

General Periodic Boundary Value Problem for Systems

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Received May 20, 2012; revised July 2, 2012; accepted July 9, 2012

ABSTRACT

The paper deals with the existence of nonzero periodic solution of systems
$$\begin{cases} u'' + k^2 u = f(t, u), & t \in (0, T), T > 0, \\ u(0) = \alpha u(T), & u'(0) = \beta u'(T), \end{cases}$$

where $k \in \left(0, \frac{\pi}{T}\right)$, α, β are $n \times n$ real nonsingular matrices, $u = (u_1, \dots, u_n)$,

$f(t, u) = (f_1(t, u), \dots, f_n(t, u)) \in C([0, T] \times \mathbb{R}_+^n, \mathbb{R}_+)$ is periodic of period T in the t variable are continuous and nonnegative functions. We determine the Green's function and prove that the existence of nonzero periodic positive solutions if one of $\lim_{\|u\| \rightarrow 0} \frac{f_i(t, u)}{\|u\|} > \lambda_1$. In addition, if all $\lim_{\|u\| \rightarrow 0} \frac{f_i(t, u)}{\|u\|} < \lambda_1$, $i = (1, \dots, n)$ where λ_1 is the principle eigenvalues of the corresponding linear systems. The proof based on the fixed point index theorem in cones. Application of our result is given to such systems with specific nonlinearities.

Keywords: Systems; Principle Eigenvalues; Positive Solutions; Green's Function; Fixed Point Index Theorem in Cones

1. Introduction

In this paper, we study the existence of nonzero positive periodic solution of systems

$$\begin{cases} u'' + k^2 u = f(t, u), & t \in (0, T), T > 0, \\ u(0) = \alpha u(T), & u'(0) = \beta u'(T), \end{cases} \quad (1.1)$$

where $k \in \left(0, \frac{\pi}{T}\right)$, α, β are $n \times n$ real nonsingular matrices, $u = (u_1, \dots, u_n)$, and

$$f(t, u) = (f_1(t, u), \dots, f_n(t, u)) \in C([0, T] \times \mathbb{R}_+^n, \mathbb{R}_+)$$

are periodic of period T in the t , are continuous and nonnegative functions and $\mathbb{R}_+ = [0, \infty)$.

Beginning with the paper of Erbe and Palamides [1], obtained the sufficient conditions for existence solution of the systems of nonlinear boundary value problem

$$\begin{cases} u'' = f(t, u, u'), & t \in (0, 1), \\ u(0) = Q_0 u(1), & u'(0) = Q_1 u'(1), \end{cases} \quad (1.2)$$

where $f: [0, 1] \times E^{2n} \rightarrow E^n$ is continuous (E^n is n -dimensional real Euclidean space) and Q_0, Q_1 are nonsingular matrices, with Q_0 orthogonal matrix, Erbe and

Palamides generalize earlier conditions of Bebernes and Schmitt [2] for periodic case. Erbe and Schmitt [3] extend the results in [1] established the sufficient conditions for existence solution of the systems (1.2). The results in [1,3] were obtained via a modifications of a degree-theoretic approach and Leray-Schuader degree and eliminates the modified function approach respectively. None of these earlier results use Green's function and the first eigenvalues of the corresponding to the linear systems of (1.1).

There has been progress in the study of the existence of positive solutions of system problem. If $\alpha = \beta = I =$ identity, then (1,1) reduces to the usual periodic boundary value problem for which the literature in both the scalar and systems versions is very extensive (We refer to [4-22] and references therein). For instance a recent paper, Wang [4] obtained the existence of periodic solution of a class of non-autonomous second-order systems

$$u'' + \mu u + f(t, u) = 0, \quad t \in (0, T), T > 0$$

where

$$u = (u_1, \dots, u_n),$$

$$f(t, u) = (f_1(t, u), \dots, f_n(t, u)) \in C([0, T] \times \mathbb{R}^n, \mathbb{R}^n)$$

are periodic of period T in the t , and μ is a constant,

if $\lim_{\|u\| \rightarrow 0} \frac{f_i(t,u)}{\|u\|} = 0, i = (1, \dots, n)$ and $f(t,u)$, is bounded

