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Limit Theorems for a Storage Process with a Random Release Rule

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ABSTRACT

We consider a discrete time Storage Process X_n with a simple random walk input S_n and a random release rule given by a family $\{U_x, x \geq 0\}$ of random variables whose probability laws $\{\mu_x, x \geq 0\}$ form a convolution semigroup of measures, that is, $\mu_x \times \mu_y = \mu_{x+y}$. The process X_n obeys the equation: $X_0 = 0, U_0 = 0, X_n = S_n - U_{S_n}, n \geq 1$. Under mild assumptions, we prove that the processes U_{S_n} and X_n are simple random walks and derive a SLLN and a CLT for each of them.

Keywords: Storage Process; Random Walk; Strong Law of Large Numbers; Central Limit Theorem

1. Introduction and Assumptions

The formal structure of a general storage process displays two main parts: the input process and the release rule. The input process, mostly a compound Poisson process $A(t)$, describes the material entering in the system during the interval $[0,t]$. The release rule is usually given by a function $r(x)$ representing the rate at which material flows out of the system when its content is x. So the state $X(t)$ of the system at time t obeys the well known equation:

$X(t) = X(0) + A(t) - \int_0^t r(X(s))\mathrm{d}s$.

Limit theorems and approximation results have been obtained for the process $X(t)$ by several authors, see [1-5] and the references therein. In this paper we study a discrete time new storage process with a simple random walk input S_n and a random release rule given by a family of random variables $\{U_x, x \geq 0\}$, where U_x has to be interpreted as the amount of material removed when the state of the system is x. Hence the evolution of the system obeys the following equation: $X_0 = 0$, $U_0 = 0$, $X_n = S_n - U_{S_n}$, $n \geq 1$. where $S_0 = 0$, $S_n = Z_1 + Z_2 + \cdots + Z_n$, for i.i.d. positive random variables Z_n, with $E(Z_1) = \alpha > 0$, and $\sigma_{Z_1}^2 = \sigma^2 < \infty$.

We will make the following assumptions:

1.1. The probability distributions $\{\mu_x, x \geq 0\}$ of the random variables $\{U_x, x \geq 0\}$ form a convolution semigroup of measures:

$$\forall x, y \geq 0, \mu_x \times \mu_y = \mu_{x+y}, \quad (1.1)$$

We will assume that for each x, μ_x is supported by the interval $[0,x]$, that is, $\forall x \in \mathrm{R}_+$, $\mu_x[0,x] = 1$. Consequently, for $x \leq y$ the distribution of $U_y - U_x$ is the same as that of U_{y-x}, (see **2.2** (ii)).

1.2. Also we will need some smoothness properties for the stochastic process $U_x, x \geq 0$. These will be achieved if we impose the following continuity condition:

$$\lim_{x \to 0_+} \mu_x = \delta_0 \quad (1.2)$$

where δ_0 is the unit mass at 0 and the limit is in the sense of the weak convergence of measures.

1.3. The two families of random variables $\{U_x, x \geq 0\}$ and $\{Z_n, n \geq 1\}$ are independent.

2. Construction of the Processes $\{Z_n, n \geq 1\}, \{U_x, x \geq 0\}$ and $\{X_n, n \geq 0\}$

2.1. Let λ be a probability measure on the Borel sets $\mathbf{B}_{\mathrm{R}_+}$ of the positive real line R_+ and form the infinite product space $(\Omega_1, \mathbf{F}_1, P_1) = \left(\mathrm{R}_+^{\mathrm{N}}, \mathbf{B}_{\mathrm{R}_+^{\mathrm{N}}}, \lambda^{\otimes \mathrm{N}}\right)$. Now, as usual define random variables Z_n on Ω_1 by:

$$Z_n(\omega_1) = \omega_1(n), \text{ if } \omega_1 = (\omega_1(k))_k \in \Omega_1.$$

Then the Z_n are independent identically distributed with common distribution λ We will assume that $E(Z_1) = \alpha > 0$, and $\sigma_{Z_1}^2 = \sigma^2 < \infty$.

2.2. Let $\{\mu_x, x \geq 0\}$ be a semigroup of convolution of probability measures on $\mathrm{R}_+, \mathbf{B}_{\mathrm{R}_+}$ with $\mu_0 = \delta_0$ and satisfying (1.2) then, it is well known, that there is a probability space $(\Omega_2, \mathbf{F}_2, P_2)$ and a family

$\{U_x,\ x\geq 0\}$ of positive random variables defined on this space such that the following properties hold:

(i). Under P_2 the distribution of U_x is μ_x, $U_0=0$.

(ii). For $x\leq y$, the random variables U_y-U_x and U_{y-x} have under P_2 the same distribution μ_{y-x}.

