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One Common Solution to the Singularity and Perihelion Problems

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ABSTRACT

With a view to surmounting the singularity problem on the one hand, as well as the moving perihelion problem of the planets on the other, as two acutely vexed questions within *Newton*'*s* gravity concept, the goal of this paper is a modification of *Newton*'*s* gravity concept itself.

Keywords: Celestial Mechanics; Planets; Rings

1. Introduction

It would be difficult to exaggerate the influence of *Newton*'*s* theory of gravitation on the subsequent development of physics. As well as explaining *Kepler*'*s* laws of planetary motion, *Newton*'*s* theory was central to the successful mathematization of physics using the newly-invented calculus and it served as a paradigm for the later theories of electrostatics and magnetostatics. However, new insights into Milky Way satellite galaxies raise awkward questions for cosmologists: Do we have to modify *Newton*'*s* theory of gravitation as it fails to explain so many observations? In other words, although *Newton*'*s* theory does, in fact, describe the everyday effects of gravity on Earth, things we can see and measure, it is conceivable that we have completely failed to comprehend the actual physics underlying the *Newton*'*s* force of gravity. In addition, *Newton*'*s* theory does not fully explain the precession of the perihelion of the orbits of the Planets, especially of planet Mercury. Namely, it has been experimentally stated that the perihelion of Mercury's orbits moves into the plane of its planetary motion around the Sun. In other words, all planetary motions of Sun's planetary system depart from elliptical orbits obtained from *Newton*'*s* gravity theory, [1]. By the strict *Schwarzshild-Droste*'*s* solution to the static gravitational field with spherical symmetry, in the general *Einstein*'*s* relativity theory, the perihelion problem has been approximately solved, [1]. On the other hand, *Einstein*'*s* theory has some difficulties hard to be overcome such as the problem of singularity, that occurs in *Newton*'*s* theory too (all relevant physical variables, such as velocity, force, kinetic and potential energy, don't exist at point of singularity). Accordingly, in order to solve simultaneously these two acutely vexed questions within *Newton*'*s* gravity theory, we present, in this research paper, an approximative modification of *Newton*'*s* gravity concept itself. The outline of this article is as follows: In the Preliminaries, the space-time continuum (the integral space), as an ambient space, is completely defined. In Section 1 we establish a causal connection between the expression for the kinetic energy of a material point and the Minkowski metric in the four-dimensional space-time continuum. In addition, in two separate subsections of this section we derive *Newton*'*s* equations of motion and the relativistic Hamilton-Jacobi equation for a free particle. Since the dynamic (*Newton*'*s*) equations of motion are formally derived from geodesic equations in Section 2, this section together with Appendix at the end of the paper provide a possibility of further work on the modification of *Newton*'*s* gravity theory in Section 3. In this last section we show that a comprehensive analysis of particle motion under the modified *Newton*'*s* gravity force leads to the perihelion motions of a *Planet*'*s* elliptical orbit.

2. Preliminaries

By a material point $\mathcal{M}$, introduced for the purpose of an useful idealization, one means a geometrical point, which is spatially no dimensional on the one hand and exactly fixed mass on the other. Closely related to the notion of a geometrical point is the set of values $\{a^\alpha\}_{\alpha=1}^{n}$ of some arbitrary n variables $\{x^\alpha\}_{\alpha=1}^{n}$ denoting the contravariant co-ordinates of the real n-dimensional configurative space. The geometrical point, defined by a set of zero values $\{a^{\alpha}=0\}_{\alpha=1}^{n}$, is the zero co-ordinate point. If one of n arbitrary variables $\{x^{\alpha}\}_{\alpha=1}^{n}$ is the time variable t, then the space aforementioned becomes the space-time continuum (shortly called the integral space), [2]. As it was noted in [3] the value of t is called moment or instant.

The set of all geometrical points of the spatial subspace of the integral space, to which the mass m can be joined in some strictly monotonous sequence of permitted instants of the time t, makes an odograph usually referring to the a trajectory (motion path) of $\mathcal{M}$. The time variable t is taken for a unique independent variable, so that all remaining spatial variables $\{x^{i}\}_{i=1}^{n-1}$ are functional variables. Across all the future text *Greek* indices take values $1,2,\cdots,n$, and *Latin* ones $1,2,\cdots,n-1$. In the space-time continuum the aforementioned trajectory of $\mathcal{M}$ blossoms into an integral curve. The vectors $\boldsymbol{\rho}\left[x^{\alpha}(t)\right]$ and $\boldsymbol{r}\left[x^{i}(t)\right]$ defined with respect to the origin are position vectors of $\mathcal{M}$ in the space-time continuum and in the spatial subspace of the integral space, respectively. The concept of a vector in vector hyper-dimensional spaces $(n>3)$ should be conditionally comprehended in the sense of its geometrical presentation in a form of segments. Hence it bears a name linear tensor, [4]. Covariant vectors $\boldsymbol{e}_{\alpha}=\partial_{\alpha}\boldsymbol{\rho}\left(x^{\beta}\right)$, where ∂_{α} denotes $\partial/\partial x^{\alpha}$, form a covariant vector basis $\{\boldsymbol{e}_{\alpha}\}_{\alpha=1}^{n}$ of the integral space. The vectors $\boldsymbol{e}^{\beta}$, such that at any point of the space $\boldsymbol{e}_{\alpha}\cdot\boldsymbol{e}^{\beta}=\delta_{\alpha}^{\beta}$, where the second order system δ_{α}^{β} (*Kronecker*'s delta-symbol, [5]) is the identity $n\times n$ matrix, form a dual basis $\{\boldsymbol{e}^{\beta}\}_{\beta=1}^{n}$ of $\{\boldsymbol{e}_{\alpha}\}_{\alpha=1}^{n}$. The differential $\mathrm{d}\boldsymbol{\rho}$ of the position vector $\boldsymbol{\rho}$ of $\mathcal{M}$ is defined by $\mathrm{d}\boldsymbol{\rho}=\mathrm{d}x^{\alpha}\boldsymbol{e}_{\alpha}=\mathrm{d}x_{\beta}\boldsymbol{e}^{\beta}$, where the so called *Einstein*'s convention is applied to a summation with respect to the repetitive indexes (uppers and lowers), herein as well as in the further text of the paper.

