 6A_1 A_4 \bar{A}_4 \right) - \frac{F_1}{2} e^{i\sigma_1 T_1} - \left[2i\omega_2 A_2 \zeta_2 + \gamma_1 A_2 + \eta_2 \left(3A_2^2 \bar{A}_2 + 6A_1 \bar{A}_1 A_2 \right) \right] e^{i\sigma_2 T_1}$$
$$- \left[2i\omega_3 A_3 \zeta_3 + \gamma_2 A_3 + \eta_3 \left(3A_3^2 \bar{A}_3 + 6A_1 \bar{A}_1 A_3 \right) \right] e^{i\sigma_3 T_1} - \left[2i\omega_4 A_4 \zeta_4 + \gamma_3 A_4 + \eta_4 \left(3A_4^2 \bar{A}_4 + 6A_1 \bar{A}_1 A_4 \right) \right] e^{i\sigma_4 T_1} \qquad (20)$$
$$+ \left[3\eta_2 \bar{A}_1 A_2^2 \right] e^{2i\sigma_2 T_1} - \left[3\eta_2 A_1^2 \bar{A}_2 \right] e^{-i\sigma_2 T_1} + \left[3\eta_3 \bar{A}_1 A_3^2 \right] e^{2i\sigma_3 T_1} - \left[3\eta_3 A_1^2 \bar{A}_3 \right] e^{-i\sigma_3 T_1}$$
$$+ \left[3\eta_4 \bar{A}_1 A_4^2 \right] e^{2i\sigma_4 T_1} - \left[3\eta_4 A_1^2 \bar{A}_4 \right] e^{-i\sigma_4 T_1} = 0$$

$$2i\omega_2 \left[D_1 A_2 + \zeta_6 A_2 \right] + \eta_5 \left(3A_2^2 \bar{A}_2 + 6A_1 \bar{A}_1 A_2 \right) - \left[3\eta_5 A_2^2 \bar{A}_1 \right] e^{i\sigma_2 T_1}$$
$$- \left[2i\omega_1 \zeta_6 A_1 + \eta_5 \left(3A_1^2 \bar{A}_1 + 6A_2 \bar{A}_2 A_1 \right) \right] e^{-i\sigma_2 T_1} + \left[3\eta_5 A_1^2 \bar{A}_2 \right] e^{-2i\sigma_2 T_1} = 0 \qquad (21)$$

$$2i\omega_3 \left[D_1 A_3 + \zeta_7 A_3 \right] + \eta_6 \left(3A_3^2 \bar{A}_3 + 6A_1 \bar{A}_1 A_3 \right) - \left[3\eta_6 A_3^2 \bar{A}_1 \right] e^{i\sigma_3 T_1}$$
$$- \left[2i\omega_1 \zeta_7 A_1 + \eta_6 \left(3A_1^2 \bar{A}_1 + 6A_3 \bar{A}_3 A_1 \right) \right] e^{-i\sigma_3 T_1} + \left[3\eta_6 A_1^2 \bar{A}_3 \right] e^{-2i\sigma_3 T_1} = 0 \qquad (22)$$

$$2i\omega_4 \left[D_1 A_4 + \zeta_8 A_4 \right] + \eta_7 \left(3A_4^2 \bar{A}_4 + 6A_1 \bar{A}_1 A_4 \right) - \left[3\eta_7 A_4^2 \bar{A}_1 \right] e^{i\sigma_4 T_1}$$
$$- \left[2i\omega_1 \zeta_8 A_1 + \eta_7 \left(3A_1^2 \bar{A}_1 + 6A_4 \bar{A}_4 A_1 \right) \right] e^{-i\sigma_4 T_1} + \left[3\eta_7 A_1^2 \bar{A}_4 \right] e^{-2i\sigma_4 T_1} = 0 \qquad (23)$$

$$\text{Putting} \quad A_n = \frac{1}{2} a_n \left(T_1 \right) e^{i\beta_n (T_1)}, \quad n = \left(1, 2, 3, 4 \right) \qquad (24)$$

where a_n and β_n are the steady state amplitudes and the phases of the motion respectively. Substituting Equation (24) into Equations (20)-(23) and separating real and imaginary part yields,