(iii). For every $0\leq x_1\leq x_2\leq\cdots\leq x_n$, the increments $U_{x_1},U_{x_2}-U_{x_1},\cdots,U_{x_n}-U_{x_{n-1}}$ are independent.

(iv). For almost all $\omega_2\in\Omega_2$ the function $x\to U_x(\omega_2)$ is right continuous with left hand limit (cadlag).

From (iv) we deduce:

(v). The function $(x,\omega_2)\to U_x(\omega_2)$ is measurable on the product space $\mathsf{R}_+\times\Omega_2$.

2.3. The basic probability space for the storage process X_n will be the product $(\Omega,\mathsf{F},P)=(\Omega_1\times\Omega_2,\mathsf{F}_1\otimes\mathsf{F}_2,P_1\otimes P_2)$. Then we define X_n by the following recipe:

$$X_0=0, \tag{2.3}$$

$$X_n(\omega)=S_n(\omega_1)-U_{S_n(\omega_1)}(\omega_2),\quad \text{if}\quad \omega=(\omega_1,\omega_2)\in\Omega,$$

$n\geq 1$. where S_n is the simple random walk with: $S_0=0,\quad S_n=Z_1+Z_2+\cdots+Z_n,\quad n\geq 1.$

2.4. Since S_n is a simple random walk, the random variables S_n-S_k and S_{n-k} have the same distribution for $k\leq n$.

3. The Main Results

The main objective is to establish limit theorems for the processes U_{S_n} and X_n. Since the behavior of S_n is well understood, we will focus attention on the structure of the process U_{S_n}. The outstanding fact is that U_{S_n} itself is a simple random walk. First we need some preparation.

3.1. Proposition: For every measurable bounded function $f:\mathsf{R}_+\to\mathsf{R}$, the function $x\to\mu_x(f)=\int_{\mathsf{R}_+} f(t)\,\mu_x(\mathrm{d}t)$ is measurable. Thus for any Borel set A of R_+ the function $x\to\mu_x(A)$ is measurable.

Proof: Assume first $f:\mathsf{R}_+\to\mathsf{R}$ continuous and bounded, then from (1.2) we have

$$\lim_{x\to 0_+}\mu_x(f)=\delta_0(f)=f(0).$$

Now by (1.1) we have

$$\begin{aligned}\mu_{x+y}(f)&=\int_{\mathsf{R}_+} f(t)\cdot\mu_x\times\mu_y(\mathrm{d}t)\\ &=\int_{\mathsf{R}_+\times\mathsf{R}_+} f(t+s)\,\mu_x(\mathrm{d}t)\cdot\mu_y(\mathrm{d}s)\\ &\to\int_{\mathsf{R}_+} f(t)\,\mu_x(\mathrm{d}t),\ y\downarrow 0.\end{aligned}$$

by (1.2) and the bounded convergence theorem. Consequently the function $x\to\mu_x(f)=\int_{\mathsf{R}_+} f(t)\,\mu_x(\mathrm{d}t)$ is right continuous for all $x\geq 0$, hence it is measurable if f is continuous and bounded. Next consider the class of functions:

$$\mathsf{H}=\left\{\begin{array}{l} f\ :\ \mathsf{R}_+\to\mathsf{R},\ \text{such that the function}\\ x\to\mu_x(f)=\int_{\mathsf{R}_+} f(t)\,\mu_x(\mathrm{d}t), is \text{ measurable.}\end{array}\right\}$$

then H is a vector space satisfying the conditions of Theorem I,T20 in [6]. Moreover, by what just proved, H contains the continuous bounded functions $f\ :\ \mathsf{R}_+\to\mathsf{R}$, therefore H contains every measurable bounded function $f\ :\ \mathsf{R}_+\to\mathsf{R}$, ■

3.2. Remark: Let E_{P_1},E_{P_2},E_P, be the expectation operators with respect to P_1,P_2,P, respectively. Since $P=P_1\otimes P_2$, we have $E_{P_1}\cdot E_{P_2}=E_{P_2}\cdot E_{P_1}$, by Fubini theorem. ■