3. The Action Metric in the Integral Space

Since the integral space is a metric affine space, whose linearly independent basis (fundamental) co-ordinate vectors reduced to the origin form an n-hedral basis, it follows that if $\mathrm{d}s$ is a line element of the metric affine space of the spatial continuum, then the expression for the kinetic energy ε of $\mathcal{M}$ can be stated in more appropriate form:

$$\frac{2}{m}\varepsilon(\mathrm{d}t)^{2}=\mathrm{d}r\cdot\mathrm{d}r=(\mathrm{d}s)^{2}, \qquad (1)$$

considering the fact that the basic mechanical (kinematics and dynamics) parameters of $\mathcal{M}$ are its velocity $v=d_{t}r$ (d_{t} denotes $\mathrm{d}/\mathrm{d}t$), quantity of motion $\boldsymbol{K}=m\boldsymbol{v}$ and kinetic energy $\varepsilon=m(\boldsymbol{v}\cdot\boldsymbol{v})/2=mv^{2}/2$. A term of $c^{2}(\mathrm{d}t)^{2}$, where c is nominally equal to the light velocity in vacuum, can be added to both sides of the previous equation, as follows

$$(\mathrm{d}s)^{2}+c^{2}(\mathrm{d}t)^{2}=\frac{2}{m}\varepsilon(\mathrm{d}t)^{2}+c^{2}(\mathrm{d}t)^{2}. \qquad (2)$$

For $k=mc^{2}/2$ let $\mathcal{K}$ be such that $\mathcal{K}=k-\varepsilon$. Then, (2) becomes

$$c^{2}(\mathrm{d}t)^{2}-(\mathrm{d}s)^{2}=\frac{2}{m}\mathcal{K}(\mathrm{d}t)^{2}. \qquad (3)$$

This means that if $\mathcal{K}=m(d_{t}\sigma)^{2}/2$, where $\mathrm{d}\sigma=\left|\sqrt{\mathrm{d}\boldsymbol{\rho}\cdot\mathrm{d}\boldsymbol{\rho}}\right|$ is a line element of the metric affine space of the space-time continuum, then the four-dimensional integral space has the Minkowski metric, [1,4]. So, in this case the Minkowski metric (3) represents the kinetic energy $\mathcal{K}$ of $\mathcal{M}$ in the integral space. Hence, the *Minkowski* metric $c^{2}(\mathrm{d}t)^{2}-(\mathrm{d}s)^{2}=(\mathrm{d}\sigma)^{2}$ is the kinetic metric of the integral space, [6].

If the *Pfaff* form $\mathrm{d}\varepsilon=\boldsymbol{F}\cdot\mathrm{d}\boldsymbol{r}$ is absolute differential, that means that there exists a scalar valued function $\mathcal{P}(\boldsymbol{r})$ such that $\boldsymbol{F}=-\mathrm{grad}\mathcal{P}(\boldsymbol{r})$, then $\mathrm{d}(\varepsilon+\mathcal{P})=0$ and

$$\mathcal{K}-\mathcal{P}=\mathbb{k}, \qquad (4)$$

where $\mathbb{k}=k-\mathcal{U}$, and $\mathcal{U}=\varepsilon+\mathcal{P}$ is the total mechanical energy of $\mathcal{M}$.

Now, we can start with the action $\mathcal{S}$ in the *Lagrange* sense along a motion path of $\mathcal{M}$ in the integral space [4, 6],

$$\mathcal{S}=2\int_{t_1}^{t_2}\mathcal{K}\mathrm{d}t. \qquad (5)$$

Since $\left|\sqrt{\mathcal{K}}\right|\mathrm{d}t=\left|\sqrt{m/2}\right|\mathrm{d}\sigma$ it follows from (4) and (5) that

$$\mathcal{S}=\left|\sqrt{2m}\right|\int_{\sigma(t_1)}^{\sigma(t_2)}\left|\sqrt{\mathcal{K}}\right|\mathrm{d}\sigma=\left|\sqrt{2m}\right|\int_{\sigma(t_1)}^{\sigma(t_2)}\left|\sqrt{\mathbb{k}+\mathcal{P}}\right|\mathrm{d}\sigma. \qquad (6)$$

Let us introduce an action line element $\mathrm{d}w$, thoroughly explained in [6], in such a way that

$$\left|\sqrt{k}\right|\mathrm{d}w=\left|\sqrt{\mathbb{k}+\mathcal{P}}\right|\mathrm{d}\sigma. \qquad (7)$$