$$a_1' = \frac{F_1}{2\omega_1}\sin\theta_1 - \Gamma_1 a_1 + \Gamma_2\cos\theta_2 + \Gamma_3\sin\theta_2 + \Gamma_4\cos\theta_3 + \Gamma_5\sin\theta_3 + \Gamma_6\cos\theta_4 + \Gamma_7\sin\theta_4$$
$$-\Gamma_8\sin 2\theta_2 - \Gamma_9\sin\theta_2 - \Gamma_{10}\sin 2\theta_3 - \Gamma_{11}\sin\theta_3 - \Gamma_{12}\sin 2\theta_4 - \Gamma_{13}\sin\theta_4 \tag{25}$$

$$a_1\beta_1' = -\frac{F_1}{2\omega_1}\cos\theta_1 + \Gamma_{14} + \Gamma_2\sin\theta_2 - \Gamma_3\cos\theta_2 + \Gamma_4\sin\theta_3 - \Gamma_5\cos\theta_3 + \Gamma_6\sin\theta_4$$
$$-\Gamma_7\cos\theta_4 + \Gamma_8\cos 2\theta_2 - \Gamma_9\cos\theta_2 + \Gamma_{10}\cos 2\theta_3 - \Gamma_{11}\cos\theta_3 + \Gamma_{12}\cos 2\theta_4 - \Gamma_{13}\cos\theta_4 \tag{26}$$

$$a_2' = -\zeta_6 a_2 + \Gamma_{15}\cos\theta_2 - \Gamma_{16}\sin\theta_2 + \Gamma_{17}\sin\theta_2 + \Gamma_{18}\sin 2\theta_2 \tag{27}$$

$$a_2\beta_2' = \Gamma_{19} - \Gamma_{15}\sin\theta_2 - \Gamma_{16}\cos\theta_2 - \Gamma_{17}\cos\theta_2 + \Gamma_{18}\cos 2\theta_2 \tag{28}$$

$$a_3' = -\zeta_7 a_3 + \Gamma_{20}\cos\theta_3 - \Gamma_{21}\sin\theta_3 + \Gamma_{22}\sin\theta_3 + \Gamma_{23}\sin 2\theta_3 \tag{29}$$

$$a_3\beta_3' = \Gamma_{24} - \Gamma_{20}\sin\theta_3 - \Gamma_{21}\cos\theta_3 - \Gamma_{22}\cos\theta_3 + \Gamma_{23}\cos 2\theta_3 \tag{30}$$

$$a_4' = -\zeta_8 a_4 + \Gamma_{25}\cos\theta_4 - \Gamma_{26}\sin\theta_4 + \Gamma_{27}\sin\theta_4 + \Gamma_{28}\sin 2\theta_4 \tag{31}$$

$$a_4\beta_4' = \Gamma_{29} - \Gamma_{25}\sin\theta_4 - \Gamma_{26}\cos\theta_4 - \Gamma_{27}\cos\theta_4 + \Gamma_{28}\cos 2\theta_4 \tag{32}$$

where: $(\Gamma_1, \Gamma_2, \cdots, \Gamma_{29})$ are defined in the appendix,

$$\theta_1 = \sigma_1 T_1 - \beta_1, \ \theta_2 = \sigma_2 T_1 + \beta_2 - \beta_1, \ \theta_3 = \sigma_3 T_1 + \beta_3 - \beta_1, \ \theta_4 = \sigma_4 T_1 + \beta_4 - \beta_1 \tag{33}$$

For steady state solutions, $a_n' = \theta_n' = 0$, $n = (1,2,3,4)$.
Then from Equation (33), we get:

$$\beta_1' = \sigma_1, \ \beta_2' = (\sigma_1 - \sigma_2), \ \beta_3' = (\sigma_1 - \sigma_3), \ \beta_4' = (\sigma_1 - \sigma_4) \tag{34}$$

