Applied Mathematics, 2012, 3, 1357-1367
http://dx.doi.org/10.4236/am.2012.330192 Published Online October 2012 (http://www.SciRP.org/journal/am)

Hybrid Extragradient-Type Methods for Finding a Common Solution of an Equilibrium Problem and a Family of Strict Pseudo-Contraction Mappings

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Received July 4, 2012; revised August 4, 2012; accepted August 11, 2012

ABSTRACT

This paper proposes a new hybrid variant of extragradient methods for finding a common solution of an equilibrium problem and a family of strict pseudo-contraction mappings. We present an algorithmic scheme that combine the idea of an extragradient method and a successive iteration method as a hybrid variant. Then, this algorithm is modified by projecting on a suitable convex set to get a better convergence property. The convergence of two these algorithms are investigated under certain assumptions.

Keywords: Equilibrium Problems; Fixed Point; Pseudo-Monotone; Lipschitz-Type Continuity; Extragradient Method; Strict Pseudo-Contraction Mapping

1. Introduction

Let $\mathcal{H}$ be a real Hilbert space endowed with an inner product $\langle\cdot,\cdot\rangle$ and a norm $\|\cdot\|$ associated with this inner product, respectively. Let C be a nonempty closed convex subset of $\mathcal{H}$, and f be a bifunction from $C \times C$ to $\mathbf{R}$ such that $f(x, x) = 0$ for all $x \in C$. An equilibrium problem in the sense of Blum and Oettli [1] is stated as follows:

$$\text{Find } x^* \in C \text{ such that } f\left(x^*, y\right) \geq 0 \text{ for all } y \in C. \qquad (1)$$

Problem of the form (1) on one hand covers many important problems in optimization as well as in nonlinear analysis such as (generalized) variational inequality, nonlinear complementary problem, nonlinear optimization problem, just to name a few. On the other hand, it is rather convenient for reformulating many practical problems in economic, transportation and engineering (see [1,2] and the references quoted therein).

The existence of solution and its characterizations can be found, for example, in [3], while the methods for solving problem (1) have been developed by many researchers [4-8].

Alternatively, the problem of finding a common fixed point element of a finite family of self-mappings

$$S := \{S_i\}_{i=1}^p \quad (p \geq 1)$$

is expressed as follows:

$$\text{Find } x^* \in C \text{ such that } x^* \in \bigcap_{i=1}^p \mathrm{Fix}\left(S_i, C\right), \qquad (2)$$

where $\mathrm{Fix}(S_i, C)$ is the set of the fixed points of the mapping S_i ($i = 1, \cdots, p$).

Problem of finding a fixed point of a mapping or a family of mappings is a classical problem in nonlinear analysis. The theory and solution methods of this problem can be found in many research papers and mono-graphs (see [9]).

Let us denote by $\mathrm{Sol}(f, C)$ and

$$\mathrm{Fix}\left(S, C\right) := \bigcap_{i=1}^p \mathrm{Fix}\left(S_i, C\right)$$

the solution sets of the equilibrium problem (1) and the fixed-point problem (2), respectively. Our aim in this paper is to address to the problem of finding a common solution of two problems (1) and (2). Typically, this problem is stated as follows:

$$\text{Find } x^* \in \mathrm{Fix}\left(S, C\right) \cap Sol\left(f, C\right). \qquad (3)$$

Our motivation originates from the following observations. Problem (3) can be on one hand considered as an extension of problem (1) by setting $S_i = I$ for all $i = 1, \cdots, p$, the identity mapping. On the other hand it is significant in many practical problems. Since the equilibrium problems have found many applications in economic, transportation and engineering, in some practical problems it may happen that the feasible set of those problems result as a fixed point solution set of one or many fixed point problems. In this case, the obtained problem can be reformulated in the form of (3). An important special case of problem (3) is that

$$f(x,y) = \langle F(x), y-x \rangle$$

and this problem is reduced to finding a common element of the solution set of variational inequalities and the solution set of a fixed point problem (see [10-12]).

In this paper, we propose a new hybrid iterative-based method for solving problem (3). This method can be considered as an improvement of the viscosity approximation method in [11] and iterative methods in [13]. The idea of the algorithm is to combine the extragradient-type methods proposed in [14] and a fixed point iteration method. Then, the algorithm is modified by projecting on a suitable convex set to obtain a new variant which possesses a better convergence property.

The rest of the paper is organized as follows. Section 2 recalls some concepts and results in equilibrium problems and fixed point problems that will be used in the sequel. Section 3 presents two algorithms for solving problem (3) and some discussion on implementation. Section 4 investigates the convergence of the algorithms presented in Section 3 as the main results of our paper.

2. Preliminaries

Associated with the equilibrium problem (1), the following definition is common used as an essential concept (see [3]).

Definition 2.1. *Let C be a nonempty closed convex subset of a real Hilbert space $\mathcal{H}$. A bifunction f: $C \times C \to \mathbf{R}$ is said to be*

1) *Monotone on C if* $f(x,y) + f(y,x) \le 0$ *for all x and y in C*;

2) *Pseudo-monotone on C if* $f(x,y) \ge 0$ *implies* $f(y,x) \le 0$ *for all x, y in C*;

3) *Lipschitz-type continuous on C with two Lipschitz constants $c_1 > 0$ and $c_2 > 0$ if*

$$\begin{aligned} &f(x,y) + f(y,z) \\ &\ge f(x,z) - c_1\|x-y\|^2 - c_2\|y-z\|^2,\ \forall x,y,z \in C. \end{aligned} \tag{4}$$

It is clear that every monotone bifunction f is pseudo-monotone. Note that the Lipschitz continuous condition (4) was first introduced by Mastroeni in [7].

The concept of strict pseudo-contraction is considered in [15], which defined as follows.

Definition 2.2. *Let C be a nonempty closed convex subset of a real Hilbert space $\mathcal{H}$. A mapping S: $C \to C$ is said to be a strict pseudo-contraction if there exists a constant $0 \le L < 1$ such that*

$$\begin{aligned} &\|S(x) - S(y)\|^2 \\ &\le \|x-y\|^2 + L\|(I-S)(x) - (I-S)(y)\|^2,\ \forall x,y \in C, \end{aligned} \tag{5}$$

where I is the identity mapping on $\mathcal{H}$. If $L = 0$ then S is called non-expansive on C.

The following proposition lists some useful properties of a strict pseudo-contraction mapping.