Accordingly, the action metric is as follows

$$\begin{aligned}k(\mathrm{d}w)^{2}&=ka_{\alpha\beta}\mathrm{d}x^{\alpha}\mathrm{d}x^{\beta}=(\mathbb{k}+\mathcal{P})\mathrm{e}_{\alpha\beta}\mathrm{d}x^{\alpha}\mathrm{d}x^{\beta}\\&=(\mathbb{k}+\mathcal{P})(\mathrm{d}\sigma)^{2}=\frac{2}{m}\mathcal{K}^{2}(\mathrm{d}t)^{2},\end{aligned} \qquad (8)$$

where

$$a_{\alpha\beta}=\boldsymbol{a}_{\alpha}\cdot\boldsymbol{a}_{\beta}\left(\boldsymbol{a}_{\alpha}=\left|\sqrt{(\mathbb{k}+\mathcal{P})/k}\right|\boldsymbol{e}_{\alpha}\right)$$

and $e_{\alpha\beta}=\boldsymbol{e}_{\alpha}\cdot\boldsymbol{e}_{\beta}$ are the metric tensors of $(\mathrm{d}w)^{2}$ and

$(\mathrm{d}\sigma)^2$, respectively.

3.1. *Newton's* Equations of Motion

By the well-known *Maupertius-Lagrange's* principle [4], the path motion of $\mathcal{M}$ is just the path along which the action is stationary, more precisely along which the following two mutually equivalent conditions

$$\left|\sqrt{2m}\right|\Delta\int_{\sigma(t_1)}^{\sigma(t_2)}\left|\sqrt{\mathbb{k}+\mathcal{P}}\right|\mathrm{d}\sigma = mc\Delta\int_{\sigma(t_1)}^{\sigma(t_2)}\left|\sqrt{1-\frac{v^2}{c^2}}\right|\mathrm{d}\sigma$$
$$= 0 \text{ and } mc^2\Delta\int_{t_1}^{t_2}\left(1-\frac{v^2}{c^2}\right)\mathrm{d}t = 0, \tag{9}$$

where Δ is the variational operator, are satisfied. By (7), the previous conditions are reduced to

$$mc\Delta\int_{w(t_1)}^{w(t_2)}\mathrm{d}w = 0. \tag{10}$$

The second condition in (9) leads to the *Euler Lagrange* equations

$$d_t\partial_{d_t x^i}\mathcal{L} - \partial_i\mathcal{L} = 0, \tag{11}$$

where $\mathcal{L} = \varepsilon - \mathcal{P}$ and

$$\mathcal{L} + 2\mathcal{K} = \mathcal{L} + mc^2\left(1 - v^2/c^2\right)$$
$$= \mathcal{L} + k\left(1 - v^2/c^2\right) + (\mathbb{k} + \mathcal{P}) = k + \mathbb{k},$$

which yield *Newton's* equations of motion

$$md_{tt}^2 x^i = -\partial_i\mathcal{P}. \tag{12}$$

3.2. The Relativistic Hamilton-Jacobi Equation for a Free Particle

Analyze (5) again, but now let $\mathcal{S}$ be a function of t, that means that $\mathcal{S}(t) = 2\int_{t_1}^{t}\mathcal{K}\mathrm{d}t$. As

$$d_t\mathcal{S} = 2\mathcal{K} = me_{\alpha\beta}d_t x^\alpha d_t x^\beta, \tag{13}$$

we introduce the functional $\mathcal{S}^H$, nominally equal to $\mathcal{S}$, such that

$$\partial_\alpha\mathcal{S}^H = me_{\alpha\beta}d_t x^\beta \text{ and } d_t\mathcal{S}^H = 2\mathcal{K}, \tag{14}$$

as well as the functional $\mathcal{Z}^H$ satisfying the condition

$$\mathcal{S}^H = (k + \mathbb{k})t - \mathcal{Z}^H, \tag{15}$$

which together with (15) yields

$$d_t\mathcal{Z}^H = \mathcal{L}, \tag{16}$$

since $\mathcal{L} + 2\mathcal{K} = k + \mathbb{k}$. Hence, $\mathcal{L} = \varepsilon - \mathcal{P}$ is *Lagrangian* of $\mathcal{M}$. Further, since $\partial_t\mathcal{S}^H = mc^2$ for $x^1 = ct$, see (14), it follows from (15) that $\mathcal{U} = -\partial_t\mathcal{Z}^H$ and

$$\partial_t\mathcal{Z}^H + \mathcal{U} = 0. \tag{17}$$

The previous equation is the *Hamilton-Jacobi* one, so that $\mathcal{Z}^H$ is the principal *Hamilton's* functional of $\mathcal{M}$. Clearly, the *Hamiltonian* $\mathcal{H}$ of $\mathcal{M}$ is equal to $\mathcal{U}$, more precisely to the integral of motion, considering the fact that the kinetic energy ε of $\mathcal{M}$ is a homogenous square function of $d_t x^\alpha$. Now, by (14) and (15), we have $\partial_i\mathcal{Z}^H = -me_{ij}d_t x^j$, so that

$$e^{kl}\partial_k\mathcal{Z}^H\partial_l\mathcal{Z}^H = m^2 e_{ij}d_t x^i d_t x^j = 2m\varepsilon. \tag{18}$$

This together with (17) leads to the second form of the *Hamilton-Jacobi* equation

$$-\partial_t\mathcal{Z}^H - \frac{1}{2m}e^{kl}\partial_k\mathcal{Z}^H\partial_l\mathcal{Z}^H = \mathcal{P}. \tag{19}$$

