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Mapping Properties of Generalized Robertson Functions under Certain Integral Operators

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ABSTRACT

In the present article, certain classes of generalized p-valent Robertson functions are considered. Mapping properties of these classes are investigated under certain p-valent integral operators introduced by Frasin recently.

Keywords: p-Valent Analytic Functions; Bounded Boundary Rotations; Bounded Radius Rotations; Integral Operators

1. Introduction

Let $A(p)$ be the class of functions $f(z)$ of the form

$$f(z) = z^p + \sum_{j=p+1}^{\infty} a_j z^j \ (p \in \mathbb{N} = \{1,2,\cdots\}),$$

which are analytic in the open unit disc $E = \{z : |z| < 1\}$. We write $A(1) = A$. A function $f(z) \in A$ is said to be spiral-like if there exists a real number λ $\left(|\lambda| < \frac{\pi}{2}\right)$ such that

$$\operatorname{Re} e^{i\lambda} \frac{zf'(z)}{f(z)} > 0 \ (z \in E).$$

The class of all spiral-like functions was introduced by L. Spacek [1] in 1933 and we denote it by S_λ^*. Later in 1969, Robertson [2] considered the class C_λ of analytic functions in E for which $zf'(z) \in S_\lambda^*$.

Let $P_k^\lambda(p,\rho)$ be the class of functions $p(z)$ analytic in E with $p(0) = 1$ and

$$\int_0^{2\pi} \left| \frac{\operatorname{Re} e^{i\lambda} p(z) - \rho \cos\lambda}{p - \rho} \right| d\theta \le k\pi \cos\lambda, \ (z = re^{i\theta}),$$

where $k \ge 2$, $0 \le \rho < 1$ and λ is real with $|\lambda| < \frac{\pi}{2}$. For $\lambda = 0$, $p = 1$, this class was introduced in [3] and for $\rho = 0$, see [4]. For $k = 2$, $\lambda = 0$ and $\rho = 0$, the class $P_k^\lambda(p,\rho)$ reduces to the class P of functions $p(z)$ analytic in E with $p(0) = 1$ and whose real part is positive.

We define the following classes

$$R_k^\lambda(p,\rho) = \left\{ f(z) : f(z) \in A(p) \text{ and } \frac{zf'(z)}{f(z)} \in P_k^\lambda(p,\rho), \ 0 \le \rho < 1 \right\},$$

$$V_k^\lambda(p,\rho) = \left\{ f(z) : f(z) \in A(p) \text{ and } \frac{(zf'(z))'}{f'(z)} \in P_k^\lambda(p,\rho), \ 0 \le \rho < 1 \right\}.$$

For $\lambda = 0$, $\rho = 0$ and $p = 1$, we obtain the well known classes R_k and V_k of analytic functions with bounded radius and bounded boundary rotations studied by Tammi [5] and Paatero [6] respectively. For details see [7-12]. Also it can easily be seen that $R_2^\lambda(0) = S_\lambda^*$ and $V_2^\lambda(0) = C_\lambda$.

Let us consider the integral operators

$$F_p(z) = \int_0^z pt^{p-1} \left(\frac{f_1(t)}{t^p} \right)^{\alpha_1} \cdots \left(\frac{f_n(t)}{t^p} \right)^{\alpha_n} dt \quad (1.1)$$

and

$$G_p(z) = \int_0^z pt^{p-1} \left[\frac{f_1'(t)}{pt^{p-1}} \right]^{\alpha_1} \cdots \left[\frac{f_n'(t)}{pt^{p-1}} \right]^{\alpha_n} dt, \quad (1.2)$$

where $f_i(z) \in A(p)$ and $\alpha_i > 0$ for all $i \in \{1,2,\cdots,n\}$.

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These operators, given by (1.1) and (1.2), are defined by Frasin [13]. If we take $p=1$, we obtain the integral operators $F_1(z)=F_n(z)$ and $G_1(z)=F_{\alpha_1\cdots\alpha_n}(z)$ introduced and studied by Breaz and Breaz [14] and Breaz *et al.* [15], for details see also [16-20]. Also for $p=n=1$, $\alpha_1=\alpha\in[0,1]$ in (1.1), we obtain the integral operator studied in [21] given as

$$\int_0^z\left(\frac{f(t)}{t}\right)^\alpha \mathrm{d}t,$$

and for $p=n=1$, $\alpha_1=\delta\in\mathbb{C}$, $|\delta|\le\frac{1}{4}$ in (1.2), we obtain the integral operator

$$\int_0^z\left(f'(t)\right)^\delta \mathrm{d}t,$$

discussed in [22,23].

In this paper, we investigate some propeties of the above integral operators $F_p(z)$ and $G_p(z)$ for the classes $V_k^\lambda(p,\rho)$ and $R_k^\lambda(p,\rho)$ respectively.