Proposition 2.3. [15] *Let C be a nonempty closed convex subset of a real Hilbert space $\mathcal{H}$, S: $C \to C$ be an L-strict pseudo-contraction and for each $i = 1, \cdots, p$, S_i: $C \to C$ is a L_i -strict pseudo-contraction for some* $0 \le L_i < 1$. *Then*:

1) *S satisfies the following Lipschitz condition*:

$$\|S(x) - S(x)\| \le \frac{1+L}{1-L}\|x-y\|,\ \forall x,y \in C;$$

2) *$I - S$ is demiclosed at* 0. *That is, if the sequence $\{x^k\}$ contains in C such that $x^k \rightharpoonup \bar{x}$ and $(I-S)(x^k) \to 0$ then $(I-S)(\bar{x}) = 0$*;

3) The set of fixed points $Fix(S)$ is closed and convex;

4) If $\lambda_i > 0$ $(i = 1, \cdots, p)$

and
$$\sum_{i=1}^{p} \lambda_i = 1$$

then
$$\sum_{i=1}^{p} \lambda_i S_i$$

is a $\bar{L}$ -strict pseudo-contraction with

$$\bar{L} := \max\{L_i \,|1 \le i \le L|\};$$

5) If λ_i is chosen as in (d) and $\{S_i \,|i = 1, \cdots, p\}$ has a common fixed point then:

$$Fix\left(\sum_{i=1}^{p} \lambda_i S_i\right) = \bigcap_{i=1}^{p} Fix(S_i, C).$$

Before presenting our main contribution, let us briefly look at the recently literature related to the methods for solving problem (3). In [11], S. Takahashi and W. Takahashi proposed an iterative scheme under the name *viscosity approximation methods* for finding a common element of set of solutions of (1) and the set of fixed points of non-expansive mapping S in a real Hilbert space $\mathcal{H}$. This method generated an iteration sequence $\{x^k\}$ starting from a given initial point $x^0 \in \mathcal{H}$ and computed x^{k+1} as

$$\begin{cases} \text{Find}\, u^k \in C \text{ such that } f(u^k, y) + \dfrac{1}{r_k}\langle y-u^k, u^k - x^k \rangle \ge 0, \\ \text{for all } y \in C, \\ \text{Compute } x^{k+1} = \alpha_k g(x^k) + (1-\alpha_k) S(u^k), \end{cases} \tag{6}$$

where g is a contraction of $\mathcal{H}$ into itself, the sequences of parameters $\{r_k\}$ and $\{\alpha_k\}$ were chosen appropriately. Under certain choice of $\{\alpha_k\}$ and $\{r_k\}$, the authors showed that two iterative sequences $\{x^k\}$ and $\{u^k\}$ converged strongly to

$$z = \mathrm{Pr}_{\mathrm{Fix}(S,C)\bigcap \mathrm{Sol}(f,C)}\left(g(z)\right),$$

where Pr_C denotes the projection onto C.

Alternatively, the problem of finding a common fixed point of a finite sequence of mappings has been studied by many researchers. For instance, Marino and Xu in [7] proposed an iterative algorithm for finding a common fixed point of p strict pseudo-contraction mapping S_i ($i = 1, \cdots, p$). The method computed a sequence $\{x^k\}$ starting from $x^0 \in \mathcal{H}$ and taking:

$$x^{k+1} = \alpha_k x^k + (1-\alpha_k)\sum_{i=1}^{p}\lambda_{k,i} S_i\left(x^k\right), \qquad (7)$$

where the sequence of parameters $\{\lambda_k\}$ was chosen in a specific way to ensure the convergence of the iterative sequence $\{x^k\}$. The authors showed that the sequence $\{x^k\}$ converged weakly to the same point

$$\bar{x} \in \bigcap_{i=1}^{p}\mathrm{Fix}\left(S_i, C\right).$$

Recently, Chen *et al.* in [13] proposed a new iterative scheme for finding a common element of the set of common fixed points of a strict pseudo-contraction sequence $\{\bar{S}_i\}$ and the set of solutions of the equilibrium problem (1) in a real Hilbert space $\mathcal{H}$. This method is briefly described as follows. Given a starting point $x^0 \in \mathcal{H}$ and generates three iterative sequences $\{x^k\}$, $\{y^k\}$ and $\{z^k\}$ using the following scheme:

$$\begin{cases} \text{Compute } y^k = \alpha_k x^k + (1-\alpha_k)\bar{S}_k\left(x^k\right), \\ \text{Find } z^k \in C \text{ such that } f\left(z^k, y\right) \\ \qquad + \dfrac{1}{r_k}\left\langle y - z^k, z^k - y^k\right\rangle \ge 0 \quad \forall y \in C, \qquad (8) \\ \text{Compute } x^{k+1} = Pr_{C_k}\left(x^0\right), \\ \text{where } C_k := \left\{v \in C \,\middle|\, \left\|z^k - v\right\| \le \left\|x^k - v\right\|\right\}. \end{cases}$$

Here, two sequences $\{\alpha_k\}$ and $\{r_k\}$ are given as control parameters. Under certain conditions imposed on $\{\alpha_k\}$ and $\{r_k\}$, the authors showed that the sequences $\{x^k\}$, $\{y^k\}$ and $\{z^k\}$ converged strongly to the same point x^* such that

$$x^* \in \mathrm{Pr}_{\mathrm{Sol}(f,C)\bigcap \mathrm{Fix}(S)}\left(x^0\right),$$

where S is a mapping of C into itself defined by

$$S(x) = \lim_{j\to\infty}\bar{S}_j(x)$$

for all $x \in C$.

The solution methods for finding a common element of the set of solutions of (1) and

$$\bigcap_{i=1}^{p}\mathrm{Fix}\left(S_i, C\right)$$

in a real Hilbert space have been recently studied in many research papers (see [8,12,16-23]). Throughout those papers, there are two essential assumptions on the function f have been used: *monotonicity* and *Lipschitz-type continuity*.

In this paper, we continue studying problem (3) by proposing a new iterative-based algorithm for finding a solution of (3). The essential assumptions that will be used in our development includes: *pseudo-monotonicity* and *Lipschitz-type continuity* of the bifunction f and the *strict pseudo-contraction* of S_i ($i = 1, \cdots, p$). The algorithm is then modified to obtain a new variant which has a better convergence property.

3. New Hybrid Extragradient Algorithms

In this section we present two algorithms for finding a solution of problem (3). Before presenting algorithmic schemes, we recall the following assumptions that will be used to prove the convergence of the algorithms.