In addition,

$$a^{\alpha\beta}\left(\partial_\alpha\mathcal{Z}^H - \frac{k+\mathbb{k}}{c}\partial_\alpha x^1\right)\left(\partial_\beta\mathcal{Z}^H - \frac{k+\mathbb{k}}{c}\partial_\beta x^1\right)$$
$$= a^{\alpha\beta}\partial_\alpha\mathcal{S}^H\partial_\beta\mathcal{S}^H = m^2c^2\frac{mk}{2\mathcal{K}^2}a_{\alpha\beta}d_t x^\alpha d_t x^\beta, \tag{20}$$

which together with (8) yields

$$a^{\alpha\beta}\partial_\alpha\mathcal{S}^H\partial_\beta\mathcal{S}^H - m^2c^2 = 0$$
$$\text{and } a^{\alpha\beta}\left(\partial_\alpha\mathcal{Z}^H - \frac{A_\alpha}{c}\right)\left(\partial_\beta\mathcal{Z}^H - \frac{A_\beta}{c}\right) - m^2c^2 = 0, \tag{21}$$

where $A_\alpha = (k + \mathbb{k}, 0, 0, 0)$. These two equations are obviously analogous to the relativistic *Hamilton-Jacobi* equation for a free particle, see [7,8].

4. The Binet Differential Equation

As is well-known from the tensorial analysis, see [6], all curves of the integral space, for which the condition (10) is satisfied, are geodesics, and the absolute *Bianchi* (covariant) derivative $d_w\boldsymbol{u}$ of the unit tangent vector $\boldsymbol{u} = d_w x^\alpha\boldsymbol{a}_\alpha$ along geodesics is equal to zero (the vector projection of $d_w\boldsymbol{u}$ onto the tangent hyper-plane of the integral space is equal to zero). Thus, the geodesic equations are as follows

$$d_w\boldsymbol{u}\cdot\boldsymbol{a}^\gamma = d_w\left(d_w x^\alpha\boldsymbol{a}_\alpha\right)\cdot\boldsymbol{a}^\gamma$$
$$= d_{ww}^2 x^\alpha\boldsymbol{a}_\alpha\cdot\boldsymbol{a}^\gamma + d_w x^\alpha d_w\boldsymbol{a}_\alpha\cdot\boldsymbol{a}^\gamma \tag{22}$$
$$= d_{ww}^2 x^\gamma + \hat{\Gamma}_{\alpha\beta}^\gamma d_w x^\alpha d_w x^\beta = 0,$$

where d_{ww}^2 denotes $d^2/(dw)^2$, and $\hat{\Gamma}_{\alpha\beta}^\gamma = \partial_\beta a_\alpha\cdot a^\gamma = a^{\gamma\delta}\left(\partial_\beta a_{\alpha\delta} + \partial_\alpha a_{\beta\delta} - \partial_\delta a_{\alpha\beta}\right)/2$ are the second kind *Christoffel* symbols with respect to the action metric space $(dw)^2$. Let $\boldsymbol{F} = -\mathrm{grad}\mathcal{P}$. Since $(\mathbb{k} + \mathcal{P})e_{\alpha\beta} = ka_{\alpha\beta}$, see (8), it follows that

$$\hat{\Gamma}_{\alpha\beta}^\gamma = \Gamma_{\alpha\beta}^\gamma$$
$$+\frac{1}{2(\mathbb{k}+\mathcal{P})}\left(\partial_\alpha\mathcal{P}\delta_\beta^\gamma + \partial_\beta\mathcal{P}\delta_\alpha^\gamma - e^{\gamma\delta}e_{\alpha\beta}\partial_\delta\mathcal{P}\right), \tag{23}$$

where $\Gamma^{\gamma}_{\alpha\beta} = e^{\gamma\delta}\left(\partial_\beta e_{\alpha\delta} + \partial_\alpha e_{\beta\delta} - \partial_\delta e_{\alpha\beta}\right)/2$ are the second kind *Christoffel* symbols with respect to the *Euclidean* metric space $(\mathrm{d}\sigma)^2 = e_{\alpha\beta}\mathrm{d}x^\alpha \mathrm{d}x^\beta$. A new form of the geodesic equations (22), for a constrained material point $\mathcal{M}(\boldsymbol{F} \neq \boldsymbol{0} \Leftrightarrow \mathcal{P} \neq \text{const.})$, is as follows

$$d^2_{ww}x^\gamma + \Gamma^\gamma_{\alpha\beta} d_w x^\alpha d_w x^\beta = -\frac{1}{\mathbb{k}+\mathcal{P}}\partial_\delta \mathcal{P} d_w x^\delta d_w x^\gamma + \frac{k}{2(\mathbb{k}+\mathcal{P})^2}\partial_\delta \mathcal{P} e^{\gamma\delta}, \quad (24)$$

which yields

$$m\left(d^2_{tt}x^\gamma + \Gamma^\gamma_{\alpha\beta} d_t x^\alpha d_t x^\beta\right) = \partial_\delta \mathcal{P} e^{\gamma\delta} = F^\gamma, \quad (25)$$

since

$$d^2_{ww}x^\gamma \left(d_t w\right)^2 + d_t x^\gamma d_w t d^2_{tt} w = d^2_{tt} x^\gamma,$$

$$d_t w = c(\mathbb{k}+\mathcal{P})/k \quad \text{and} \quad d^2_{tt} w d_w t = \partial_\delta \mathcal{P}/(\mathbb{k}+\mathcal{P}) d_t x^\delta .$$

So, (25) represents the *Euler-Lagrange* differential equations of the extremal curve in the explicit form, and at the same time *Newton*'*s* second law of motion under the action of a potential force $F^k = \partial_l \mathcal{P} e^{kl}$ in the contravariant form:

$$m\left(d^2_{tt}x^k + \Gamma^k_{ij} d_t x^i d_t x^j\right) = \partial_l \mathcal{P} e^{kl} = F^k. \quad (26)$$

Accordingly, one may conclude that the dynamic (*Newton*'*s*) Equations (26) of motion are formally derived from the geometric Equations (22).