Then it follows from Equations (25)-(32) that the steady state solutions are given by:

$$\frac{F_1}{2\omega_1}\sin\theta_1 - \Gamma_1 a_1 + \Gamma_2\cos\theta_2 + \Gamma_3\sin\theta_2 + \Gamma_4\cos\theta_3 + \Gamma_5\sin\theta_3 + \Gamma_6\cos\theta_4 + \Gamma_7\sin\theta_4$$
$$-\Gamma_8\sin 2\theta_2 - \Gamma_9\sin\theta_2 - \Gamma_{10}\sin 2\theta_3 - \Gamma_{11}\sin\theta_3 - \Gamma_{12}\sin 2\theta_4 - \Gamma_{13}\sin\theta_4 = 0 \tag{35}$$

$$a_1\sigma_1 = -\frac{F_1}{2\omega_1}\cos\theta_1 + \Gamma_{14} + \Gamma_2\sin\theta_2 - \Gamma_3\cos\theta_2 + \Gamma_4\sin\theta_3 - \Gamma_5\cos\theta_3 + \Gamma_6\sin\theta_4 - \Gamma_7\cos\theta_4$$
$$+\Gamma_8\cos 2\theta_2 - \Gamma_9\cos\theta_2 + \Gamma_{10}\cos 2\theta_3 - \Gamma_{11}\cos\theta_3 + \Gamma_{12}\cos 2\theta_4 - \Gamma_{13}\cos\theta_4 \tag{36}$$

$$-\zeta_6 a_2 + \Gamma_{15}\cos\theta_2 - \Gamma_{16}\sin\theta_2 + \Gamma_{17}\sin\theta_2 + \Gamma_{18}\sin 2\theta_2 = 0 \tag{37}$$

$$a_2(\sigma_1 - \sigma_2) = \Gamma_{19} - \Gamma_{15}\sin\theta_2 - \Gamma_{16}\cos\theta_2 - \Gamma_{17}\cos\theta_2 + \Gamma_{18}\cos 2\theta_2 \tag{38}$$

$$-\zeta_7 a_3 + \Gamma_{20}\cos\theta_3 - \Gamma_{21}\sin\theta_3 + \Gamma_{22}\sin\theta_3 + \Gamma_{23}\sin 2\theta_3 = 0 \tag{39}$$

$$a_3(\sigma_1 - \sigma_3) = \Gamma_{24} - \Gamma_{20}\sin\theta_3 - \Gamma_{21}\cos\theta_3 - \Gamma_{22}\cos\theta_3 + \Gamma_{23}\cos 2\theta_3 \tag{40}$$

$$-\zeta_8 a_4 + \Gamma_{25}\cos\theta_4 - \Gamma_{26}\sin\theta_4 + \Gamma_{27}\sin\theta_4 + \Gamma_{28}\sin 2\theta_4 = 0 \tag{41}$$

$$a_4(\sigma_1 - \sigma_4) = \Gamma_{29} - \Gamma_{25}\sin\theta_4 - \Gamma_{26}\cos\theta_4 - \Gamma_{27}\cos\theta_4 + \Gamma_{28}\cos 2\theta_4 \tag{42}$$

From Equations (35)-(42) we have the following case: (practical case) $a_1 \neq 0, a_2 \neq 0, a_3 \neq 0, a_4 \neq 0$

Table 3 gives the final results of the frequency response equations (in **Table 3**), where K_1, K_2, K_3, K_4, K_5, K_6, K_7 and K_8 are real functions. The stability of the linear solution of the obtained fixed points will be determined as follows. Consider A_n in the form:

$$A_n = \frac{1}{2}[p_n - iq_n]e^{iv_n T_1}, n = (1,2,3,4) \tag{43}$$

Table 3. Frequency response equations.

Case	Frequency response equations (FRE)
$a_1 \neq 0, a_2 \neq 0, a_3 \neq 0, a_4 \neq 0$	$\sigma_1^2 + K_1\sigma_1 + K_2 = 0,\ \sigma_2^2 + K_3\sigma_2 + K_4 = 0,\ \sigma_1^2 + K_5\sigma_1 + K_6 = 0,\ \sigma_2^2 + K_7\sigma_2 + K_8 = 0$

where p_n and q_n are real and $v_1 = \sigma_1$, $v_2 = (\sigma_1 - \sigma_2)$, $v_3 = (\sigma_1 - \sigma_3)$ and $v_4 = (\sigma_1 - \sigma_3)$. Substituting Equation (43) into the linear part of Equations (20)-(23) and separating real and imaginary part yields,