Assumption 3.1. *The bifunction f satisfies the following conditions*:

1) *f is pseudo-monotone and continuous on C*;

2) *f is Lipschitz-type continuous on C*;

3) *For each $x \in C$, $f(x,\cdot)$ is convex and subdifferentiable on C*.

Assumption 3.2. *For each $i = 1, \cdots, p$, S_i is L_i-strict pseudo-contraction for some $0 \le L_i < 1$.*

Assumption 3.3. *The solution set of* (3) *is nonempty, i.e.*

$$\mathrm{Fix}(S,C) \cap \mathrm{Sol}(f,C) \ne \varnothing. \qquad (9)$$

Note that if

$$C \subseteq \mathrm{ri}\left(\mathrm{dom}\left(f(x,\cdot)\right)\right),$$

where

$$\mathrm{ri}\left(\mathrm{dom}\left(f(x,\cdot)\right)\right)$$

is the set of relative interior points of the domain of $f(x,\cdot)$, then Assumption 3.1 3) is automatically satisfied. The first algorithm is now described as follows.

Algorithm 3.4

Initialization: *Given a tolerance $\varepsilon > 0$. Choose three positive sequences $\{\lambda_k\}$, $\{\lambda_{k,i}\}$ and $\{\alpha_k\}$ satisfy the conditions*:

$$\begin{cases} \{\lambda_k\} \subset [a,b] \text{ for some } a,b \in \left(0, \frac{1}{L}\right), \\ \text{where } L := \max\{2c_1, 2c_2\}, \\ \{\alpha_k\} \subset [\alpha,\beta] \text{ for some } \alpha,\beta \in \left(\bar{L},1\right), \\ \text{where } \bar{L} := \max\left\{L_i \middle| 1 \le i \le p\right\}, \\ \sum_{i=1}^{p} \lambda_{k,i} = 1 \text{ for all } k \ge 1 \end{cases} \quad (10)$$

Find an initial point $x^0 \in C$. *Set* $k := 0$.

***Iteration k*:** *Perform the two steps below*:

Step 1: *Solve two strongly convex programs*:

$$\begin{cases} y^k := \operatorname{argmin}\left\{\lambda_k f\left(x^k, y\right) + \frac{1}{2}\left\|y - x^k\right\|^2 \middle| y \in C\right\}, \\ t^k := \operatorname{argmin}\left\{\lambda_k f\left(y^k, y\right) + \frac{1}{2}\left\|y - x^k\right\|^2 \middle| y \in C\right\}. \end{cases} \quad (11)$$

Step 2: *Set*

$$x^{k+1} := \alpha_k t^k + (1-\alpha_k)\sum_{i=1}^{p}\lambda_{k,i} S_i\left(t^k\right).$$

Set $k := k+1$ *and go back to Step* 1.

The main task of Algorithm 3.4 is to solve two strongly convex programming problems at Step 1. Since these problems are strongly convex and *C* is nonempty, they are uniquely solvable. To terminal the algorithm, we can use the condition on step size by checking $\left\|x^{k+1} - x^k\right\| \le \varepsilon$ for a given tolerance $\varepsilon > 0$. Step 1 of Algorithm 3.4 is known as an extragradient-type step for solving equilibrium problem (1) proposed in [14]. Step 2 is indeed the iteration (7) of the iterative method proposed in [24].

As we will see in the next section, Algorithm 3.4 generates a sequence $\{x^k\}$ that converges weakly to a solution x^* of (3). Recently, a modification of Mann's algorithm for finding a common element of *p* strict pseudo-contractions was proposed [15]. The authors proved that this algorithm converged strongly to a common fixed point of the *p* strict pseudo-contractions. In the next algorithm, we extended the algorithm in [15] for finding a common solution of the set of common fixed points of *p* strict pseudo-contractions and the equilibrium problems to obtain a strongly convergence algorithm. This algorithm is similar to the iterative scheme (8), where an augmented step will be added to Algorithm 3.4 and obtain a new variant of Algorithm 3.4.

For a given closed, convex set *X* in $\mathcal{H}$ and $x \in \mathcal{H}$, $Pr_X(x)$ denotes the projection of *x* onto *X*. The algorithm is described as follows.

Algorithm 3.5

***Initialization*:** *Given a tolerance* $\varepsilon > 0$. *Choose positive sequences* $\{\lambda_k\}$, $\{\lambda_{k,i}\}$ *and* $\{\alpha_k\}$ *satisfy the conditions* (10) *and the following addition condition*:

$$\inf_{k\ge1}\lambda_{k,i} > 0 \text{ for all } i = 1,\cdots,p. \quad (12)$$

Find an initial point $x^0 \in C$. *Set* $k := 0$.

***Iteration k*:** *Perform the three steps below*:

Step 1: *Solve two strongly convex programs*:

$$\begin{cases} y^k := \operatorname{argmin}\left\{\lambda_k f\left(x^k, y\right) + \frac{1}{2}\left\|y - x^k\right\|^2 \middle| y \in C\right\}, \\ t^k := \operatorname{argmin}\left\{\lambda_k f\left(y^k, y\right) + \frac{1}{2}\left\|y - x^k\right\|^2 \middle| y \in C\right\}. \end{cases} \quad (13)$$

Step 2: *Set*

$$z^k := \alpha_k t^k + (1-\alpha_k)\sum_{i=1}^{p}\lambda_{k,i} S_i\left(t^k\right).$$

Step 3: *Compute*

$$x^{k+1} := \Pr_{P_k \cap Q_k}\left(x^0\right), \quad (14)$$

where

$$P_k := \left\{x \in C \middle| \left\|z^k - x\right\|^2 \le \left\|t^k - x\right\|^2 - r_k\right\},$$

$$Q_k := \left\{x \in C \middle| \left\langle x^k - x, x^0 - x^k\right\rangle \ge 0\right\}, \quad (15)$$

$$r_k := (1-\alpha_k)\left(\alpha_k - \bar{L}\right)\left\|t^k - \sum_{i=1}^{p}\lambda_{k,i} S_i\left(t\right)^k\right\|^2.$$

Set $k := k+1$ *and go back to Step* 1.

The augmented step needed in Algorithm 3.5 is a simple projection on the intersect of two half-planes. The projection is cheap to compute in implementation.