below or above for appropriate ranges of μ , via fixed point theorem in cones. Franco and Webb [5] established the existence of T -periodic solutions for systems of (1.1) with $\alpha = \beta = I = \text{identity}$, in the boundary conditions, where $k \in \mathbb{R}$ and $f(t,u)$ is a continuous vector valued function, periodic in t with period T , and f is allowed to have a singularity when $u = 0$. Non-singular systems, which are included in the same framework that we study here (*i.e.*, they can be reduced to a Hammerstein integral system with positive kernel), have been considered using some other approaches based on fixed point theorems in conical shells, but previously it has always been assumed that the nonlinearity $f(t,u)$ has a constant sign behaviour: $f_i(t,u)\theta_i \geq 0$ where $\theta_i \in \{1, -1\}$ for $0 \leq i \leq n$ (see [6]) with $\alpha = \beta = I = \text{identity}$, in the boundary conditions. For systems problem see also [7-12] and references therein.

Even in the scalar case the existence of periodic solutions for problems with nonsingular and singular case has commanded much attention in recent years (see [13-22] and references therein. In particular, in [13-15] fixed point theorems in conical shells are used to obtain existence and multiplicity results, some of these are improved in this paper. In this notes, we prove result in the case where f has no singularity. In scalar case problem see [16-22] and references therein.

Motivated by these problems mentioned above, we study the existence of nonzero positive solution of (1.1) while we assume that if one components satisfy

$\lim_{\|u\| \rightarrow 0} \frac{f_{i_0}(t,u)}{\|u\|} > \lambda_1, i_0 \in \{1, \dots, n\}$, and all components of

nonlinearity are $\lim_{\|u\| \rightarrow 0} \frac{f_i(t,u)}{\|u\|} < \lambda_1, i = \{1, \dots, n\}$, where

λ_1 , is the largest characteristic value of the linear system corresponding to (1.1). The approach is to use the theory of fixed point index for compact maps defined on cones [23]. To apply this theory one needs to find the Green's function. Our purpose here is prove that (1.1) has non-trivial nonnegative solution, assuming the following conditions:

$$(H1) \quad 0 < k < \frac{\pi}{T},$$

$$(H2) \quad \alpha, \beta \geq 0, \alpha \beta \geq 0, \alpha - \beta > 0, \alpha + \beta > 0$$

and

$$G(t, S) = \begin{cases} 2\alpha\beta \sin k(t-s) + (\alpha - \beta) \sin k(T-s-t) + (\alpha + \beta) \sin k(T+s-t), & 0 \leq s \leq t \leq T, \\ 2\alpha\beta \sin k(s-t) + (\alpha - \beta) \sin k(T-s-t) + (\alpha + \beta) \sin k(T+t-s), & 0 \leq t \leq s \leq T. \end{cases} \tag{2.1^*}$$

$$2k(1 + \alpha\beta - (\alpha + \beta) \cos kT) > 0,$$

$$(H3) \quad f(t,u) = (f_1(t,u), \dots, f_n(t,u)) \in C([0, T] \times \mathbb{R}_+^n, \mathbb{R}_+)$$

are continuous and periodic of period T in the t variable and $f_i(t,u) \neq 0, i = (1, \dots, n)$ on any subinterval of $[0, T]$.

(H4) There exists $i_0 \in \{1, \dots, n\}$ such that

$$\lim_{\|u\| \rightarrow 0} \frac{f_{i_0}(t,u)}{\|u\|} > \lambda_1,$$

where $u = (u_1, \dots, u_n) \in \mathbb{R}_+^n$ and $\|u\| = \sum_{i=1}^n \sup |u_i|$, and

λ_1 is the largest characteristic value of the linear system corresponding to (1.1),

(H5) For all $i \in (1, \dots, n)$,

$$\lim_{\|u\| \rightarrow 0} \frac{f_i(t,u)}{\|u\|} < \lambda_1,$$

where $u = (u_1, \dots, u_n) \in \mathbb{R}_+^n$ and $\|u\| = \sum_{i=1}^n \sup |u_i|$, and

λ_1 is the largest characteristic value of the linear system corresponding to (1.1).

Remark 1.1. The assumptions (H4) and (H5) appeared in Lan [24].

Remark 1.2. The nonzero positive solution has been studied by Lan [24] and Hai and Wang [25].

Throughout this paper, we will use the notation

$\mathbb{R}_+ = [0, \infty), \mathbb{R}_+^n = \prod_{i=1}^n \mathbb{R}_+$, and denote by $|u| = \sum_{i=1}^n |u_i|$

the usual norm of \mathbb{R}_+^n for $u = (u_1, \dots, u_n) \in \mathbb{R}_+^n$, $f(t,u) = (f_1(t,u), \dots, f_n(t,u))$ and $i = (1, \dots, n)$.