$$p_1' + (\zeta_1 + \zeta_2 + \zeta_3 + \zeta_4) p_1 + \left(v_1 - \frac{(\gamma_1 + \gamma_2 + \gamma_3)}{2\omega_1}\right) q_1 - \frac{\omega_2 \zeta_2}{\omega_1} p_2 + \frac{\gamma_1}{2\omega_1} q_2 - \frac{\omega_3 \zeta_3}{\omega_1} p_3$$
$$+ \frac{\gamma_2}{2\omega_1} q_3 - \frac{\omega_4 \zeta_4}{\omega_1} p_4 + \frac{\gamma_3}{2\omega_1} q_4 = 0 \tag{44}$$

$$q_1' + (\zeta_1 + \zeta_2 + \zeta_3 + \zeta_4) q_1 - \left(v_1 - \frac{(\gamma_1 + \gamma_2 + \gamma_3)}{2\omega_1} + \frac{F_1}{2\omega_1 p_1}\right) p_1 - \frac{\omega_2 \zeta_2}{\omega_1} q_2 - \frac{\gamma_1}{2\omega_1} p_2 - \frac{\omega_3 \zeta_3}{\omega_1} q_3$$
$$- \frac{\gamma_2}{2\omega_1} p_3 - \frac{\omega_4 \zeta_4}{\omega_1} q_4 - \frac{\gamma_3}{2\omega_1} p_4 = 0 \tag{45}$$

$$p_2' + \zeta_6 p_2 + v_2 q_2 - \frac{\omega_1 \zeta_6}{\omega_2} p_1 = 0 \tag{46}$$

$$q_2' + \zeta_6 q_2 - v_2 p_2 - \frac{\omega_1 \zeta_6}{\omega_2} q_1 = 0 \tag{47}$$

$$p_3' + \zeta_7 p_3 + v_3 q_3 - \frac{\omega_1 \zeta_7}{\omega_3} p_1 = 0 \tag{48}$$

$$q_3' + \zeta_7 q_3 - v_3 p_3 - \frac{\omega_1 \zeta_7}{\omega_3} q_1 = 0 \tag{49}$$

$$p_4' + \zeta_8 p_4 + v_4 q_4 - \frac{\omega_1 \zeta_8}{\omega_4} p_1 = 0 \tag{50}$$

$$q_4' + \zeta_8 q_4 - v_4 p_4 - \frac{\omega_1 \zeta_8}{\omega_4} q_1 = 0 \tag{51}$$

The eigenvalues of the above system of equations are given by the equation

$$\lambda^8 + r_1\lambda^7 + r_2\lambda^6 + r_3\lambda^5 + r_4\lambda^4 + r_5\lambda^3 + r_6\lambda^2 + r_7\lambda + r_8 = 0 \tag{52}$$

where, $(r_1, r_2, \cdots, r_8)$ are functions in the parameters $(a_1, a_2, a_3, a_4, \omega_1, \omega_2, \omega_3, \omega_3, \sigma_1, \sigma_2, \sigma_3, \sigma_4, \zeta_1, \zeta_2, \zeta_3, \zeta_4, \gamma_1, \gamma_2, \gamma_3\ F_1\ \theta_1, \theta_2, \theta_3, \theta_4)$. According to the Routh-Hurtwitz criterion, the necessary and sufficient conditions for all the roots of Equation (52) to possess negative real parts if, and only if,

$$D = \begin{vmatrix} K_9 & K_{10} & K_{11} & K_{12} & K_{13} & K_{14} & K_{15} & K_{16} \\ -K_{17} & K_9 & -K_{12} & K_{11} & -K_{14} & K_{13} & -K_{16} & -K_{15} \\ K_{18} & 0 & \lambda+\zeta_6 & v_2 & 0 & 0 & 0 & 0 \\ 0 & K_{18} & -v_2 & \lambda+\zeta_6 & 0 & 0 & 0 & 0 \\ K_{19} & 0 & 0 & 0 & \lambda+\zeta_7 & v_3 & 0 & 0 \\ 0 & K_{19} & 0 & 0 & -v_3 & \lambda+\zeta_7 & 0 & 0 \\ K_{20} & 0 & 0 & 0 & 0 & 0 & \lambda+\zeta_8 & v_4 \\ 0 & K_{20} & 0 & 0 & 0 & 0 & -v_4 & \lambda+\zeta_8 \end{vmatrix} \tag{53}$$

and all its principle minors are positive. $K_9 \cdots, K_{20}$ are real functions.