4. The Convergence of the Algorithms

This section investigates the convergence of the algorithms 3.4 and 3.5. For this purpose, let us recall the following technical lemmas which will be used in the sequel.

Lemma 4.1 [25]. *Let C be a nonempty closed convex subset of a real Hilbert space* $\mathcal{H}$. *Suppose that*, *for all* $u \in C$, *the sequence* $\{x^k\}$ *satisfies*

$$\left\|x^{k+1} - u\right\| \le \left\|x^k - u\right\|, \ \forall k \ge 0.$$

Then the sequence $\left\{\Pr_C\left(x^k\right)\right\}$ *converges strongly to some* $x \in C$.

Using the well-known necessary and sufficient condition for optimality in convex programming, we have the following result.

Lemma 4.2 [2]. *Let C be a nonempty closed convex subset of a real Hilbert space* $\mathcal{H}$ *and g*: $C \to \mathbf{R}$ *be subdifferentiable on C. Then* x^* *is a solution to the following convex problem*

$$\min\{g(x) | x \in C\}$$

if and only if

$$0 \in \partial g(x^*) + N_C(x^*),$$

where $\partial g(\cdot)$ *denotes the subdifferential of* g *and* $N_C(x^*)$ *is the (outward) normal cone of C at* $x^* \in C$.

The next lemma is regarded to the property of a projection mapping.

Lemma 4.3 [9]. *Let C be a nonempty closed convex subset of a real Hilbert space* $\mathcal{H}$ *and* $x^0 \in \mathcal{H}$. *Let the sequence* $\{x^k\}$ *be bounded such that every weakly cluster point* $\bar{x}$ *of* $\{x^k\}$ *belongs to C and*

$$\|x^k - x^0\| \le \|x^0 - Pr_C(x^0)\|, \ \forall k \ge 0.$$

Then x^k *converges strongly to* $\mathrm{Pr}_C(x^0)$ *as* $k \to \infty$.

Lemma 4.4 (*see* [14], *Lemma* 3.1). *Let C be a nonempty closed convex subset of a real Hilbert space* $\mathcal{H}$. *Let* $f: C\times C \to \mathfrak{R}$ *be a pseudomonotone, Lipschitz- type continuous bifunction with constants* $c_1 > 0$ *and* $c_2 > 0$. *For each* $x \in C$, *let* $f(x,\cdot)$ *be convex and subdifferentiable on C. Suppose that the sequences* $\{x^k\}$, $\{y^k\}$, $\{t^k\}$ *generated by Scheme* (13) *and* $x^* \in Sol(f,C)$. *Then*

$$\|t^k - x^*\|^2 \le \|x^k - x^*\|^2 - (1-2\lambda_k c_1)\|x^k - y^k\|^2 - (1-2\lambda_k c_2)\|y^k - t^k\|^2, \ \forall k \ge 0.$$

Now, we prove the main convergence theorem.

Theorem 4.5. *Suppose that Assumptions* 3.1-3.3 *are satisfied. Then the sequences* $\{x^k\}$, $\{y^k\}$ *and* $\{z^k\}$ *generated by Algorithm* 3.4 *converge weakly to the same point*

$$x^* \in \bigcap_{i=1}^p \mathrm{Fix}(S_i, C) \cap \mathrm{Sol}(f,C),$$

where

$$x^* = \lim_{k\to\infty} Pr_{\bigcap_{i=1}^p \mathrm{Fix}(S_i,C)\cap \mathrm{Sol}(f,C)}(x^k) \tag{16}$$

Proof. The proof of this theorem is divided into several steps.

Step 1. *We claim that*

$$\lim_{k\to\infty}\|x^k - y^k\| = \lim_{k\to\infty}\|x^k - t^k\| = 0.$$

Indeed, for each $k \ge 1$, we denote by

$$\bar{S}_k := \sum_{i=1}^p \lambda_{k,i} S_i.$$

By statement 4) of Proposition 2.3, we see that $\bar{S}_k$ is a $\bar{L}$ -strict pseudo-contraction on C and then

$$x^{k+1} := \alpha_k t^k + (1-\alpha_k)\bar{S}_k(t^k).$$

Now, we suppose that

$$x^* \in \bigcap_{i=1}^p \mathrm{Fix}(S_i, C) \cap \mathrm{Sol}(f,C).$$

Then, using Lemma 4.4 and the relation

$$\|\lambda x + (1-\lambda)y\|^2 = \lambda\|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)\|x-y\|^2,$$

for all $\lambda \in [0,1]$ and $x, y \in \mathcal{H}$, we have

$$\begin{aligned} 2\|x^{k+1} - x^*\|^2 &= \|\alpha_k t^k + (1-\alpha_k)\bar{S}_k(t^k) - x^*\|^2 \\ &= \|\alpha_k(t^k - x^*) + (1-\alpha_k)(\bar{S}_k(t^k) - x^*)\|^2 \\ &= \alpha_k\|t^k - x^*\|^2 + (1-\alpha_k)\|\bar{S}_k(t^k) - \bar{S}_k(x^*)\|^2 \\ &\quad - \alpha_k(1-\alpha_k)\|\bar{S}_k(t^k) - t^k\|^2 \\ &\le \alpha_k\|t^k - x^*\|^2 \\ &\quad + (1-\alpha_k)\left(\|t^k - x^*\|^2 + \bar{L}\|(I-\bar{S}_k)(t^k) - (I-\bar{S}_k)(x^*)\|^2\right) \\ &\quad - \alpha_k(1-\alpha_k)\|\bar{S}_k(t^k) - t^k\|^2 \\ &= \|t^k - x^*\|^2 + (1-\alpha_k)(\bar{L}-\alpha_k)\|\bar{S}_k(t^k) - t^k\|^2 \end{aligned} \tag{17}$$

$$\le \|t^k - x^*\|^2$$

$$\le \|x^k - x^*\|^2 - (1-2\lambda_k c_1)\|x^k - y^k\|^2 \tag{18}$$

$$- (1-2\lambda_k c_2)\|y^k - t^k\|^2$$

$$\le \|x^k - x^*\|. \tag{19}$$

Therefore, there exists

$$c := \lim_{k\to\infty}\|x^k - x^*\|.$$

It follows from (18) that

$$(1-2\lambda_k c_1)\|x^k - y^k\|^2 \le \|x^k - x^*\|^2 - \|x^{k+1} - x^*\|^2.$$

So we have

$$2\|x^k - y^k\|^2 \le \frac{1}{1-2\lambda_k c_1}\left(\|x^k - x^*\|^2 - \|x^{k+1} - x^*\|^2\right) \le \frac{1}{1-Lb}\left(\|x^k - x^*\|^2 - \|x^{k+1} - x^*\|\right)^2.$$