2. Preliminaries

In this section, we shall introduce some basic lemmas which are used throughout this paper.

Lemma 2.1. Let (H1) and (H2) holds. Let $\Delta = 2k(1 + \alpha\beta - (\alpha + \beta) \cos kT)$ then for $v_i \in C[0, T]$, the periodic boundary value problems problem

$$\begin{cases} u_i'' + k^2 u_i = v_i, & t \in (0, T), T > 0, \\ u_i(0) = \alpha u_i(T), & u_i'(0) = \beta u_i'(T), \end{cases} \tag{2.1}$$

has a unique solution

$$u_i = \int_0^T G_i(t, s) v_i ds,$$

where

where $i = (1, \dots, n)$.

Proof. Consider the scalar periodic boundary value problems of (2.1) and let $u_1(t)$ and $u_2(t)$ be linearly distinct solutions of the scalar equation of (2.1) and consider the function

$$F(t, s) = Au_1(t) + Bu_2(t) \pm \left(\frac{u_1(t)u_2(s) - u_1(s)u_2(t)}{2(u_1(s)u_2'(s) - u_1'(s)u_2(s))} \right)$$

where the positive sign is taken when $0 \leq t \leq s$, and the negative sign when $s \leq t \leq T$ we can obtain this result by routine substitutions of scalar boundary conditions, we do not state it here. \square

Lemma 2.2. Let conditions (H1) hold, then $G_i(t, s)$, $i = (1, \dots, n)$, is continuous and positive on $[0, T] \times [0, T]$, and we can find its positive minimum value A_i and maximum value B_i of $G_i(t, s)$, $i = (1, \dots, n)$ by

$$0 < A_i = \min_{t,s \in [0,T]} G_i(t, s), \quad B_i = \max_{t,s \in [0,T]} G_i(t, s),$$

and

$$0 < \sigma_i = \frac{A_i}{B_i}, \quad i = (1, \dots, n).$$

Proof. It is easy to check that $G_i(t, s)$, is continuous and positive on $[0, T] \times [0, T]$, $i = (1, \dots, n)$. \square

It is clear that the problem (1.1) has a solution u if and only if u solves the operator equation

$$Au(t) := \int_0^T G_i(t, s) f_i(s, u(s)) ds \tag{I}$$

It is easy to verify that the operator A is completely continuous.

We define $L : C([0, T]; \mathbb{R}^n) \rightarrow C([0, T]; \mathbb{R}^n)$ corresponding to linear equation of (1.1) by

$$Lu(t) = \int_0^T G_1(t, s) u_1(s) ds, \dots, \int_0^T G_n(t, s) u_n(s) ds, \tag{2.2}$$

$t \in [0, T]$

where $u(t) = (u_1(t), \dots, u_n(t)) \in \mathbb{R}^n$ and $G_i(t, s)$ is Green's function define in (2.1), and define

$$(\Gamma u_1)(t) = \int_0^T G_1(t, s) u_1(s) ds$$

and

$$(\Gamma u_n)(t) = \int_0^T G_n(t, s) u_n(s) ds,$$

where $\Gamma : C([0, T]) \rightarrow C([0, T])$ is completely continuous.

Remark 2.1. Equations (2.2) appeared in [26].

It is known that $\Gamma^* : C([0, T]) \rightarrow C([0, T])$, is a

bounded and surjective linear operator and has a unique extension, denoted by Γ , to $C([0, T])$. We write

$$e = \Gamma^* v_0, v_0(t) \equiv 1, v \in C([0, T]). \tag{2.3}$$

It is known that e is an interior point of the positive cone P_1 in $C([0, T])$, where

$$P_1 = \{u \in C([0, T]) : u(t) \geq 0, t \in [0, T]\}. \tag{2.4}$$

Lemma 2.3. [24] $\Gamma : C([0, T]) \rightarrow C([0, T])$ is a compact linear operator such that $\Gamma(P_1) \subset P_1$ and for each $v \in P_1 \setminus \{0\}$ there exists $\alpha_v > 0$ such that $\Gamma v \geq \alpha_v e$.