In the case of the free motion of $\mathcal{M}$, when $\mathcal{P} = \text{const.}$, both the kinetic and action metric form of the integral space are *pseudo-euclidean*, while integral curves are straight-lines (see Appendix), as it was thoroughly explained in the monograph by [6]. On the other hand, the well-known *Binet* differential equation for central force motion of $\mathcal{M}$

$$d^2_{\theta\theta}\frac{1}{r} + \frac{1}{r} = -\frac{1}{2k\alpha^2} d_{\frac{1}{r}} \mathcal{P} \quad (27)$$

is obtained by differentiating (58) (see Appendix).

5. Modified *Newton*'*s* Gravity Concept

For the conservative *Newton*'*s* gravity force $\boldsymbol{F}_N = -\text{grad}\mathcal{P}$ the expression $\mathbb{k}+\mathcal{P}$ is as follows $\mathbb{k}+\mathcal{P} = \mathbb{k} + k(1-2\rho/r)$, where $\rho = \gamma M/c^2$ is the gravitational radius, so that (27) is reduced to

$$d^2_{\theta\theta}\frac{1}{r} + \frac{1}{r} = \frac{\rho}{\alpha^2}, \quad (28)$$

where $\alpha = S/c$ and $S = r^2 d_t \theta = \text{const}$.

Since, in the limit as $r \to 0^+$, *Newton*'*s* gravitational potential $\mathcal{P} = k(1-2\rho/r)$ tends to infinity, it is logical to assume that $\mathcal{P}$ is the first-order *MacLaurin* series approximation of the exponential function $ke^{-2\rho/r}$, so that $\mathcal{P}_{N_e} = ke^{-2\rho/r}$ and

$$k(\mathrm{d}w)^2 = \left(\mathbb{k} + ke^{-\frac{2\rho}{r}}\right)(\mathrm{d}\sigma)^2. \quad (29)$$

Accordingly, the modified *Binet* differential equation for the modified central *Newton*'*s* gravity force

$$\boldsymbol{F}_{N_e} = -\text{grad}\mathcal{P}_{N_e} = -\frac{2k\rho}{r^3}e^{-\frac{2\rho}{r}}\boldsymbol{r}, \quad (30)$$

is as follows

$$d^2_{\theta\theta}\frac{1}{r} + \frac{1}{r} = \frac{\rho}{\alpha^2}e^{-\frac{2\rho}{r}}. \quad (31)$$

5.1. A motion Under the Action of $\boldsymbol{F}_{N_e}$

Start with *Newton*'*s* second law of motion

$$md^2_{tt}\boldsymbol{r} = -\frac{2k\rho}{r^3}e^{-\frac{2\rho}{r}}\boldsymbol{r}. \quad (32)$$

Multiply (32) on the right by the sector velocity vector $\boldsymbol{S} = \boldsymbol{r}\times\boldsymbol{v}$ as follows

$$md^2_{tt}\boldsymbol{r}\times\boldsymbol{S} = -\frac{2k\rho}{r^3}e^{-\frac{2\rho}{r}}\boldsymbol{r}\times\boldsymbol{S}. \quad (33)$$

Since $d_t\boldsymbol{S} = \boldsymbol{0}$ it follows from (33) that

$$\mathrm{d}(\boldsymbol{v}\times\boldsymbol{S}) = \gamma M e^{-\frac{2\rho}{r}}\mathrm{d}\frac{\boldsymbol{r}}{r} \text{ and } \frac{k}{\gamma M}\mathrm{d}\boldsymbol{L} = -\frac{2k\rho}{r^2}e^{-\frac{2\rho}{r}}\boldsymbol{r}_0\mathrm{d}r = \boldsymbol{F}_{N_e}\mathrm{d}r, \quad (34)$$

where the vector $\boldsymbol{L} = \boldsymbol{v}\times\boldsymbol{S} - \gamma M e^{-\frac{2\rho}{r}}\boldsymbol{r}_0$ satisfying the relation $\boldsymbol{L}\cdot\boldsymbol{S} = 0$ is no longer an element of *Milankovic*'*s* constant vector elements, more precisely is no longer *Laplace*'*s* integration vector constant, see [9]. If we now multiply $\boldsymbol{L}$ by $\boldsymbol{r}$, we get

$$\boldsymbol{L}\cdot\boldsymbol{r} = (\boldsymbol{v}\times\boldsymbol{S})\cdot\boldsymbol{r} - \gamma M e^{-\frac{2\rho}{r}}r. \quad (35)$$

Since $(\boldsymbol{v}\times\boldsymbol{S})\cdot\boldsymbol{r} = S^2$, it follows from (35) that

$$r = \frac{1}{\gamma M}\frac{S^2}{e^{-\frac{2\rho}{r}} + \frac{L}{\gamma M}\cos\varphi}, \quad (36)$$

where φ is an angle between $\boldsymbol{r}$ and $\boldsymbol{L}$. This equation describes the motion of $\mathcal{M}$ under the action of the modified *Newton*'*s* gravity force $\boldsymbol{F}_{N_e}$. Conditionally speaking, there is no formal difference between (36) and its analog in the ordinary *Newton*'*s* gravity theory. The key difference lies in the fact that $\boldsymbol{L}$ is no longer constant vector.