3.3. Proposition: Let Y be a positive random variable on $(\Omega_1,\mathsf{F}_1,P_1)$ with probability distribution γ. Then the function U_Y defined on (Ω,F,P) by:

$$\omega=(\omega_1,\omega_2)\to U_Y(\omega)=U_{Y(\omega_1)}(\omega_2) \tag{3.3}$$

is a random variable such that

$$E_P\left(f(U_Y)\right)=\int_{\mathsf{R}_+\times\mathsf{R}_+} f(t)\mu_y(\mathrm{d}t)\cdot\gamma(\mathrm{d}y)$$

for every measurable positive function $f\ :\ \mathsf{R}_+\to\mathsf{R}$. In particular the probability distribution of U_Y is given by:

$$A\in\mathsf{B}_{\mathsf{R}_+} \tag{3.5}$$

$$P(U_Y\in A)=\int_{\mathsf{R}_+}\mu_y(A)\gamma(\mathrm{d}y)$$

and its expectation is equal to

$$E_P(U_Y)=\int_{\mathsf{R}_+\times\mathsf{R}_+} t\mu_y(\mathrm{d}t)\cdot\gamma(\mathrm{d}y) \tag{3.6}$$

Proof: Define $T\ :\ \Omega_1\times\Omega_2\to\mathsf{R}_+\times\Omega_2$ by $T(\omega_1,\omega_2)=(Y(\omega_1),\omega_2)$ and $S\ :\ \mathsf{R}_+\times\Omega_2\to\mathsf{R}_+$ by $S(x,\omega_2)=U_x(\omega_2)$. It is clear that T is measurable. Also S is measurable by **2.2** (v), so $S\circ T=U_Y$ is measurable.

(3.4) is a simple change of variable formula since $E_P=E_{P_1}\cdot E_{P_2}$. ■

3.7. Proposition: For all $1\leq k\leq n$, the random variables $U_{S_n}-U_{S_k}$, $U_{S_n-S_k}$, $U_{S_{n-k}}$ have the same probability distribution.

Proof: It is enough to show that for every positive measurable function $f:\mathsf{R}_+\to\mathsf{R}$, we have:

$$\begin{aligned}&E_P\left(f\left(U_{S_n}-U_{S_k}\right)\right)\\ &=E_P\left(f\left(U_{S_n-S_k}\right)\right)=E_P\left(f\left(U_{S_{n-k}}\right)\right).\end{aligned} \tag{3.4}$$

Since $E_P = E_{P_1} \cdot E_{P_2}$, we can write:

$$E_P\left(f\left(U_{S_n} - U_{S_k}\right)\right)$$
$$= \int_{\Omega_1}\int_{\Omega_2} f\left(U_{S_n(\omega_1)}(\omega_2) - U_{S_k(\omega_1)}(\omega_2)\right) P_1(\mathrm{d}\omega_1)\cdot P_2(\mathrm{d}\omega_2).$$

But for each fixed $\omega_1 \in \Omega_1$ we get from **2.2**(ii):

$$\int_{\Omega_2} f\left(U_{S_n(\omega_1)}(\omega_2) - U_{S_k(\omega_1)}(\omega_2)\right) P_2(\mathrm{d}\omega_2)$$
$$= \int_{\Omega_2} f\left(U_{S_n(\omega_1)-S_k(\omega_1)}(\omega_2)\right) P_2(\mathrm{d}\omega_2)$$
$$= \mu_{S_n(\omega_1)-S_k(\omega_1)}(f)$$

Applying E_{P_1} to both sides of this formula we get the first equality of (3.7). To get the second one, observe that the function $\omega_1 \to \mu_{S_n(\omega_1)-S_k(\omega_1)}(f)$ is measurable (Proposition **3.1**) and use the fact that under P_1, the random variables $S_n - S_k$ and S_{n-k} have the same probability distribution by **2.4**. ■

3.8. Theorem: The process $U_{S_n}, n \ge 0$ is a simple random walk with:
$U_{S_0} = U_0 = 0$
and $P\left(U_{S_1} \in A\right) = P\left(U_{Z_1} \in A\right) = \int_{\mathrm{R}_+} \mu_z(A)\lambda(\mathrm{d}z)$

Proof: We prove that for all integers $1 \le i \le j \le k \le n$, and all positive measurable functions $f, g, h : \mathrm{R}_+ \to \mathrm{R}$ we have:

$$E_P\left(f\left(U_{S_n} - U_{S_k}\right)\cdot g\left(U_{S_k} - U_{S_j}\right)\cdot h\left(U_{S_j} - U_{S_i}\right)\right)$$
$$= E_P f\left(U_{S_n} - U_{S_k}\right)\cdot E_P\left(g\left(U_{S_k} - U_{S_j}\right)\right) \quad (3.8)$$
$$\cdot E_P\left(h\left(U_{S_j} - U_{S_i}\right)\right)$$