By Lemma 2.3 and the well-known Krein-Rutman theorem (see [23, Theorem 3.1] or [27], it is easy to see that $\mu_1 \in (0, \infty)$ and there exists $\varphi_1 \in P_1 \setminus \{0\}$ such that

$$\varphi_1 = \mu_1 \Gamma \varphi_1 \tag{2.5}$$

where $\mu_1 = 1/r(\Gamma)$ and $r(\Gamma)$ is the spectral radius of Γ .

We use the following maximum norm in \mathbb{R}^n :

$$|u| = \max \{|u_i| : i \in (1, \dots, n)\} \tag{2.6}$$

where $u(t) = (u_1(t), \dots, u_n(t))$. We denote by $(C([0, T]; \mathbb{R}^n))$ the Banach space of continuous functions from $[0, T]$ into \mathbb{R}^n with norm

$$\|u\| = \max \{\|u_i\| : i \in (1, \dots, n)\},$$

where $u(t) = (u_1(t), \dots, u_n(t))$ for $t \in [0, T]$.

We use the standard positive cone in $(C([0, T]); \mathbb{R}^n)$ defined by

$$P = C([0, T]; \mathbb{R}^n) \tag{2.7}$$

We can write $L : C([0, T]; \mathbb{R}^n) \rightarrow C([0, T]; \mathbb{R}^n)$ defined in (2.2) as operator equations

$$(Lu)(t) := ((\Gamma u_1)(t), \dots, (\Gamma u_n)(t)) \tag{2.8}$$

where $(\Gamma u_1)(t)$ and $(\Gamma u_n)(t)$ are define above and define a Nemytskii operator

$$(Fu)(t) := (f_1(t, u(t)), \dots, f_n(t, u(t))). \tag{2.9}$$

It is easy to verify that (1.1) is equivalent to the following fixed point equation:

$$u(t) = (LF)(t) := Au(t), t \in [0, T] \tag{2.10}$$

Note that (2.10) same as (I).

Recall that a solution $u \in (C([0, T]); \mathbb{R}^n)$ of (1.1) is said to be a nonzero positive solution if $u \in P \setminus \{0\}$; that is, $u \in (C([0, T]); \mathbb{R}^n)$ and $u(t) = (u_1(t), \dots, u_n(t))$ satisfies $u_i(t) \geq 0$ for $t \in [0, T]$ and $i = (1, \dots, n)$ and there exists such that $k \in (1, \dots, n)$ $u_k \neq 0$ on $[0, T]$.

Let $\rho > 0$ and let

$$P_\rho = \{u \in P : \|u\| < \rho\}, \quad \bar{P}_\rho = \{u \in P : \|u\| \leq \rho\}$$

and

$$\partial P_\rho = \{u \in P : \|u\| = \rho\}.$$

We need some results from the theory of the fixed point index for compact maps defined on cones in a Banach space X (see [23]).

Lemma 2.4. Assume that $A: \bar{P}_\rho \rightarrow P$ is a compact map. Then the following results hold:

- 1) If there exists $x_0 \in P \setminus \{0\}$ such that $u \neq Au + vx_0$ for $u \in \partial P_\rho$ and $v \geq 0$, then $i_\rho(A, P_\rho) = 0$.
- 2) If $u \neq \zeta Au$ for $u \in \partial P_\rho$ and $\zeta \in (0, 1]$, then $i_\rho(A, P_\rho) = 1$.
- 3) If $i_\rho(A, P_\rho) = 1$ and $i_\rho(A, P_{\rho_0}) = 0$ for some $\rho_0 \in (0, \rho)$, then A has a fixed point in $P_\rho \setminus \bar{P}_{\rho_0}$.

Now, we are in a position to give our main result and proof analogous results were established in [24].

Theorem 2.1. Assume that (H1) - (H5) holds. $\mu_1 \in (0, \infty)$ be the same as in (2.5). Assume that the following conditions hold:

$\left((f_{i_0})_0 \right)_{\rho_0}$. There exist $i_0 \in (1, \dots, n)$, $\varepsilon > 0$ and $\rho_0 > 0$ such that $f_{i_0}(t, u) \geq (\mu_1 + \varepsilon)u_{i_0}$ for $t \in [0, T]$ and all $u \in \mathbb{R}_+^n$ with $|u| \in [0, \rho_0]$.

$\left((f_i^\infty)_{\rho_1} \right)$. There exist $\varepsilon > 0$ and $\rho_1 > 0$ such that for $i \in (1, \dots, n)$, $f_i(t, u) \leq (\mu_1 - \varepsilon)|u|$ for $t \in [0, T]$ and all $u \in \mathbb{R}_+^n$ with $|u| \geq \rho_1$. Then (1.1) has a nonzero positive solution in P .