Using this and

$$c = \lim_{k\to\infty}\|x^k - x^*\|,$$

we have

$$\lim_{k\to\infty}\left\|x^k - y^k\right\| = 0.$$

Similarly, we also have

$$\lim_{k\to\infty}\left\|y^k - t^k\right\| = 0.$$

Then, since

$$\left\|x^k - t^k\right\| \le \left\|x^k - y^k\right\| + \left\|y^k - t^k\right\|,$$

we also have

$$\lim_{k\to\infty}\left\|x^k - t^k\right\| = 0.$$

Step 2. We show that

$$\lim_{k\to\infty}\left\|x^k - \bar{S}_k\left(x^k\right)\right\| = \lim_{k\to\infty}\left\|t^k - \bar{S}_k\left(t^k\right)\right\| = 0.$$

It follows from (17) that

$$(1-\alpha_k)\left(\alpha_k - \bar{L}\right)\left\|\bar{S}_k\left(t^k\right) - t^k\right\|^2 \le \left\|t^k - x^*\right\|^2 - \left\|x^{k+1} - x^*\right\|^2.$$

Combining this and Lemma 4.4, we get

$$\begin{aligned} &2\left\|\bar{S}_k\left(t^k\right) - t^k\right\|^2 \\ &\le \frac{1}{(1-\beta)\left(\alpha - \bar{L}\right)}\left(\left\|t^k - x^*\right\|^2 - \left\|x^{k+1} - x^*\right\|^2\right) \\ &\le \frac{1}{(1-\beta)(\alpha - \bar{L})}\left(\left\|x^k - x^*\right\|^2 - \left\|x^{k+1} - x^*\right\|^2\right) \to 0 \text{ as } k \to \infty. \end{aligned}$$

Then, using (*a*) of Proposition 2.3 and Step 2, we obtain

$$\begin{aligned} &\le \left\|x^k - t^k\right\| + \left\|t^k - \bar{S}_k\left(t^k\right)\right\| + \left\|\bar{S}_k\left(t^k\right) - \bar{S}_k\left(x^k\right)\right\| \\ &\le \left\|x^k - t^k\right\| + \left\|t^k - \bar{S}_k\left(t^k\right)\right\| + \frac{1+\bar{L}}{1-\bar{L}}\left\|t^k - x^k\right\| \\ &= \frac{2}{1-\bar{L}}\left\|x^k - t^k\right\| + \left\|t^k - \bar{S}_k\left(t^k\right)\right\| \to 0 \text{ as } k \to \infty. \end{aligned}$$

In Step 3 and Step 4 of this theorem, we will consider weakly clusters of $\{x^k\}$. It follows from (19) that the sequence $\{x^k\}$ is bounded and hence there exists a subsequence $\left\{x^{k_j}\right\}$ converges weakly to $\bar{x}$ as $j \to \infty$.

Step 3. We show that

$$\bar{x} \in \bigcap_{i=1}^{p} \mathrm{Fix}\left(S_i, C\right).$$

Indeed, for each $i = 1, \cdots, p$, we suppose that $\lambda_{k_j,i}$ converges λ_i as $j \to \infty$ such that

$$\sum_{i=1}^{p} \lambda_i = 1.$$

Then we have

$$S_{k_j}(x) \to S(x) := \sum_{i=1}^{p} \lambda_i S_i(x) (\text{as } j \to \infty)\ \forall x \in C.$$

Since $\sum_{i=1}^{p} \lambda_i = 1$, from Step 2 and

$$\begin{aligned} &2\left\|x^{k_j} - S\left(x^{k_j}\right)\right\| \\ &\le \left\|x^{k_j} - \bar{S}_{k_j}\left(x^{k_j}\right)\right\| + \left\|\bar{S}_{k_j}\left(x^{k_j}\right) - S\left(x^{k_j}\right)\right\| \\ &= \left\|x^{k_j} - \bar{S}_{k_j}\left(x^{k_j}\right)\right\| + \left\|\sum_{i=1}^{p} \lambda_{k_j,i} S_i\left(x^{k_j}\right) - \sum_{i=1}^{p} \lambda_i S_i\left(x^{k_j}\right)\right\| \\ &= \left\|x^{k_j} - \bar{S}_{k_j}\left(x^{k_j}\right)\right\| + \left\|\sum_{i=1}^{p}\left(\lambda_{k_j,i} - \lambda_i\right) S_i\left(x^{k_j}\right)\right\| \\ &\le \left\|x^{k_j} - \bar{S}_{k_j}\left(x^{k_j}\right)\right\| + \sum_{i=1}^{p}\left|\lambda_{k_j,i} - \lambda_i\right|\left\|S_i\left(x^{k_j}\right)\right\|, \end{aligned}$$

we obtain that

$$\lim_{k\to\infty}\left\|x^{k_j} - S\left(x^{k_j}\right)\right\| = 0.$$

By 2) of Proposition 2.3, we have

$$\bar{x} \in \mathrm{Fix}(S) = \mathrm{Fix}\left(\sum_{i=1}^{p} \lambda_i S_i\right).$$

Then, it implies from 5) of Proposition 2.3 that

$$\bar{x} \in \bigcap_{i=1}^{p} \mathrm{Fix}\left(S_i, C\right).$$

Step 4. When $x^{k_j} \rightharpoonup \bar{x}$ as $j \to \infty$, we show that $\bar{x} \in \mathrm{Sol}(f, C)$. Indeed, since y^k is the unique strongly convex problem

$$\min\left\{\frac{1}{2}\left\|x - x^k\right\|^2 + f\left(x^k, y\right) \middle| \in C\right\}$$

and Lemma 4.2, we have

$$0 \in \partial_2\left(\lambda_k f\left(x^k, y\right) + \frac{1}{2}\left\|y - x^k\right\|^2\right)\left(y^k\right) + N_C\left(y^k\right).$$

This follows that

$$0 = \lambda_k w + y^k - x^k + \bar{w},$$

where $w \in \partial_2 f\left(x^k, y^k\right)$ and $\bar{w} \in N_C\left(y^k\right)$. By the definition of the normal cone N_C we imply that

$$\left\langle y^k - x^k, y - y^k\right\rangle \ge \lambda_k\left\langle w, y^k - y\right\rangle\ \forall y \in C. \quad (20)$$