3.2. Numerical Results

Figures 3(a) and **4(a)** shows that the effects of the detuning parameters $\sigma_1, \sigma_2, \sigma_3$ and σ_4 on the steady state amplitudes of the main system a_1 and absorbers a_2, a_3 and a_4 for the stability of the practical case where $a_1 \neq 0, a_2 \neq 0, a_3 \neq 0$ and $a_4 \neq 0$. For different values of the damping coefficients ζ_n, the non-linear parameters η_n, (n = 1, 2, 3,4) and the non-linear parameters γ_m, (m = 2, 3, 4) the effects are trivial as shown in **Figure 3(b)**. From **Figure 3(c)** we find that the steady state amplitude of the tool holder is a monotonic increasing function in its excitation amplitude F_1 with an increase
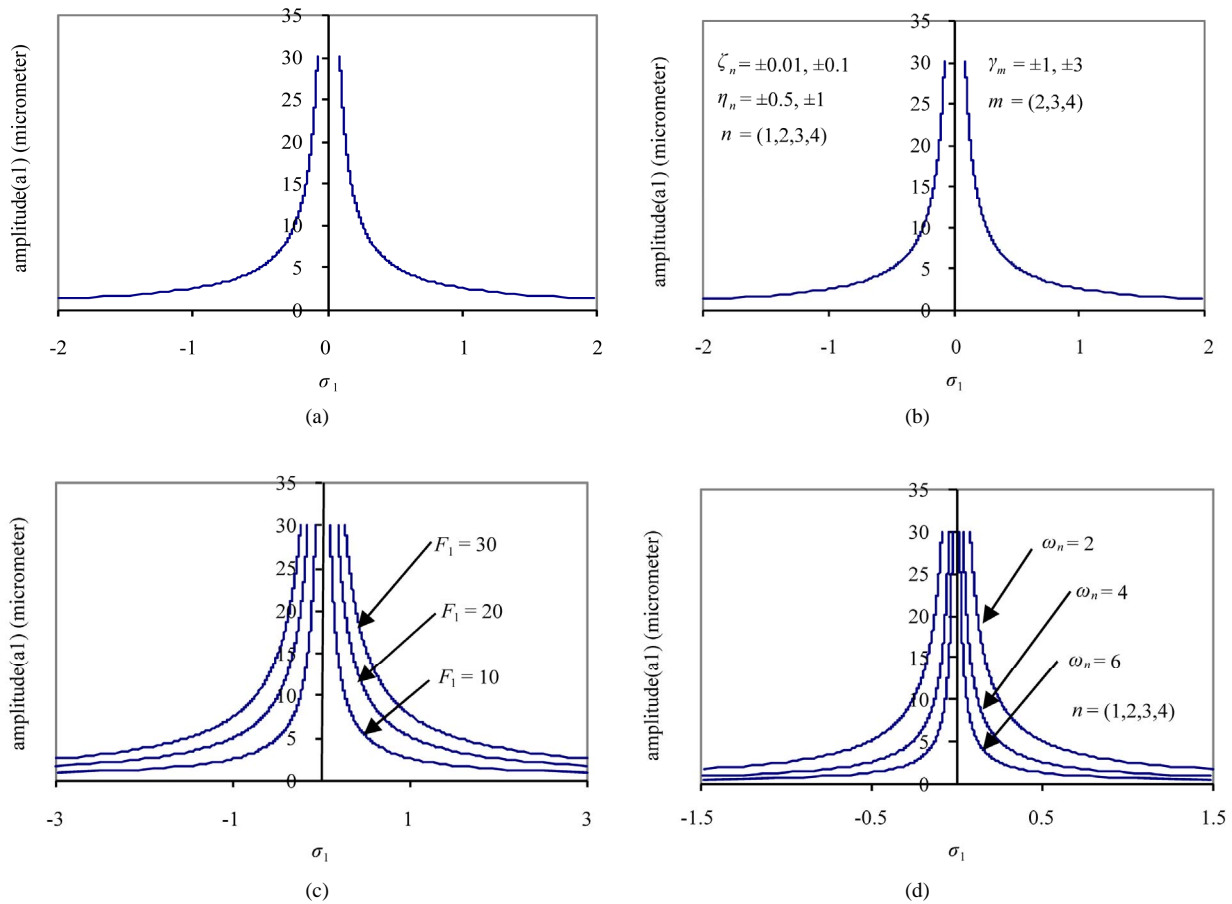
Figure 3. Response curves (different parameters against σ_1).

in the unstable region of the solution.