Proof. By Lemma 2.1, Lemma 2.2 and Lemma 2.3, $L: C([0, T]; \mathbb{R}^n) \rightarrow C([0, T]; \mathbb{R}^n)$ is compact and satisfies $L(P) \subset P$.

This, together with the continuity of f_i in (H3), implies that $A: P \rightarrow P$ is compact. Without loss of generalization, we assume that $u \neq Au$ for $u \in \partial P_{\rho_0}$. Let $\varphi = (\varphi_1, \dots, \varphi_1)$, where φ_1 is the same as in (2.5). We prove that

$$u \neq Au + v\varphi, \forall u \in \partial P_{\rho_0}, v \geq 0. \tag{2.11}$$

In fact, if not, there exist $u \in \partial P_{\rho_0}$ and $v \geq 0$ such that $u = Au + v\varphi$. Then

$$u_{i_0}(t) = \Gamma(f_{i_0}(t, u(t))) + v\varphi_1, \forall t \in [0, T] \tag{2.12}$$

It follows that $u_{i_0}(t) \geq v\varphi_1$ for $\forall t \in [0, T]$. Let

$$\tau = \sup \{ \tau_0 \geq 0 : u_{i_0}(t) \geq \tau_0\varphi_1, \forall t \in [0, T] \}.$$

Then $0 < v \leq \tau < \infty$ and $u_{i_0}(t) \geq \tau\varphi_1$, $\forall t \in [0, T]$.

This, together with (2.12), $\left((f_{i_0})_0 \right)_{\rho_0}$ and (2.5), implies that for all $t \in [0, T]$

$$\begin{aligned} u_{i_0}(t) &= \Gamma((\mu_1 + \varepsilon)u_{i_0}(t)) \geq (\mu_1 + \varepsilon)\tau\Gamma\varphi_1(t) \\ &\geq (\mu_1 + \varepsilon)\tau(\varphi_1(t)/\mu_1). \end{aligned}$$

Hence, we have $\tau \geq (\mu_1 + \varepsilon)(\tau/\mu_1)$, a contradiction. It follows from (2.11) and Lemma 2.4 (1) $i_\rho(A, P_\rho) = 0$.

For each $i \in (1, \dots, n)$, by the continuity of f_i , there exists $b_i > 0$ such that $f_i(t, u) \leq b_i$ for $t \in [0, T]$, $u \in \mathbb{R}_+^n$ with $|u| \leq \rho_1$.

This, together with $\left((f_i^\infty)_{\rho_1} \right)$ implies that, for each $i \in (1, \dots, n)$ $f_i(t, u) \leq b_i + (\mu_1 - \varepsilon)|u|$ for $t \in [0, T]$ and all

$$u \in \mathbb{R}_+^n \tag{2.13}$$

Since

$$(r(\mu_1 - \varepsilon)\Gamma) = (\mu_1 - \varepsilon)r(\Gamma) < 1,$$

$(I - (\mu_1 - \varepsilon)\Gamma)^{-1}$ exists and is bounded and satisfies $(I - (\mu_1 - \varepsilon)\Gamma)^{-1}P_1 \subset P_1$.

Let $\hat{b}_i = b_i$ for $t \in [0, T]$,

$$\rho_1 = \max \{ \| \Gamma \hat{b}_i \| : i \in (1, \dots, n) \}$$

and

$$\rho^* = \| (I - (\mu_1 - \varepsilon)\Gamma)^{-1} \hat{\rho}_1 \|,$$

where $\hat{\rho}_1(t) = \rho_1$ for $t \in [0, T]$. Let $\rho > \rho^*$. We prove

$$u \neq \zeta Au, u \in \partial P_\rho, \zeta \in (0, 1].$$

Indeed, if not, there exist $u \in \partial P_\rho$ and $\zeta \in (0, 1]$ such that $u = \zeta Au$. By (2.13), we have for each $i \in (1, \dots, n)$,

$$u_i(t) \leq \rho_1 + (\mu_1 - \varepsilon)(\Gamma|u|)(t) \text{ for } t \in [0, T],$$

where $|u|(t) = \max \{ |u_i(t)| : i \in (1, \dots, n) \}$. Taking the maximum in the above inequality implies that

$$|u|(t) \leq \rho_1 + (\mu_1 - \varepsilon)(\Gamma|u|)(t)$$

for $t \in [0, T]$, and

$$(I - (\mu_1 - \varepsilon)\Gamma)|u|(t) \leq \rho_1$$

for $t \in [0, T]$.