Let ω_1 be fixed in Ω_1. By **2.2** $(ii),(iii)$, under P_2 the random variables

$$U_{S_n(\omega_1)} - U_{S_k(\omega_1)}, U_{S_k(\omega_1)} - U_{S_j(\omega_1)},$$
$$U_{S_j(\omega_1)} - U_{S_i(\omega_1)},$$

are independent. Therefore, applying first E_{P_2} in the L.H.S of (3.8), we get the formula:

$$E_{P_2}\left(\left(f\left(U_{S_n(\omega_1)} - U_{S_k(\omega_1)}\right)\right)\cdot\left(g\left(U_{S_k(\omega_1)} - U_{S_j(\omega_1)}\right)\right)\right.$$
$$\left.\cdot h\left(U_{S_j(\omega_1)} - U_{S_i(\omega_1)}\right)\right) \quad \textbf{(*)}$$
$$= E_{P_2}\left(f\left(U_{S_n(\omega_1)} - U_{S_k(\omega_1)}\right)\right)\cdot E_{P_2}\left(g\left(U_{S_k(\omega_1)} - U_{S_j(\omega_1)}\right)\right)$$
$$\cdot E_{P_2}\left(h\left(U_{S_j(\omega_1)} - U_{S_i(\omega_1)}\right)\right)$$

But $U_{S_n(\omega_1)} - U_{S_k(\omega_1)}, U_{S_k(\omega_1)} - U_{S_j(\omega_1)}, U_{S_j(\omega_1)} - U_{S_i(\omega_1)}$ have distributions $\mu_{S_n(\omega_1)-S_k(\omega_1)}$, $\mu_{S_k(\omega_1)-S_j(\omega_1)}$, $\mu_{S_j(\omega_1)-S_i(\omega_1)}$, respectively. Thus:

$$E_{P_2}\left(f\left(U_{S_n(\omega_1)} - U_{S_k(\omega_1)}\right)\right) = \mu_{S_n(\omega_1)-S_k(\omega_1)}(f)$$
$$E_{P_2}\left(g\left(U_{S_k(\omega_1)} - U_{S_j(\omega_1)}\right)\right) = \mu_{S_k(\omega_1)-S_j(\omega_1)}(g)$$
$$E_{P_2}\left(h\left(U_{S_j(\omega_1)} - U_{S_i(\omega_1)}\right)\right) = \mu_{S_j(\omega_1)-S_i(\omega_1)}(h)$$

By Proposition **3.1**, the R.H.S of these equalities are random variables of ω_1, independent under P_1 since they are measurable functions of the independent random variables $S_n - S_k$, $S_k - S_j$, $S_j - S_i$. Therefore, applying E_{P_1} to both sides of formula **(*)** we get the proof of (3.8):

$$E_{P_1}E_{P_2}\left(\left[f\left(U_{S_n} - U_{S_k}\right)\cdot g\left(U_{S_k} - U_{S_j}\right)\cdot h\left(U_{S_j} - U_{S_i}\right)\right]\right)$$
$$= E_{P_1}\left[E_{P_2}\left(f\left(U_{S_n(\omega_1)} - U_{S_k(\omega_1)}\right)\right)\right.$$
$$\left.\cdot E_{P_2}\left(g\left(U_{S_k(\omega_1)} - U_{S_j(\omega_1)}\right)\right)\cdot E_{P_2}\left(h\left(U_{S_j(\omega_1)} - U_{S_i(\omega_1)}\right)\right)\right]$$

$$E_{P_1}E_{P_2}\left(f\left(U_{S_n(\omega_1)} - U_{S_k(\omega_1)}\right)\right)$$
$$\cdot E_{P_1}E_{P_2}\left(g\left(U_{S_k(\omega_1)} - U_{S_j(\omega_1)}\right)\right)$$
$$\cdot E_{P_1}E_{P_2}\left(h\left(U_{S_j(\omega_1)} - U_{S_i(\omega_1)}\right)\right)$$
$$= E_P\left(f\left(U_{S_n} - U_{S_k}\right)\right)\cdot E_P\left(g\left(U_{S_k} - U_{S_j}\right)\right)$$
$$\cdot E_P\left(h\left(U_{S_j} - U_{S_i}\right)\right).$$

To achieve the proof, write U_{S_n} as follows: $U_{S_n} = \sum_1^n\left(U_{S_k} - U_{S_{k-1}}\right)$, where the $U_{S_k} - U_{S_{k-1}}$ are independent with the same distribution given by

$$P\left(U_{Z_k} \in A\right) = \int_{\mathrm{R}_+} \mu_z(A)\lambda(\mathrm{d}z)$$

according to (3.5). ■

3.9. Proposition: For every positive measurable function $f : \mathrm{R}_+ \to \mathrm{R}$, we have:

$$E_P\left(f\left(U_{S_n}\right)\right) = \int_{\mathrm{R}_+\times\mathrm{R}_+} f(t)\cdot\mu_s(\mathrm{d}t)\cdot\lambda^{*n}(\mathrm{d}s) \quad (3.9)$$