The steady state amplitude of the tool holder is a monotonic decreasing function in the natural frequencies ω_n, $(n = 1, 2, 3, 4)$ with a decrease in the unstable region of the solution as shown in **Figure 3(d)**.

Now the effect of the detuning parameters σ_2, σ_3 and σ_4 on the steady state amplitude of the tools a_2, a_3 and a_4 is shown in **Figure 4(a)**.

For different values of the damping coefficients ζ_i, $(i = 6, 7, 8)$, the effects on the steady state amplitudes of the tools are trivial as shown in **Figure 4(b)**. For different values of the non-linear parameters η_s, $(s = 5, 6, 7)$, the steady state amplitude of the tools are monotonic increasing as shown in **Figure 4(c)**. **Figure 4(d)** shows that the steady state amplitude of the tools is a monotonic decreasing function in the natural frequencies ω_m, $(m = 2, 3, 4)$ and the region of unstable solution is decreasing. For all figures no jump phenomena was observed.

4. Conclusions

The vibrations of a four-degree-of-freedom non-linear mechanical system and absorbers are investigated. The physical motivation for the system stems from applica-

tions in ultrasonic machining in which an exciter (machine head) drives tuned blades (absorbers) having both linear and cubic non-linearities. In the present work, we considered multi-tools which allow the machining of different materials and different shapes in different or one workpiece the vibration of ultrasonic machine head can be controlled via non-linear absorbers. Multiple time scale perturbation technique is applied to determine semi-closed form solutions for the coupled deferential equations describing the system up to the second order approximations. To study the stability of the system, both the frequency response equations and the phase-plane technique are applied. From the above study the following may be concluded.

1) Optimum working conditions at $\Omega \cong \omega_1$, $\omega_1 \cong \omega_2 \cong \omega_3 \cong \omega_4$, where the vibration of the tool holder is suppressed to about 12.5% of the original amplitude, and the three tools have reasonable amplitudes.

2) For different values of the damping coefficients ζ_n, the non-linear parameters η_n, $(n = 1,2,3,4)$ and the non-linear parameters γ_m, $(m = 2, 3, 4)$ we find the effects of these parameters on the steady state amplitude of the tool holder are trivial and same effects have been
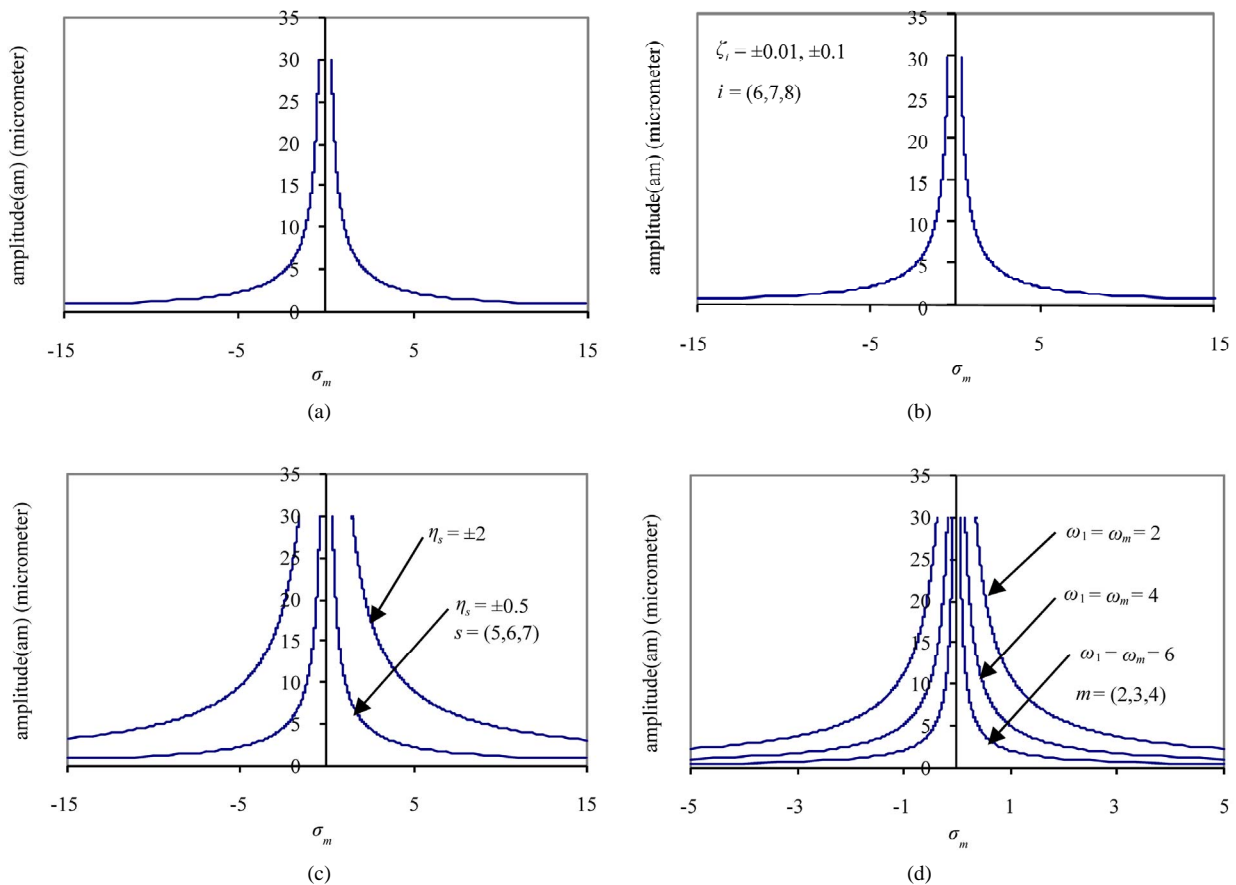
Figure 4. Response curves (different parameters against σ_m, $m = (2, 3, 4)$).