2. Main Result

Theorem 2.1. Let $f_i(z)\in R_k^\lambda(p,\rho)$ for $1\le i\le n$ with $0\le\rho<1$. Also let λ is real with $|\lambda|<\frac{\pi}{2}$, $\alpha_i>0$, $1\le i\le n$. If

$$0\le(\rho-p)\sum_{i=1}^n\alpha_i+p<1,$$

then $F_p(z)\in V_k^\lambda(p,\lambda_1)$ with

$$\lambda_1=(\rho-p)\sum_{i=1}^n\alpha_i+p. \tag{2.1}$$

Proof. From (1.1), we have

$$\frac{zF_p''(z)}{F_p'(z)}=(p-1)+\sum_{i=1}^n\alpha_i\left(\frac{zf_i'(z)}{f_i(z)}-p\right), \tag{2.2}$$

or, equivalently

$$\mathrm{e}^{i\lambda}\left(1+\frac{zF_p''(z)}{F_p'(z)}\right)=\mathrm{e}^{i\lambda}p\left(1-\sum_{i=1}^n\alpha_i\right)+\mathrm{e}^{i\lambda}\sum_{i=1}^n\alpha_i\frac{zf_i'(z)}{f_i(z)}. \tag{2.3}$$

Subtracting and adding $\rho\cos\lambda\sum_{i=1}^n\alpha_i$ on the right hand side of (2.3), we have

$$\begin{aligned}\mathrm{e}^{i\lambda}\left(1+\frac{zF_p''(z)}{F_p'(z)}\right)&=p\mathrm{e}^{i\lambda}+\left(\rho\cos\lambda-p\mathrm{e}^{i\lambda}\right)\sum_{i=1}^n\alpha_i\\&\quad+\sum_{i=1}^n\alpha_i\left[\mathrm{e}^{i\lambda}\frac{zf_i'(z)}{f_i(z)}-\rho\cos\lambda\right],\end{aligned} \tag{2.4}$$

Taking real part of (2.4) and then simple computation gives

$$\begin{aligned}&\int_0^{2\pi}\left|\mathrm{Re}\left[\mathrm{e}^{i\lambda}\left(1+\frac{zF_p''(z)}{F_p'(z)}\right)-\lambda_1\cos\lambda\right]\right|\mathrm{d}\theta\\&\le\sum_{i=1}^n\alpha_i\int_0^{2\pi}\left|\mathrm{Re}\left[\mathrm{e}^{i\lambda}\frac{zf_i'(z)}{f_i(z)}-\rho\cos\lambda\right]\right|\mathrm{d}\theta,\end{aligned} \tag{2.5}$$

where λ_1 is given by (2.1). Since $f_i(z)\in R_k^\lambda(p,\rho)$ for $1\le i\le n$, we have

$$\int_0^{2\pi}\left|\mathrm{Re}\left[\mathrm{e}^{i\lambda}\frac{zf_i'(z)}{f_i(z)}-\rho\cos\lambda\right]\right|\mathrm{d}\theta\le(p-\rho)\cos\lambda k\pi. \tag{2.6}$$

Using (2.6) and (2.1) in (2.5), we obtain

$$\int_0^{2\pi}\left|\mathrm{Re}\left[\mathrm{e}^{i\lambda}\left(1+\frac{zF_p''(z)}{F_p'(z)}\right)-\lambda_1\cos\lambda\right]\right|\mathrm{d}\theta\le(p-\lambda_1)\cos\lambda k\pi.$$

Hence $F_n(z)\in V_k^\lambda(p,\lambda_1)$ with λ_1 is given by (2.1).

By setting $p=1$ and $\lambda=0$ in Theorem 2.1, we obtain the following result proved in [9].

Corollory 2.2. Let $f_i(z)\in R_k(\rho)$ for $1\le i\le n$ with $0\le\rho<1$. Also let $\alpha_i>0$, $1\le i\le n$. If

$$0\le(\rho-1)\sum_{i=1}^n\alpha_i+1<1,$$

then $F_n(z)\in V_k(\lambda_1)$ and λ_1 is given by (2.1).

Now if we take $k=2$ and $\lambda=0$ in Theorem 2.1, we obtain the following result.

Corollory 2.3. Let $f_i(z)\in S_p^*(\rho)$ for $1\le i\le n$ with $0\le\rho<1$. Also let $\alpha_i>0$, $1\le i\le n$. If

$$0\le(\rho-p)\sum_{i=1}^n\alpha_i+p<1,$$

then $F_p(z)\in C_p(\lambda_1)$ and λ_1 is given by (2.1).

Letting $p=n=1$, $\lambda=0$, $\alpha_1=\alpha$ and $f_1(z)=f(z)$ in Theorem 2.1, we have.