λ^{*n} being the n-fold convolution of the probability λ. In particular the distribution law of the process U_{S_n} is given by:

$$B \in \mathbf{B}_{\mathrm{R}_+}, \quad P\left(U_{S_n} \in B\right) = \int_{\mathrm{R}_+} \mu_s(B)\lambda^{*n}(\mathrm{d}s)$$

and its expectation is:

$$E_P\left(U_{S_n}\right) = \int_{\mathrm{R}_+\times\mathrm{R}_+} t\mu_s(\mathrm{d}t)\cdot\lambda^{*n}(\mathrm{d}s)$$

Proof: We have:

$$E_P\left(f\left(U_{S_n}\right)\right)=E_{P_1}E_{P_2}\left(f\left(U_{S_n(\omega_1)}\left(\omega_2\right)\right)\right)$$
$$=E_{P_1}\int_{\mathsf{R}_+} f\left(t\right)\mu_{S_n(\omega_1)}\left(\mathrm{d}t\right)$$

and, by Proposition **3.1**, the function $\omega_1\to\int_{\mathsf{R}_+} f\left(t\right)\mu_{S_n(\omega_1)}\left(\mathrm{d}t\right)$ is a measurable function of $S_n\left(\omega_1\right)$. Since $S_n=Z_1+Z_2+\cdots+Z_n$ is a simple random walk with the Z_n having distribution λ, the random variable S_n has the distribution λ^{*n}. So, by a simple change of variable we get: $E_{P_1}\int_{\mathsf{R}_+} f\left(t\right)\mu_{S_n(\omega_1)}\left(\mathrm{d}t\right)=\int_{\mathsf{R}_+}\int_{\mathsf{R}_+} f\left(t\right)\mu_s\left(\mathrm{d}t\right).\lambda^{*n}\left(\mathrm{d}s\right)$. So formula (3.9) is proved. To get the distribution law of the process U_{S_n}, take f equal to the characteristic function of some Borel set B. ■

3.10. Remark: Let ν be the distribution of U_{Z_1}, that is $\nu\left(A\right)=\int_{\mathsf{R}_+}\mu_z\left(A\right)\lambda\left(\mathrm{d}z\right)$ and let $\beta=E_P\left(U_{Z_1}\right)=\int_{\mathsf{R}_+\times\mathsf{R}_+} t\mu_z\left(\mathrm{d}t\right)\cdot\lambda\left(\mathrm{d}z\right)$, then as a direct consequence of theorem **3.8**,

$$P\left(U_{S_n}\in B\right)=\nu^{*n}\left(B\right)$$

$$E_P\left(U_{S_n}\right)=n\cdot\beta \qquad ■$$

Now we turn to the structure of the process X_n. We need the following technical lemma:

3.11. Lemma: For every Borel positive function $F:\mathsf{R}_+\times\mathsf{R}_+\to\mathsf{R}$, the function $\varphi:s\to\int_{\mathsf{R}_+}F\left(s,t\right)\mu_s\left(\mathrm{d}t\right)$ is measurable.

Proof: Start with $F=I_{A\times B}$, the characteristic function of the measurable rectangle $A\times B$, in which case we have $\varphi\left(s\right)=I_A\left(s\right)\mu_s\left(B\right)$. Since by proposition **3.1**, the function $s\to\mu_s\left(B\right)$ is measurable we deduce that φ is measurable in this case. Next consider the family

$$\Gamma=\left\{B\in\mathbf{B}_{\mathsf{R}_+\times\mathsf{R}_+}:s\to\int_{\mathsf{R}+}I_B\left(s,t\right)\mu_s\left(\mathrm{d}t\right),\text{is measurable.}\right\}$$

It is easy to check that Γ is a monotone class closed under finite disjoint unions. Since it contains the measurable rectangles, we deduce that $\Gamma=\mathbf{B}_{\mathsf{R}_+\times\mathsf{R}_+}$. Finally consider the following class of Borel positive functions

$$\mathbf{P}=\left\{F:\mathsf{R}_+\times\mathsf{R}_+\to\mathsf{R},\ \varphi(s)=\int_{\mathsf{R}_+}F\left(s,t\right)\mu_s\left(\mathrm{d}t\right)\text{ is Borel}\right\}$$

It is clear that $\mathbf{P}$ is closed under addition and, by the step above, it contains the simple Borel positive functions. By the monotone convergence theorem, $\mathbf{P}$ is exactly the class of all Borel positive functions. ■

3.12. Theorem: The random variables $Z_k-\left(U_{S_k}-U_{S_{k-1}}\right),k=1,2,\cdots,$ are independent with the same distribution given by: for $B\in\mathbf{B}_{\mathsf{R}_+}$

$$P\left(\left(Z_k-\left(U_{S_k}-U_{S_{k-1}}\right)\right)\in B\right)=\int_{\mathsf{R}_+}\mu_s\left(s-B\right)\cdot\lambda\left(\mathrm{d}s\right) \tag{3.12}$$

Consequently the storage process $X_n=S_n-U_{S_n}=\sum_1^n\left(Z_k-\left(U_{S_k}-U_{S_{k-1}}\right)\right)$, is a simple random walk with the basic distribution (3.12).