On the other hand, since $f\left(x^k, \cdot\right)$ is subdifferentiable on C, by the well known Moreau-Rockafellar theorem, there exists $w \in \partial_2 f\left(x^k, y^k\right)$ such that

$$f\left(x^k, y\right) - f\left(x^k, y^k\right) \ge \left\langle w, y - y^k\right\rangle\ \forall y \in C.$$

Combining this with (20), we have

$$\lambda_k\left(f\left(x^k, y\right) - f\left(x^k, y^k\right)\right) \ge \left\langle y^k - x^k, y^k - y\right\rangle\ \forall y \in C.$$

Hence

$$\lambda_{k_j}\left(f\left(x^{k_j},y\right)-f\left(x^{k_j},y^{k_j}\right)\right) \geq \left\langle y^{k_j}-x^{k_j},y^{k_j}-y\right\rangle \ \forall y\in C.$$

Then, using

$$\{\lambda_k\}\subset[a,b]\subset\left(0,\frac{1}{L}\right),$$

Step 2, $x^{k_j} \rightharpoonup \bar{x}$ as $j\to\infty$ and continuity of *f*, we have

$$f(\bar{x},y)\geq 0 \ \forall y\in C.$$

This means that $\bar{x}\in \mathrm{Sol}(f,C)$.

Step 5. We claim that the sequences $\{x^k\}$, $\{y^k\}$ and $\{t^k\}$ convergence weakly to $\bar{x}$ as $k\to\infty$, where

$$\bar{x}=\lim_{k\to\infty}\mathrm{Pr}_{\bigcap_{i=1}^{p}\mathrm{Fix}(S_i,C)\cap\mathrm{Sol}(f,C)}\left(x^k\right).$$

Indeed, since $\{x^k\}$ is bounded, we suppose that there exists two sequences $\left\{x^{k_j}\right\}$ and $\left\{x^{n_j}\right\}$ such that

$$x^{k_j}\rightharpoonup\bar{x},\ x^{n_j}\rightharpoonup\hat{x} \ \text{as } j\to\infty.$$

By Step 3 and Step 4, we have

$$\bar{x},\hat{x}\in\bigcap_{i=1}^{p}\mathrm{Fix}(S_i,C)\cap\mathrm{Sol}(f,C).$$

Then, using $\qquad c=\lim_{k\to\infty}\left\|x^k-\bar{x}\right\|$

and the following equality (see [7])

$$\limsup_{k\to\infty}\left\|z^k-y\right\|^2=\limsup_{k\to\infty}\left\|z^k-x\right\|^2+\|x-y\|^2 \ \forall y\in H,$$

where $z^k\rightharpoonup x$ as $k\to\infty$, we have

$$\begin{aligned}2c^2&=\lim_{k\to\infty}\left\|x^k-\bar{x}\right\|^2\\&=\lim_{j\to\infty}\left\|x^{n_j}-\bar{x}\right\|^2\\&=\limsup_{j\to\infty}\left\|x^{n_j}-\bar{x}\right\|^2\\&=\limsup_{j\to\infty}\left\|x^{n_j}-\hat{x}\right\|^2+\|\bar{x}-\hat{x}\|^2\\&=\limsup_{j\to\infty}\left\|x^{k_j}-\hat{x}\right\|^2+\|\bar{x}-\hat{x}\|^2\\&=\limsup_{j\to\infty}\left\|x^{k_j}-\bar{x}\right\|^2+2\|\bar{x}-\hat{x}\|^2\\&=\limsup_{j\to\infty}\left\|x^{k}-\bar{x}\right\|^2+2\|\bar{x}-\hat{x}\|^2\\&=c^2+2\|\bar{x}-\hat{x}\|^2.\end{aligned}$$

Hence, we have $\bar{x}=\hat{x}$. This implies that

$$x^k\rightharpoonup\bar{x}\in\bigcap_{i=1}^{p}\mathrm{Fix}(S_i,C)\cap\mathrm{Sol}(f,C) \ \text{as } k\to\infty.$$

Then, it follows from Step 2 that $y^k,t^k\rightharpoonup\bar{x}$ as $k\to\infty$.

Now, we suppose that

$$z^k:=\mathrm{Pr}_{\bigcap_{i=1}^{p}\mathrm{Fix}(S_i,C)\cap\mathrm{Sol}(f,C)}\left(x^k\right)$$

and $x^k\rightharpoonup\bar{x}$ as $k\to\infty$. By the definition of $\mathrm{Pr}_C(\cdot)$, we have

$$\left\langle z^k-x^k,z^k-x\right\rangle\leq 0 \ \forall x\in\bigcap_{i=1}^{p}\mathrm{Fix}(S_i,C)\cap\mathrm{Sol}(f,C). \quad (21)$$

It follows from Step 1 that

$$\left\|x^{k+1}-x^*\right\|\leq\left\|x^k-x^*\right\| \ \forall k\geq 0,$$

$$x^*\in\bigcap_{i=1}^{p}\mathrm{Fix}(S_i,C)\cap\mathrm{Sol}(f,C).$$

Then, by Lemma 4.1, we have

$$\mathrm{Pr}_{\bigcap_{i=1}^{p}\mathrm{Fix}(S_i,C)\cap\mathrm{Sol}(f,C)}\left(x^k\right)\to x_1\in\bigcap_{i=1}^{p}\mathrm{Fix}(S_i,C)\cap\mathrm{Sol}(f,C) \ \text{as } k\to\infty. \quad (22)$$

Pass the limit in (21) and combining this with (22), we have

$$\left\langle x_1-\bar{x},x_1-x\right\rangle\leq 0 \ \forall x\in\bigcap_{i=1}^{p}\mathrm{Fix}(S_i,C)\cap\mathrm{Sol}(f,C).$$

This means that $\bar{x}=x_1$ and

$$\bar{x}=\mathrm{Pr}_{\bigcap_{i=1}^{p}\mathrm{Fix}(S_i,C)\cap\mathrm{Sol}(f,C)}\left(x^k\right).$$

It follows from Step 2 that the sequences $\{x^k\}$, $\{y^k\}$ and $\{t^k\}$ converge weakly to $\bar{x}$ as $k\to\infty$, where

$$\bar{x}=\lim_{k\to\infty}\mathrm{Pr}_{\bigcap_{i=1}^{p}\mathrm{Fix}(S_i,C)\cap\mathrm{Sol}(f,C)}\left(x^k\right).$$

The proof is completed. □

The next theorem proves the strong convergence of Algorithm 3.5.