For $L = |\boldsymbol{L}|$ let $\boldsymbol{L}_0 = \boldsymbol{L}/L$ and $\mathrm{d}\boldsymbol{L}_0 = \mathrm{d}\omega\boldsymbol{k}$, where

$\boldsymbol{k}$ is the unit vector orthogonal to $\boldsymbol{L}_0$. Then, from (34) we get

$$\frac{kL}{\gamma M}\mathrm{d}\boldsymbol{L}_0\cdot\boldsymbol{k}=\boldsymbol{F}_{N_e}\mathrm{d}r\cdot\boldsymbol{k}, \tag{37}$$

which together with (36) yields

$$\left(\frac{e^{\frac{2\rho}{r}}}{r}-\frac{1}{\tilde r}\right)\mathrm{d}\omega=\frac{\rho}{\tilde r}\sin(2\varphi)\mathrm{d}\left(\frac{1}{r}\right), \tag{38}$$

where $\tilde r=\alpha^2/\rho=S^2/\gamma M$ and

$$e^{2\rho/r}/r-1/\tilde r=(L/\gamma M)\left(e^{2\rho/r}/\tilde r\right)\cos\varphi.$$

In addition, the dot product $\boldsymbol{L}\cdot\boldsymbol{L}=L^2$ leads to

$$\left(\frac{L}{\gamma M}\right)^2=\left(\frac{\tilde r}{r}-e^{-\frac{2\rho}{r}}\right)^2-\left(\frac{\tilde r}{r}\right)^2+\left(\frac{\tilde r v}{S}\right)^2. \tag{39}$$

Thus,

$$\left(\frac{L}{\gamma M}\frac{e^{\frac{2\rho}{r}}}{\tilde r}\right)^2=\left(\frac{e^{\frac{2\rho}{r}}}{r}-\frac{1}{\tilde r}\right)^2-\frac{e^{\frac{4\rho}{r}}}{r^2}+\left(e^{\frac{2\rho}{r}}\frac{v}{S}\right)^2. \tag{40}$$

If ϑ is the angle between $\boldsymbol{r}$ and $\boldsymbol{v}$, it follows from (36) and (40) that

$$\left(\frac{e^{\frac{2\rho}{r}}}{r}-\frac{1}{\tilde r}\right)^2\tan^2\varphi=\frac{e^{\frac{4\rho}{r}}}{r^2}\frac{1}{\tan^2\vartheta}. \tag{41}$$

Hence,

$$1-\frac{r}{\tilde r}e^{-\frac{2\rho}{r}}=\frac{1}{\tan\varphi\tan\vartheta}\ \text{and}\ \frac{r}{\tilde r}e^{-\frac{2\rho}{r}}=-\frac{\cos(\varphi+\vartheta)}{\sin\varphi\sin\vartheta}. \tag{42}$$

Note that $\vartheta=\pi/2$, whenever $\varphi=k\pi(k=0,1,2,\cdots)$. If we now multiply (38) by $\tan\varphi$ we get

$$\left(\frac{e^{\frac{2\rho}{r}}}{r}-\frac{1}{\tilde r}\right)\tan\varphi\mathrm{d}\omega=2\frac{\rho}{\tilde r}\sin^2\varphi\mathrm{d}\left(\frac{1}{r}\right), \tag{43}$$

which together with (41) and (42) yields

$$\begin{aligned}\mathrm{d}\omega&=2\frac{\rho}{\tilde r}\frac{r}{\tilde r}e^{-\frac{2\rho}{r}}\tan\vartheta\sin^2\varphi\mathrm{d}\left(\frac{\tilde r}{r}\right)\\&=-2\frac{\rho}{\tilde r}\cos(\varphi+\vartheta)\frac{\sin\varphi}{\cos\vartheta}\mathrm{d}\left(\frac{\tilde r}{r}\right),\end{aligned} \tag{44}$$

whenever $0<\vartheta\le\pi/2$. If $\varphi=\pi/2\left(re^{-2\rho/r}\Big|_{\varphi=\pi/2}=\tilde r\right)$, then

$$\tilde r\,d_r\omega\Big|_{\varphi=\frac{\pi}{2}}=-2\frac{\rho}{\tilde r}e^{-\frac{4\rho}{r}}\tan\vartheta\Big|_{\varphi=\frac{\pi}{2}}. \tag{45}$$

Since $L/\gamma M\big|_{\varphi=\pi/2}=e^{-2\rho/r}/\tan\vartheta\big|_{\varphi=\pi/2}$, see (38), it follows from (36) that

$$\frac{1}{\tilde r}d_\varphi r\Big|_{\varphi=\pi/2}=\frac{e^{\frac{4\rho}{r}}}{1+\frac{2\rho}{r}}\frac{L}{\gamma M}\Bigg|_{\varphi=\pi/2}=\frac{e^{\frac{2\rho}{r}}}{1+\frac{2\rho}{r}}\frac{1}{\tan\vartheta}\Bigg|_{\varphi=\pi/2}, \tag{46}$$

which together with (45) finally yields

$$d_\varphi\omega\Big|_{\varphi=\pi/2}=-2\frac{\rho}{\tilde r}\frac{e^{-\frac{2\rho}{r}}}{1+\frac{2\rho}{r}}\Bigg|_{\varphi=\pi/2}=-2\rho\frac{1}{r+2\rho}\Bigg|_{\varphi=\pi/2}. \tag{47}$$