Proof: For each integer $r\geq 0$, and each $\left(\omega_1,\omega_2\right)\in\Omega_1\times\Omega_2$, put:

$$W_r\left(\omega_1,\omega_2\right)=Z_r\left(\omega_1\right)-\left(U_{S_r(\omega_1)}\left(\omega_2\right)-U_{S_{r-1}(\omega_1)}\left(\omega_2\right)\right)$$

So it is enough to prove that for all $0\leq i\leq j\leq k$ and all Borel positive functions $f,g,h:\mathsf{R}_+\to\mathsf{R}$, we have:

$$E_P\left(f\left(W_k\right)\cdot g\left(W_j\right)\cdot h\left(W_i\right)\right)=E_P\left(f\left(W_k\right)\right)\cdot E_P\left(g\left(W_j\right)\right)\cdot E_P\left(h\left(W_i\right)\right) \tag{3.13}$$

From the construction of the process U_{S_n}, we know that for ω_1 fixed, the random variables $W_r\left(\omega_1,\omega_2\right)$, $r=i,j,k$, are independent under P_2 (see **2.2** (*iii*)). So, applying E_{P_2} to $f\left(W_k\right)\cdot g\left(W_j\right)\cdot h\left(W_i\right)$, we get:

$$E_{P_2}\left(f\left(W_k\right)\cdot g\left(W_j\right)\cdot h\left(W_i\right)\right)=E_{P_2}\left(f\left(W_k\right)\right)\cdot E_{P_2}\left(g\left(W_j\right)\right)\cdot E_{P_2}\left(h\left(W_i\right)\right) \tag{3.14}$$

Now, since under P_2, the distribution of $U_{S_r(\omega_1)}\left(\omega_2\right)-U_{S_{r-1}(\omega_1)}\left(\omega_2\right)$ is the same as that of $U_{S_r(\omega_1)-S_{r-1}(\omega_1)}=U_{Z_r(\omega_1)}$ (ω_1 fixed), we have for each Borel positive function $\psi:\mathsf{R}_+\to\mathsf{R}$

$$E_{P_2}\left(\psi\left(W_r\right)\right)=\int_{\mathsf{R}_+}\psi\left(Z_r\left(\omega_1\right)-t\right)\mu_{Z_r(\omega_1)}\left(\mathrm{d}t\right),\quad r=i,j,k$$

From lemma **3.11**, the functions $\omega_1\to\int_{\mathsf{R}_+}\psi\left(Z_r\left(\omega_1\right)-t\right)\mu_{Z_r(\omega_1)}\left(\mathrm{d}t\right),r=i,j,k,$ are Borel functions of the random variables Z_r, thus they are independent under the probability P_1. Therefore, applying E_{P_1} to both sides of (3.14) we get (3.13). ■

As for the process X_n, the counterpart of proposition **3.9** is the following:

3.15. Proposition: If $f:\mathsf{R}_+\to\mathsf{R}$ is positive measurable and if $B\in\mathbf{B}_{\mathsf{R}_+}$, then we have:

$$E_P\left(f\left(X_n\right)\right)=\int_{\mathsf{R}_+\times\mathsf{R}_+}f\left(s-t\right)\mu_s\left(\mathrm{d}t\right)\cdot\lambda^{*n}\left(\mathrm{d}s\right)$$

$$P(X_n \in B) = \int_{\mathsf{R}_+} \mu_s(s-B)\lambda^{*n}(\mathrm{d}s)$$

$$E_P(X_n) = n \cdot (\alpha - \beta)$$

For the proof, use the formula $X_n = S_n - U_{S_n}$ and routine integration.

3.16. Example: Let $0 < c < 1$ and let us take as measure μ_s the unit mass at the point cs, that is, the Dirac measure $\mu_s = \delta_{cs}$, $s \in \mathsf{R}_+$. It easy to check that $\mu_{s+t} = \mu_s \times \mu_t$ for all s,t in R_+ Then for every probability measure λ on R_+ we have: $P(U_{Z_1} \in B) = \int_{\mathsf{R}_+} \mu_s(B)\lambda(\mathrm{d}s) = \lambda(c^{-1}B)$. This gives the distribution of the release process in this case: $P(U_{S_n} \in B) = \int_{\mathsf{R}_+} \mu_s(B)\lambda^{*n}(\mathrm{d}s) = \lambda^{*n}(c^{-1}B)$.