Theorem 4.6. *Suppose that Assumptions* 3.1-3.3 *are satisfied. Then the sequences* $\{x^k\}$ *and* $\{y^k\}$ *generated by Algorithm* 3.5 *converge strongly to the same point* x^*,

where $$x^*=\mathrm{Pr}_{\bigcap_{i=1}^{p}\mathrm{Fix}(S_i,C)\cap\mathrm{Sol}(f,C)}\left(x^0\right),$$

provided $$\lim_{k\to\infty}\left\|x^k-y^k\right\|=0.$$

Proof. We also divide the proof of this theorem into several steps.

Step 1. We claim that

$$\bigcap_{i=1}^{p}\text{Fix}\left(S_i,C\right)\cap\text{Sol}\left(f,C\right)\subset P_k\cap Q_k\quad\forall k\geq 0.$$

Indeed, for each

$$x^*\in\bigcap_{i=1}^{p}\text{Fix}\left(S_i,C\right)\cap\text{Sol}\left(f,C\right),$$

it implies from (17) that

$$\begin{aligned}&2\left\|x^{k+1}-x^*\right\|\\&\leq\left\|t^k-x^*\right\|^2+\left(1-\alpha_k\right)\left(\bar{L}-\alpha_k\right)\left\|\bar{S}_k\left(t^k\right)-t^k\right\|^2\\&\leq\left\|x^k-x^*\right\|^2+\left(1-\alpha_k\right)\left(\bar{L}-\alpha_k\right)\left\|\bar{S}_k\left(t^k\right)-t^k\right\|^2,\end{aligned}$$

where

$$\bar{S}_k=\sum_{i=1}^{p}\lambda_{k,i}S_i.$$

This means that $x^*\in P_k$ and hence

$$\bigcap_{i=1}^{p}\text{Fix}\left(S_i,C\right)\cap\text{Sol}\left(f,C\right)\subset P_k$$

for all $k\geq 1$. We prove

$$\bigcap_{i=1}^{p}\text{Fix}\left(S_i,C\right)\cap\text{Sol}\left(f,C\right)\subset Q_k\quad\forall k\geq 0\tag{23}$$

by induction. For $k = 0$, we have

$$\bigcap_{i=1}^{p}\text{Fix}\left(S_i,C\right)\cap\text{Sol}(f,C)\subset C=Q_0.$$

As $\bigcap_{i=1}^{p}\text{Fix}\left(S_i,C\right)\cap\text{Sol}\left(f,C\right)\subset Q_k$

by the induction assumption and hence

$$\bigcap_{i=1}^{p}\text{Fix}\left(S_i,C\right)\cap\text{Sol}\left(f,C\right)\subset P_k\cap Q_k.$$

Since $x^{k+1}=\text{Pr}_{P_k\cap Q_k}\left(x^0\right)$,

we have $\left\langle x^{k+1}-x,x^0-x^{k+1}\right\rangle\geq 0\;\;\forall x\in P_k\cap Q_k$.

Thus

$$\left\langle x^{k+1}-x,x^0-x^{k+1}\right\rangle\geq 0\;\;\forall x\in\bigcap_{i=1}^{p}\text{Fix}\left(S_i,C\right)\cap\text{Sol}\left(f,C\right).$$

By the definition of Q_{k+1}, we have

$$\bigcap_{i=1}^{p}\text{Fix}\left(S_i,C\right)\cap\text{Sol}\left(f,C\right)\subset Q_{k+1}.$$

Hence (23) holds for all $k\geq 0$.

Step 2. We show that

$$\begin{cases}\lim_{k\to\infty}\left\|x^k-t^k\right\|=0,\\\lim_{k\to\infty}\left\|x^k-\bar{S}_k\left(x^k\right)\right\|=0.\end{cases}$$

Indeed, from $x^{k+1}=\text{Pr}_{P_k\cap Q_k}\left(x^0\right)$

it follows $x^{k+1}\in Q_k$, *i.e.*,

$$\left\langle x^{k+1}-x^k,x^k-x^0\right\rangle\geq 0.$$

Then, we have

$$\begin{aligned}&2\left\|x^{k+1}-x^k\right\|^2\\&=\left\|x^{k+1}-x^0+x^0-x^k\right\|^2\\&=\left\|x^{k+1}-x^0\right\|^2-\left\|x^k-x^0\right\|^2-2\left\langle x^{k+1}-x^k,x^k-x^0\right\rangle\\&\leq\left\|x^{k+1}-x^0\right\|^2-\left\|x^k-x^0\right\|^2\\&\to 0\text{ as }k\to\infty,\end{aligned}$$

the last claim holds by Step 1 of Theorem 4.5. From

$$x^{k+1}=\text{Pr}_{P_k\cap Q_k}\left(x^0\right),$$

it follows that $x^{k+1}\in P_k$, *i.e.*

$$\begin{aligned}&\left\|z^k-x^{k+1}\right\|^2\\&\leq\left\|t^k-x^{k+1}\right\|^2-(1-\alpha_k)(\alpha_k-\bar{L})\left\|t^k-\bar{S}_k(t^k)\right\|^2.\end{aligned}$$

Using this and the following equality

$$\begin{aligned}&\left\|\lambda x+\left(1-\lambda\right)y\right\|^2\\&=\lambda\left\|x\right\|^2+\left(1-\lambda\right)\left\|y\right\|^2-\lambda\left(1-\lambda\right)\left\|x-y\right\|^2\\&\forall\lambda\in\left[0,1\right],x,y\in H,\end{aligned}$$

we obtain

$$\begin{aligned}&2\left\|x^{k+1}-z^k\right\|^2\\&=\left\|\alpha_k t^k+\left(1-\alpha_k\right)\bar{S}_k\left(t^k\right)-x^{k+1}\right\|^2\\&=\left\|\alpha_k\left(t^k-x^{k+1}\right)+\left(1-\alpha_k\right)\left(\bar{S}_k\left(t^k\right)-x^{k+1}\right)\right\|^2\\&=\alpha_k\left\|x^{k+1}-t^k\right\|^2+\left(1-\alpha_k\right)\left\|\bar{S}_k\left(t^k\right)-x^{k+1}\right\|^2\\&\quad-\alpha_k(1-\alpha_k)\left\|\bar{S}_k\left(t^k\right)-t^k\right\|^2\\&\geq\alpha_k\left\|x^{k+1}-t^k\right\|^2+\left(1-\alpha_k\right)\left\|\bar{S}_k\left(t^k\right)-x^{k+1}\right\|^2\\&\quad+\frac{\alpha_k}{\alpha_k-\bar{L}}\left\|z^k-x^{k+1}\right\|^2-\frac{\alpha_k}{\alpha_k-\bar{L}}\left\|t^k-x^{k+1}\right\|^2.\end{aligned}$$