Since

$$(I - (\mu_1 - \varepsilon)\Gamma)^{-1}P_1 \subset P_1,$$

$$|u|(t) \leq (I - (\mu_1 - \varepsilon)\Gamma)^{-1} \hat{\rho}_1(t)$$

for $t \in [0, T]$.

Hence, we have

$$\rho = \|u\| = \max \{ |u|(t) : t \in [0, T] \} \leq \rho^* < \rho.$$

a contradiction. By (2.14) and Lemma 2.3 (2),

$i_\rho(A, P_\rho) = 1$. By Lemma 2.4 (3), (1.1) has a solution in $P_\rho \setminus P_{\rho_0}$. \square

3. Application

Let the systems

$$\begin{cases} u'' + k^2 u = (a_i u_i^{1/3} + b_i u_i^{1/3}) h_i(\hat{u}_i(t)), t \in (0, T), T > 0, \\ u(0) = \alpha u(T), \quad u'(0) = \beta u'(T), \end{cases} \quad (3.1)$$

where $k \in \left(0, \frac{\pi}{T}\right)$, $\hat{u}_i = (u_1, \dots, u_{i-1}, u_{i+1}, \dots, u_n)$ and

$i \in (1, \dots, n)$. Assume that the following conditions hold:

- 1) For each $i \in (1, \dots, n)$, $a_i, b_i \geq 0$ and $h_i : \mathbb{R}_+^{n-1} \rightarrow \mathbb{R}_+$ is continuous and let

$$\omega := \sup \{ h_i(\hat{u}_i(t)) : \hat{u}_i(t) \in \mathbb{R}_+^{n-1}, i \in (1, \dots, n) \} < \infty.$$

- 2) There exists $i_0 \in (1, \dots, n)$ such that $a_{i_0} + b_{i_0} > 0$

and $\xi := \min \{ h_i(\hat{u}_i(t)) : \hat{u}_i(t) \in \mathbb{R}_+^{n-1}, i \in (1, \dots, n) \} > 0$.

Then equation (3.1) have a nonzero positive solution in P .

Proof. For each $i \in (1, \dots, n)$, we define a function $f_i(t, u) : [0, T] \times \mathbb{R}_+^n \rightarrow \mathbb{R}_+$ by

$$f_i(t, u) = (a_i u_i^{1/3} + b_i u_i^{1/3}) h_i(\hat{u}_i(t)).$$

Let $\varepsilon > 0$ and

$$0 < \rho_0 \leq \min \left\{ 1, \left(\frac{(a_{i_0} + b_{i_0}) \xi}{\mu_1 + \varepsilon} \right)^{3/2} \right\}.$$

Then for $t \in [0, T]$ and $u \in \mathbb{R}_+^n$ with $|u| \in [0, \rho_0]$ and $u_{i_0} \neq 0$,

$$\begin{aligned} f_{i_0}(t, u) &\geq \left(\frac{a_{i_0}}{u_{i_0}^{3/2}} + \frac{b_{i_0}}{u_{i_0}^{3/2}} \right) \xi u_{i_0} \geq \left(\frac{a_{i_0}}{\rho_0^{3/2}} + \frac{b_{i_0}}{\rho_0^{3/2}} \right) \xi u_{i_0} \\ &\geq (\mu_1 + \varepsilon) u_{i_0}. \end{aligned}$$

Hence, $\left((f_{i_0})_{\rho_0} \right)$ holds. Let $\rho_1 > \rho_0$,

$$\rho_1 \geq \max \left\{ \left(\frac{(a_{i_0} + b_{i_0}) \xi}{\rho} \right)^{3/2}, i \in (1, \dots, n) \right\}.$$

Then for $t \in [0, T]$ and $u \in \mathbb{R}_+^n$ with $|u| \in [\rho_1, \infty]$,

$$f_i(t, u) \leq [a_i |u_i^{1/3}| + b_i |u_i^{1/3}|] u_i \leq a_i |u_i^{1/3}| + b_i |u_i^{1/3}| \omega$$

for $i \in (1, \dots, n)$ it follows that $\left(f_i^\infty \right)_{\rho_1}$ holds. The result follows from Theorem 2.1. \square

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