This result we can also get explicitly from (46). Namely, if θ is the polar angle, then $\theta=\varphi+\omega$. Therefore, it follows from (46) that

$$\frac{1}{r}d_\theta r\left(1+d_\varphi\omega\right)\Big|_{\varphi=\pi/2}=\frac{1}{1+\frac{2\rho}{r}}\frac{1}{\tan\vartheta}\Bigg|_{\varphi=\pi/2}. \tag{48}$$

Since $d_\theta r/r=1/\tan\vartheta$ we have

$$d_\varphi\omega\Big|_{\varphi=\pi/2}=\frac{r}{r+2\rho}\Bigg|_{\varphi=\pi/2}-1=-2\rho\frac{1}{r+2\rho}\Bigg|_{\varphi=\pi/2}, \tag{49}$$

that is just the same as (47). Hence,

$$\begin{aligned}&d_\theta\omega\left(1+d_\varphi\omega\right)\Big|_{\varphi=\pi/2}=-2\rho\frac{1}{r+2\rho}\Bigg|_{\varphi=\pi/2}\ \text{and}\ d_\theta\omega\Big|_{\varphi=\pi/2}\\&=-2\rho\frac{1}{r+2\rho}\frac{r+2\rho}{r}\Bigg|_{\varphi=\pi/2}=-2\rho\frac{1}{r}\Bigg|_{\varphi=\pi/2}.\end{aligned} \tag{50}$$

So, as $r\big|_{\varphi=\pi/2}\simeq a\left(1-e^2\right)$, where a and e are the semimajor axis and the eccentricity of the orbit, the following angle value

$$\tilde\omega=-\frac{2\pi\rho}{a\left(1-e^2\right)} \tag{51}$$

is a very good approximation for the perihelion regression ω per one revolution $(\varphi=2\pi)$ of the *Planets*.

5.2. The Modified Perturbing Force

If, in addition to the modified *Newton*'s gravity force $\boldsymbol{F}_{N_e}$, we include the modified perturbing force [10]

$$\hat{\boldsymbol{F}}=\frac{\hat F}{\hat r}\hat{\boldsymbol{r}}=\frac{2k\hat\rho}{\hat r^3}e^{-\frac{2\hat\rho}{\hat r}}\hat{\boldsymbol{r}},$$

where $\hat{\boldsymbol{r}}$ and $\hat\rho=\gamma M_p/c^2$ are the radial vector between the *Planet* and the perturbing planet (whose orbit is assumed to be circular and coplanar with *Mercury*'*s* orbit) and the gravitational radius for the perturbing planet, respectively, then

$$\frac{k}{\gamma M}\frac{\hat{\boldsymbol{F}}\times\boldsymbol{S}}{m}=-\frac{S}{2\rho}\left(\hat{F}\cos\gamma\boldsymbol{t}+\hat{F}\sin\gamma\boldsymbol{r}_0\right),$$

where γ is the angle between $\boldsymbol{F}_{N_e}$ and $\hat{\boldsymbol{F}}$, and $\boldsymbol{t}$ is the unit vector perpendicular to $\boldsymbol{r}_0$. Thus, the second equation of (35) becomes

$$\frac{k}{\gamma M}\mathrm{d}\hat{\boldsymbol{L}}=\left(\boldsymbol{F}_{N_e}+\frac{r}{\rho}\hat{\boldsymbol{F}}_r\right)\mathrm{d}r+\frac{r}{\rho}\frac{\mathrm{d}\hat{F}\cos\gamma-\hat{F}\sin\gamma\mathrm{d}\left(\theta+\gamma\right)}{2}\boldsymbol{r},$$

where $\hat{\boldsymbol{F}}_r=\hat{F}\cos\gamma\boldsymbol{r}_0$ and

$$\hat{\boldsymbol{L}}=\boldsymbol{v}\times\boldsymbol{S}-\gamma Me^{-\frac{2\rho}{r}}\left[1-\left(\frac{r}{\hat{r}}\right)^2\frac{\hat{\rho}}{\rho}\cos\gamma e^{-\frac{2\hat{\rho}}{\hat{r}}\left(1-\frac{\hat{r}}{r}\frac{\rho}{\hat{\rho}}\right)}\right]\boldsymbol{r}_0.$$

This vector is the modified *Laplace*'s integration vector (or more precisely, the modified *Laplace-Runge-Lenz* vector). Their original versions come from the ordinary *Newton*'s gravity theory. If we denote

$$r\left[d_r\hat{F}\cos\gamma-\hat{F}\sin\gamma d_r\left(\theta+\gamma\right)\right]\boldsymbol{r}_0/2$$

by $\Delta\hat{\boldsymbol{F}}_r$, then we have

$$\frac{k}{\gamma M}\mathrm{d}\hat{\boldsymbol{L}}=\left[\boldsymbol{F}_{N_e}+\frac{r}{\rho}\left(\hat{\boldsymbol{F}}_r+\Delta\hat{\boldsymbol{F}}_r\right)\right]\mathrm{d}r.$$

6. Conclusion

The mathematical model of a material point motion in the three-dimensional spatial subspace of the four-dimensional space-time continuum and in the field of the action of a conservative active force $\boldsymbol{F}$ is analogous to *Newton*'s mathematical model of the classical mechanics. In addition, the metric $\left(\mathrm{d}\sigma\right)^2$ of the integral space, which represents *the kinetic energy* of a material point from the viewpoint of that space, is the *Minkowski* metric from *Einstein*'s relativity theory. Accordingly, it can be said that in the paper a new connection has been established, in contrast to an approximative one, between the classical *Newton*'s mathematical model and the relativistic *Einstein*'s mathematical model.