obtained for the damping coefficients ζ_i, $(i = 6, 7, 8)$ on the steady state amplitude of the three tools.

3) The steady state amplitude of the tool holder is a monotonic decreasing function in the natural frequencies ω_n, $(n =1, 2, 3, 4)$ with decreasing in the region of unstable solution.

4) The steady state amplitude of the tool holder is a monotonic increasing function in its excitation amplitude F_1 with increasing in the region of unstable solution.

5) The steady state amplitude of the tools is a monotonic decreasing function in the non-linear parameters η_s, $(s = 5, 6, 7)$ and the natural frequencies ω_m, $(m = 2, 3, 4)$ and the region of unstable solution is decreasing.

6) To make use of machine capability, multi-tools are used to save both time and power.

7) The reported results are in a good agreement with References [3,4] regarding the amplitude reduction.

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Nomenclature

c_n, (n = 1, 2, 3, 4). The damping coefficients of the system and the absorber.

k_m, (m = 1, 2, 3, 4). The stiffness of the system and the absorbers.

h_m, (m = 1, 2, 3, 4). The non-linear parameters of the system and the absorber.

F_j, Ω_j (j = 1, 2, 3). The excitation amplitudes and frequencies.

$m_1, m_2 m_3, m_4$. The masses of the system and the absorber.

$\zeta_n = c_n/2m_1$, (n = 1, 2, 3, 4). The linear damping factors of the system.

$\zeta_5 = c_5/m_1$. The quadratic damping factors of the system.

$\zeta_6 = c_2/2m_2, \zeta_7 = c_3/2m_3, \zeta_8 = c_4/2m_4$. The damping factors of the absorbers.

$\eta_m = h_m/m_1$, (m = 1, 2, 3, 4). The coupling non-linear parameters of the system.

$\eta_5 = h_2/m_2, \eta_6 = h_3/m_3, \eta_7 = h_4/m_4$. The non-linear parameters of the absorbers.

$\omega_s^2 = k_s/m_s$, (s = 1, 2, 3, 4). The natural frequencies of the system and absorbers.

$\gamma_1 = k_1/m_1, \gamma_2 = k_2/m_1, \gamma_3 = k_3/m_1$. The stiffness of the system.

x_i, i = (1, 2, 3, 4). Displacement of both system and absorber.