Corollory 2.4. Let $f(z)\in R_k(\rho)$ with $0\le\rho<1$. Also let $\alpha>0$. If

$$0\le(\rho-1)\alpha+1<1,$$

then

$$\int_0^z\left(\frac{f(t)}{t}\right)^\alpha\mathrm{d}t\in V_k(\lambda_1)$$

with $\lambda_1=(\rho-1)\alpha+1$.

Theorem 2.5. Let $f_i(z)\in V_k^\lambda(p,\rho)$ for $1\le i\le n$ with $0\le\rho<1$. Also let λ is real is real with $|\lambda|<\frac{\pi}{2}$, $\alpha_i>0$, $1\le i\le n$. If

$$0 \le (\rho - p)\sum_{i=1}^{n}\alpha_i + p < 1,$$

then $G_p(z) \in V_k^\lambda(p,\lambda_1)$ and λ_1 is given by (2.1).

Proof. From (1.2), we have

$$1+\frac{zG_p''(z)}{G_p'(z)} = p+\sum_{i=1}^{n}\alpha_i\left(\frac{zf_i''(z)}{f_i'(z)}+1\right)-p\sum_{i=1}^{n}\alpha_i,$$

or, equivalently

$$e^{i\lambda}\left(1+\frac{zG_p''(z)}{G_p'(z)}\right) = pe^{i\lambda}\left(1-\sum_{i=1}^{n}\alpha_i\right)+\sum_{i=1}^{n}\alpha_i e^{i\lambda}\left(1+\frac{zf_i''(z)}{f_i'(z)}\right).$$

This relation is equivalent to

$$e^{i\lambda}\left(1+\frac{zG_p''(z)}{G_p'(z)}\right) = pe^{i\lambda}+\left(\rho\cos\lambda - pe^{i\lambda}\right)\sum_{i=1}^{n}\alpha_i + \sum_{i=1}^{n}\alpha_i\left[e^{i\lambda}\left(1+\frac{zf_i''(z)}{f_i'(z)}\right)-\rho\cos\lambda\right]. \quad (2.7)$$

Taking real part of (2.7) and then simple computation gives us

$$\int_0^{2\pi}\left|\mathrm{Re}\left[e^{i\lambda}\left(1+\frac{zG_p''(z)}{G_p'(z)}\right)-\lambda_1\cos\lambda\right]\right|d\theta \le \sum_{i=1}^{n}\alpha_i\int_0^{2\pi}\left|\mathrm{Re}\left[e^{i\lambda}\left(1+\frac{zf_i''(z)}{f_i'(z)}\right)-\rho\cos\lambda\right]\right|d\theta, \quad (2.8)$$

where λ_1 is given by (2.1). Since $f_i(z) \in V_k^\lambda(p,\rho)$ for $1 \le i \le n$, we have

$$\int_0^{2\pi}\left|Re\left[e^{i\lambda}\left(1+\frac{zf_i''(z)}{f_i'(z)}\right)-\rho\cos\lambda\right]\right|d\theta \le (p-\rho)\cos\lambda k\pi. \quad (2.9)$$

Using (2.9) in (2.8), we obtain

$$\int_0^{2\pi}\left|Re\left[e^{i\lambda}\left(1+\frac{zG_p''(z)}{G_p'(z)}\right)-\lambda_1\cos\lambda\right]\right|d\theta \le (p-\lambda_1)\cos\lambda k\pi.$$

Hence $G_p(z) \in V_k^\lambda(p,\lambda_1)$ with λ_1 is given by (2.1).

By setting $k = 2$ and $\lambda = 0$ in Theorem 2.5, we obtain the following result.

Corollory 2.6. Let $f_i(z) \in C_p(\rho)$ for $1 \le i \le n$ with $0 \le \rho < 1$. Also let $\alpha_i > 0$, $1 \le i \le n$. If

$$0 \le (\rho - p)\sum_{i=1}^{n}\alpha_i + p < 1,$$

then $G_p(z) \in C_p(\lambda_1)$ with λ_1 is given by (2.1).

Letting $p = n = 1$, $\lambda = 0$, $\alpha_1 = \delta$ and $f_1(z) = f(z)$ in Theorem 2.5, we have.

Corollory 2.7. Let $f(z) \in V_k(\rho)$ with $0 \le \rho < 1$. Also let $\delta > 0$. If $0 \le (\rho - 1)\delta + 1 < 1$, then

$$\int_0^z (f'(t))^\delta \, dt \in V_k(\lambda_1)$$

with $\lambda_1 = (\rho - 1)\delta + 1$.

REFERENCES

[1] L. Spacek, "Prispěvek k Teorii Funkei Prostych," *Časopis pro pěstováni matematiky a fysiky*, Vol. 62, No. 2, 1933, pp. 12-19.

[2] M. S. Robertson, "Univalent Functions *f*(*z*) for wich *zf*′(*z*) Is Spiral-Like," *Michigan Mathematical Journal*, Vol. 16, No. 2, 1969, pp. 97-101.