On the other hand the approximately modified *Newton*'s gravity concept is not, from any point of view, in collision with old *Newton*'s one. At the same time it solves the acutely vexed questions within old *Newton*'s gravity concept (the singularity and perihelion problems). Furthermore, analyzing the analytical expression for the modified *Newton*'s gravity force $\boldsymbol{F}_{N_e}$, we can separate the four indicative domains of its field of the action (see **Figure 1**). The first one is a domain of the weak action on finitely small distances. The second one is a domain of the strong action in a neighborhood of the gravitational radius $\rho=\gamma M/c^2\left(\partial_r F\big|_{r=\rho}=0\text{ and }\partial_{rr}^2 F\big|_{r=\rho}<0\right)$. The third one is a domain of the action on finitely large distances relative to the gravitational radius ρ and with the relatively small velocities relative to the light velocity, and the fourth on finitely large distances relative to the gravitational radius ρ and with velocities that are comparable to the light velocity. Previously separated domains of the field of the action of the modified *Newton*'s gravity force $\boldsymbol{F}_{N_e}$ it would be desirable to compare to the fields of the action of the four so far non-unified fundamental forces (weak and strong nuclear interactions, gravity and *Lorenz*'s electromagnetism). Clearly, all of these facts aforementioned could be subject of further analyses. Note at the end that a correction to *Newton*'s gravity law in the form of the functional dependence $r^{-3}e^{-r/r_c}\ \boldsymbol{r}$ irresistibly reminding of the modified *Newton*'s gravity force, and obviously wrongly called the fifth force, has been revealed by a reexamination of the old attraction data and careful new force measurements presented in [11].

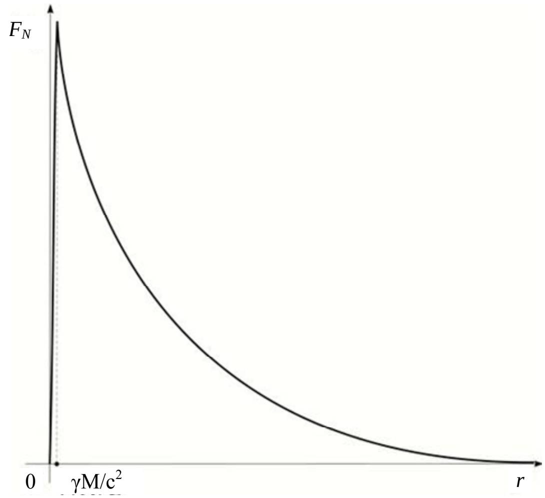

Figure 1. Modified *Newton*'s gravity force.

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Appendix: The Freee Motion of $\mathcal{M}$ in the Integral Space

Let us start with the *Euler-Lagrange* equations

$$d_w\left(\partial_{d_w x^\beta}\mathcal{W}\right)-\partial_\beta\mathcal{W}=0, \tag{52}$$

where

$$\mathcal{W}=\frac{\Bbbk+\mathcal{P}}{k}e_{\alpha\beta}d_w x^\alpha d_w x^\beta, \tag{53}$$

as the condition for the action (12) to be stationary. The geodesic Equations (13) are explicitly obtained from it in a known way. If spatial co-ordinates are spherical ones (r,θ,φ), then the components of $e_{\alpha\beta}$ depend only on r and φ, so that it follows from (52) that

$$d_w\left(\frac{\Bbbk+\mathcal{P}}{k}e_{11}cd_w t\right)=0 \tag{54}$$

and

$$d_w\left(\frac{\Bbbk+\mathcal{P}}{k}e_{33}d_w\theta\right)=0, \tag{55}$$

that leads to

$$\left(\Bbbk+\mathcal{P}\right)cd_w t=k \tag{56}$$

and

$$\left(\Bbbk+\mathcal{P}\right)\left(r^2\cos^2\varphi\right)d_w\theta=k\alpha. \tag{57}$$

Let the polar extension r and the polar angle θ be intensities of $\boldsymbol{r}$ and an angle between the position vector $\boldsymbol{r}$ and the polar axis p passing through the origin and the perihelial point, respectively. Then, since $S=r^2\dot{\theta}=\text{const.}$, where $\boldsymbol{S}=\boldsymbol{r}\times\boldsymbol{v}$ is the so-called sector velocity vector, it follows from the condition (57) that the motion is the plane one $(\varphi=0)$ and $S=\alpha c$. As $(\mathrm{d}s)^2=(\mathrm{d}r)^2+r^2(\mathrm{d}\theta)^2$ then we obtain finally from (5), (10) and (57) that

$$(\mathrm{d}r)^2=r^4\left[\left(\frac{1}{\alpha}\right)^2-\left(\frac{1}{r}\right)^2-\frac{\Bbbk+\mathcal{P}}{k\alpha^2}\right](\mathrm{d}\theta)^2, \tag{58}$$

that just leads to the *Binet* differential equation for free motion in plane polar co-ordinates

$$d^2_{\theta\theta}\frac{1}{r}+\frac{1}{r}=0. \tag{59}$$

The solution $r_0=r\cos\theta$, where r_0 is the perihelial distance, to this differential equation, defines a straight-line in plane polar co-ordinates.