Appendix

$$\Gamma_1 = (\zeta_1 + \zeta_2 + \zeta_3 + \zeta_4), \ \Gamma_2 = \frac{\omega_2 \zeta_2 a_2}{\omega_1}, \ \Gamma_3 = \left[\frac{\gamma_1 a_2}{2\omega_1} + \frac{3\eta_2 a_2^3}{8\omega_1} + \frac{3\eta_2 a_1^2 a_2}{4\omega_1} \right],$$

$$\Gamma_4 = \frac{\omega_3 \zeta_3 a_3}{\omega_1}, \ \Gamma_5 = \left[\frac{\gamma_2 a_3}{2\omega_1} + \frac{3\eta_3 a_3^3}{8\omega_1} + \frac{3\eta_3 a_1^2 a_3}{4\omega_1} \right], \ \Gamma_6 = \frac{\omega_4 \zeta_4 a_4}{\omega_1},$$

$$\Gamma_7 = \left[\frac{\gamma_3 a_4}{2\omega_1} + \frac{3\eta_4 a_4^3}{8\omega_1} + \frac{3\eta_4 a_1^2 a_4}{4\omega_1} \right], \ \Gamma_8 = \frac{3\eta_2 a_1 a_2^2}{8\omega_1}, \ \Gamma_9 = \frac{3\eta_2 a_2 a_1^2}{8\omega_1},$$

$$\Gamma_{10} = \frac{3\eta_3 a_1 a_3^2}{8\omega_1}, \ \Gamma_{11} = \frac{3\eta_3 a_3 a_1^2}{8\omega_1}, \ \Gamma_{12} = \frac{3\eta_4 a_1 a_4^2}{8\omega_1}, \ \Gamma_{13} = \frac{3\eta_2 a_4 a_1^2}{8\omega_1},$$

$$\Gamma_{14} = \frac{(\gamma_1 + \gamma_2 + \gamma_3)a_1}{2\omega_1} + \frac{3\eta_1 a_1^3}{8\omega_1} + \frac{3\eta_2 a_1^3}{8\omega_1} + \frac{3\eta_2 a_1 a_2^2}{4\omega_1} + \frac{3\eta_3 a_1^3}{8\omega_1} + \frac{3\eta_3 a_1 a_3^2}{4\omega_1} + \frac{3\eta_4 a_1^3}{8\omega_1} + \frac{3\eta_4 a_1 a_4^2}{4\omega_1},$$

$$\Gamma_{15} = \frac{\omega_1 \zeta_6 a_1}{\omega_2}, \ \Gamma_{16} = \left[\frac{3\eta_5 a_1^3}{8\omega_2} + \frac{3\eta_5 a_1 a_2^2}{4\omega_2} \right], \ \Gamma_{17} = \frac{3\eta_5 a_2^2 a_1}{8\omega_2}, \ \Gamma_{18} = \frac{3\eta_5 a_1^2 a_2}{8\omega_2},$$

$$\Gamma_{19} = \left[\frac{3\eta_5 a_2^3}{8\omega_2} + \frac{3\eta_5 a_2 a_1^2}{4\omega_2} \right], \ \Gamma_{20} = \frac{\omega_1 \zeta_7 a_1}{\omega_3}, \ \Gamma_{21} = \left[\frac{3\eta_6 a_1^3}{8\omega_3} + \frac{3\eta_6 a_1 a_3^2}{4\omega_3} \right],$$

$$\Gamma_{22} = \frac{3\eta_6 a_3^2 a_1}{8\omega_3}, \ \Gamma_{23} = \frac{3\eta_6 a_1^2 a_3}{8\omega_3}, \ \Gamma_{24} = \left[\frac{3\eta_6 a_3^3}{8\omega_3} + \frac{3\eta_6 a_3 a_1^2}{4\omega_3} \right], \ \Gamma_{25} = \frac{\omega_1 \zeta_8 a_1}{\omega_4},$$

$$\Gamma_{26} = \left[\frac{3\eta_7 a_1^3}{8\omega_4} + \frac{3\eta_7 a_1 a_4^2}{4\omega_4} \right], \ \Gamma_{27} = \frac{3\eta_7 a_4^2 a_1}{8\omega_4}, \ \Gamma_{28} = \frac{3\eta_7 a_1^2 a_4}{8\omega_4}, \ \Gamma_{29} = \left[\frac{3\eta_7 a_4^3}{8\omega_4} + \frac{3\eta_7 a_4 a_1^2}{4\omega_4} \right]$$