[3] K. S. Padmanabhan and R. Parvatham, "Properties of a Class of Functions with Bounded Boundary Rotation," *Annales Polonici Mathematici*, Vol. 31, No. 1, 1975, pp. 311-323.

[4] B. Pinchuk, "Functions with Bounded Boundary Rotation," *Israel Journal of Mathematics*, Vol. 10, No. 1, 1971, pp. 7-16. doi:10.1007/BF02771515

[5] O. Tammi, "On the Maximization of the Coefficients of Schlicht and Related Functions," *Annales Academiae Scientiarum Fennicae. Series A I. Mathematica*, Vol. 114, No. 1, 1952, p. 51

[6] V. Paatero, "Uber Gebiete von Beschrankter Randdrehung," *Annales Academiae Scientiarum Fennnicae*, Vol. 37-39, No. 9, 1933.

[7] M. Arif, M. Ayaz and S. I. Ali Shah, "Radii Problems for Certain Classes of Analytic Functions with Fixed Second Coefficients," *World Applied Sciences Journal*, Vol. 13, No. 10, 2011, pp. 2240-2243.

[8] K. I. Noor, "On Some Subclasses of Fuctions with Bounded Boundary and Bounded Radius Rotation," *Pan American Mathematical Journal*, Vol. 6, No. 1, 1996, pp. 75-81.

[9] K. I. Noor, M. Arif and W. Haq, "Some Properties of Certain Integral Opertors," *Acta Universitatis Apulensis*, Vol. 21, 2010, pp. 89-95.

[10] K. I. Noor, M. Arif and A. Muhammad, "Mapping Properties of Some Classes of Analytic Functions under an Integral Operator," *Journal of Mathematical Inequalities*, Vol. 4, No. 4, 2010, pp. 593-600.

[11] K. I. Noor, W. Haq, M. Arif and S. Mustafa, "On Bounded Boundary and Bounded Radius Rotations," *Journal of Inequalities and Applications*, 2009, Article ID: 813687.

[12] K. I. Noor, S. N. Malik, M. Arif and M. Raza, "On Bounded Boundary and Bounded Radius Rotation Related with Janowski Function," *World Applied Sciences Journal*, Vol. 12, No. 6, 2011, pp. 895-902.

[13] B. A. Frasin, "New General Integral Operators of *p*-Valent Functions," *Journal of Inequatilies Pure and Applied Mathematics*, Vol. 10, No. 4, 2009

[14] D. Breaz and N. Breaz, "Two Integral Operators," *Studia Universitatis Babes-Bolyai*, *Mathematica*, *Clunj-Napoca*,

Vol. 47, No. 3, 2002, pp. 13-21.

[15] D. Breaz, S. Owa and N. Breaz, "A New Integral Univalent Operator," *Acta Universitatis Apulensis*, Vol. 16, 2008, pp. 11-16.

[16] N. Breaz, V. Pescar and D. Breaz, "Univalence Criteria for a New Integral Operator," *Mathematical and Computer Modelling*, Vol. 52, No. 1-2, 2010, pp. 241-246. doi:10.1016/j.mcm.2010.02.013

[17] B. A. Frasin, "Convexity of Integral Operators of *p*-Valent Functions," *Mathematical and Computer Modelling*, Vol. 51, No. 5-6, 2010, pp. 601-605.

[18] B. A. Frasin, "Some Sufficient Conditions for Certain Integral Operators," *Journal of Mathematics and Inequalities*, Vol. 2. No. 4, 2008, pp. 527-535.

[19] G. Saltik, E. Deniz and E. Kadioglu, "Two New General *p*-Valent Integral Operators," *Mathematical and Computer Modelling*, Vol. 52, No. 9-10, 2010, pp. 1605-1609. doi:10.1016/j.mcm.2010.06.025

[20] R. M. Ali and V. Ravichandran, "Integral Operators on Ma-Minda Type Starlike and Convex Functions," *Mathematical and Computer Modelling*, Vo. 53, No. 5-6, 2011, pp. 581-586. doi:10.1016/j.mcm.2010.09.007

[21] S. S. Miller, P. T. Mocanu and M. O. Reade, "Starlike Integral Operators," *Pacific Journal of Mathematics*, Vol. 79, No. 1, 1978, pp.157-168.

[22] Y. J. Kim and E. P. Merkes, "On an Integral of Powers of a Spirallike Function," *Kyungpook Mathematical Journal*, Vol. 12, No. 2, 1972, pp. 249-252.

[23] N. N. Pascu and V. Pescar, "On the Integral Operators of Kim-Merkes and Pfaltzgraff," *Mathematica*, *Universitatis Babes-Bolyai Cluj-Napoca*, Vol. 32, No. 2, 1990, pp. 185-192.