Since we have $\lambda^{*n}(c^{-1}B) = P(cS_n \in B)$, we deduce that the release rule consists in removing from S_n the quantity cS_n.

Likewise it is straightforward, from Proposition **3.14**, that

$$\begin{aligned} P(X_n \in B) &= \int_{\mathsf{R}_+} \mu_s(s-B)\cdot\lambda^{*n}(\mathrm{d}s) \\ &= \int_{\mathsf{R}_+} \delta_{cs}(s-B)\lambda^{*n}(\mathrm{d}s) \\ &= \int_{\mathsf{R}_+} 1_{(1-c)^{-1}B}(s)\lambda^{*n}(\mathrm{d}s) \\ &= \lambda^{*n}\left((1-c)^{-1}B\right) \end{aligned}$$

from which we deduce that the distribution of the storage process is

$$P(X_n \in B) = P((1-c)S_n \in B).$$

One can give more examples in this way by choosing the distribution λ or/and the semigoup $\{\mu_x,\ x \geq 0\}$. Consider the following simple example:

3.17. Example: Take λ the 0 - 1 Bernoulli distribution with probability of success p. In this case the semigroup $\{\mu_x,\ x \geq 0\}$ is a sequence μ_n of probabilities with μ_n supported by $\{1,2,\cdots,n\}$ for $n \geq 1$ and λ^{*n} is the Binomial distribution. So we get from proposition **3.9**

$$\begin{aligned} P(U_{S_n} \in B) &= \int_{\mathsf{R}_+} \mu_s(B)\lambda^{*n}(\mathrm{d}s) \\ &= \sum_{k=0}^{n} \binom{n}{k} p^k (1-p)^{n-k} \mu_k(B) \end{aligned}$$

Likewise we get the distribution of X_n from proposition **3.15** as :

$$\begin{aligned} P(X_n \in B) &= \int_{\mathsf{R}_+} \mu_s(s-B)\lambda^{*n}(\mathrm{d}s) \\ &= \sum_{k=0}^{n} \binom{n}{k} p^k (1-p)^{n-k} \mu_k(s-B) \end{aligned} . \blacksquare$$

4. Limit Theorems

Due to the simple structure of the processes U_{S_n} and X_n (Theorems **3.8**, **3.12**), it is not difficult to establish a SLLN and a CLT for them.

4.1. Theorem: For the storage process X_n and the release rule process U_{S_n}, we have:

$$\lim_n \frac{X_n}{n} = \alpha - \beta = E_P(X_1)$$

and

$$\lim_n \frac{U_{S_n}}{S_n} = \frac{\beta}{\alpha}$$

Proof: Since S_n and U_{S_n} are simple random walks with $E_P(Z_1) = \alpha$ and $E_P(U_{S_1}) = \beta$, we have:

$\lim_n \frac{S_n}{n} = \alpha$ and $\lim_n \frac{U_{S_n}}{n} = \beta$, by the classical S.L.L.N. So we deduce:

$$\lim_n \frac{X_n}{n} = \lim_n \frac{S_n - U_{S_n}}{n} = \alpha - \beta$$

and

$$\lim_n \frac{U_{S_n}}{S_n} = \lim_n \frac{\frac{U_{S_n}}{n}}{\frac{S_n}{n}} = \frac{\beta}{\alpha}.$$

4.2. Proposition: Under the conditions:
$\int_{\mathsf{R}_+\times\mathsf{R}_+} t^2\mu_s(\mathrm{d}t)\cdot\lambda(\mathrm{d}s) < \infty$ and
$\int_{\mathsf{R}_+\times\mathsf{R}_+} s\cdot t\mu_s(\mathrm{d}t)\cdot\lambda(\mathrm{d}s) < \infty$, the variances σ_U^2 and $\sigma_{X_1}^2$ of the random variables U_Z and X_1 are finite. The conditions can respectively be written as

$$\int_{\mathsf{R}_+} E(U_s^2)\cdot\lambda(\mathrm{d}s) < \infty$$

and

$$\int_{\mathsf{R}_+\times\mathsf{R}_+} s\cdot E(U_s)\cdot\lambda(\mathrm{d}s) < \infty .$$

Proof: We have
$\sigma_U^2 = \int_{\mathsf{R}_+\times\mathsf{R}_+} t^2\mu_s(\mathrm{d}t)\cdot\lambda(\mathrm{d}s) - \beta^2$, so the first condition gives $\sigma_U^2 < \infty$. On the other hand we have

$$\sigma_{X_1}^2 = \int_{R_+ \times R_+} (s-t)^2 \mu_s(\mathrm{dt}) \cdot \lambda(\mathrm{ds}) - (\alpha - \beta)^2$$

and

$$\begin{aligned} &\int_{R_+ \times R_+} (s-t)^2 \mu_s(\mathrm{dt}) \cdot \lambda(\mathrm{ds}) \\ &= \int_{R_+ \times R_+} \left(s^2 + t^2\right) \mu_s(\mathrm{dt}) \cdot \lambda(\mathrm{ds})\Big) \\ &-2 \int_{R_+ \times R_+} s \cdot t \mu_s(\mathrm{dt}) \cdot \lambda(\mathrm{ds}) \end{aligned}$$

Since the variance σ^2 of Z_n is finite we have $\int_{R_+ \times R_+} s^2 \mu_s(\mathrm{dt}) \cdot \lambda(\mathrm{ds}) = \int_{R_+} s^2 \cdot \lambda(\mathrm{ds}) < \infty$, so the conclusion follows. ■

Finally we get under the conditions of proposition **4.2**:

4.3. Theorem: Assume the conditions of proposition **4.2.** Then the normalized sequences of random variables:

$$T_n = \frac{U_{S_n} - n \cdot \beta}{\sigma_U \sqrt{n}} \quad \text{and} \quad R_n = \frac{X_n - n \cdot (\alpha - \beta)}{\sigma_{X_1} \sqrt{n}}$$

both converge in distribution to the Normal law $N(0,1)$.

Proof: The condition of the theorem insures the finiteness of the variances σ_U^2 and $\sigma_{X_1}^2$. Now the conclusion results from the fact that U_{S_n} and X_n are simple random walks and the Lindberg Central Limit Theorem. To see this, we use the method of characteristic functions. Let us denote by f_θ the characteristic function of the random variable θ. Since by Theorem **3.8** the components $U_{S_k} - U_{S_{k-1}}$ of U_{S_n} have the same distribution as U_{Z_1}, we have

$$\begin{aligned} &f_{T_n}(t) = \exp(\mathrm{it}\ T_n) \\ &= \left(f_{U_{Z_1}-\beta}\left(\frac{t}{\sigma_U \sqrt{n}}\right) \right)^n \\ &= \left\{ 1 + \frac{i^2 \sigma_U{}^2}{2} \left(\frac{t}{\sigma_U \sqrt{n}} \right)^2 + o\left(\frac{|t|}{\sigma_U \sqrt{n}} \right)^2 \right\}^n \\ &= \left\{ 1 - \frac{t^2}{2n} + o\left(\frac{t^2}{n} \right) \right\}^n \to \exp\left(-\frac{t^2}{2} \right) \end{aligned}$$

where the second equality comes from the Taylor expansion of $f_{U_{Z_1}-\beta}$. It is well known that this limit is the characteristic function of the random variable $N(0,1)$. The same proof works for R_n, using the components of the process X_n as given in Theorem **3.12**. ■

In some storage systems, the changes due to supply and release do not take place regularly in time. So it would be more realistic to consider the time parameter n as random. We will do so in what follows and will consider the asymptotic distributions of the processes U_{S_n}, and X_n, when properly normalized and randomized. First let us put for each k,

$$A_k = \frac{U_{S_k} - U_{S_{k-1}} - \beta}{\sigma_U}, \text{ and}$$

$$B_k = \frac{Z_k - \left(U_{S_k} - U_{S_{k-1}}\right) - (\alpha - \beta)}{\sigma_{X_1}}.$$

Then we have:

4.4. Theorem: Let $\{N_n : n \ge 1\}$ be a sequence of integral valued random variables, independent of the A_k and B_k.

If $\frac{N_n}{n}$ converges in probability to 1, as $n \to \infty$, then the randomized processes:

$$\frac{\sum_1^{N_n} A_k}{\sqrt{n}} \quad \text{and} \quad \frac{\sum_1^{N_n} B_k}{\sqrt{n}}$$

both converge in distribution to the Normal law $N(0,1)$.

Proof: It is a simple adaptation of [7], VIII.4, Theorem 4, p. 265. ■

5. Conclusion

In this paper, we presented a simple stochastic storage process X_n with a random walk input S_n and a natural release rule U_{S_n}. Realistic conditions are prescribed which make this process more tractable when compared to those models studied elsewhere (see Introduction). In particular the conditions led to a simple structure of random walk for the processes U_{S_n} and X_n, which has given explicitly their distributions, and a rather good insight on their asymptotic behavior since a SLLN and a CLT has been easily established for each of them. Moreover, a slightly more general limit theorem has been obtained when time is adequately randomized and both processes U_{S_n} and X_n properly normalized.

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