Hence

$$\left\|x^{k+1}-\bar{S}_k\left(t^k\right)\right\|^2 \le \left\|x^{k+1}-t^k\right\|^2+\bar{L}\left\|t^k-\bar{S}_k\left(t^k\right)\right\|^2.$$

Combining this and

$$\begin{aligned}&\left\|x^{k+1}-\bar{S}_k\left(t^k\right)\right\|^2\\ &=\left\|x^{k+1}-t^k\right\|^2+2\left\langle x^{k+1}-t^k,t^k-\bar{S}_k\left(t^k\right)\right\rangle\\ &\quad+\left\|t^k-\bar{S}_k\left(t^k\right)\right\|^2,\end{aligned}$$

we have

$$\begin{aligned}2\left\|t^k-\bar{S}_k\left(t^k\right)\right\| &\le \frac{2}{1-\bar{L}}\left\|x^{k+1}-t^k\right\|\\ &\le \frac{2}{1-\bar{L}}\left(\left\|x^{k+1}-x^k\right\|+\left\|x^k-t^k\right\|\right).\end{aligned} \tag{24}$$

Since $f\left(y^n,\cdot\right)$ is subdifferentiable on C, by the well known Moreau-Rockafellar theorem, there exists $w\in\partial_2 f\left(y^n,t^n\right)$ such that

$$f\left(y^n,t\right)-f\left(y^n,t^n\right)\ge\left\langle w,t-t^n\right\rangle \quad \forall t\in C.$$

With $t=x^*$, this inequality becomes

$$f\left(y^n,x^*\right)-f\left(y^n,t^n\right)\ge\left\langle w,x^*-t^n\right\rangle. \tag{25}$$

By the definition of the normal cone N_C we have, from the latter inequality, that

$$\left\langle t^n-x^n,t-t^n\right\rangle\ge\lambda_n\left\langle w,t^n-t\right\rangle \, \forall t\in C. \tag{26}$$

With $t=y^k$, we have

$$\left\langle t^k-x^k,y^k-t^k\right\rangle\ge\lambda_k\left\langle w,t^k-y^k\right\rangle \, \forall w\in\partial_2 f\left(y^k,t^k\right).$$

Combining $f(x, x) = 0$ for all $x\in C$, the last inequality and the definition of w,

$$f\left(y^k,t\right)-f\left(y^k,t^k\right)\ge\left\langle w,t-t^k\right\rangle \, \forall t\in C,$$

we have

$$\begin{aligned}&2\left\langle t^k-x^k,y^k-t^k\right\rangle\\ &\ge-\lambda_k\left\langle w,y^k-t^k\right\rangle\\ &\ge\lambda_k\left(f\left(y^k,t^k\right)-f\left(y^k,y^k\right)\right)\\ &=\lambda_k f\left(y^k,t^k\right).\end{aligned} \tag{27}$$

Since y^n is the unique solution to the strongly convex problem

$$\min\left\{\frac{1}{2}\left\|y-x^n\right\|^2+\lambda_n f\left(x^n,y\right):y\in C\right\},$$

we have

$$\lambda_n\left\{f\left(x^n,y\right)-f\left(x^n,y^n\right)\right\}\ge\left\langle y^n-x^n,y^n-y\right\rangle \, \forall y\in C. \tag{28}$$

Substituting $y=t^n\in C$, we obtain

$$\lambda_n\left\{f\left(x^n,t^n\right)-f\left(x^n,y^n\right)\right\}\ge\left\langle y^n-x^n,y^n-t^n\right\rangle. \tag{29}$$

From (29), it follows

$$\left\langle y^k-x^k,t^k-y^k\right\rangle\ge\lambda_k\left\{f\left(x^k,y^k\right)-f\left(x^k,t^k\right)\right\}. \tag{30}$$

Adding two inequalities (27) and (30), we obtain

$$\begin{aligned}&\left\langle t^k-y^k,y^k-x^k-t^k+x^k\right\rangle\\ &\ge\lambda_k\left(f\left(x^k,y^k\right)+f\left(y^k,t^k\right)-f\left(x^k,t^k\right)\right).\end{aligned}$$

Then, since f is Lipschitz-type continuous on C, we have

$$-\left\|t^k-y^k\right\|^2\ge\lambda_k\left(-c_1\left\|x^k-y^k\right\|^2-c_2\left\|y^k-t^k\right\|^2\right),$$

which follows that

$$\left(1-\lambda_k c_2\right)\left\|t^k-y^k\right\|^2\le\lambda_k c_1\left\|x^k-y^k\right\|^2.$$

From $\lim_{k\to\infty}\left\|x^k-y^k\right\|=0$,

it follows that $\lim_{k\to\infty}\left\|t^k-y^k\right\|=0$. Then, we have

$$2\left\|x^k-t^k\right\|\le\left\|x^k-y^k\right\|+\left\|y^k-t^k\right\|\to0 \text{ as } k\to\infty. \tag{31}$$

Since (24), (31) and $\bar{S}_k$ is Lipschitz continuous, we have

$$\begin{aligned}&2\left\|x^k-\bar{S}_k\left(x^k\right)\right\|\\ &\le\left\|x^k-t^k\right\|+\left\|t^k-\bar{S}_k\left(t^k\right)\right\|+\left\|\bar{S}_k\left(t^k\right)-\bar{S}_k\left(x^k\right)\right\|\\ &\le\left\|t^k-\bar{S}_k\left(t^k\right)\right\|+\frac{2}{1-\bar{L}}\left\|t^k-x^k\right\|\\ &\le\frac{2}{1-\bar{L}}\left(\left\|x^{k+1}-x^k\right\|+2\left\|x^k-t^k\right\|\right)\\ &\to0 \text{ as } k\to\infty.\end{aligned}$$

Step 3. We show that the sequence $\{x^k\}$ converges strongly to x^*, where

$$x^*=Pr_{\bigcap_{i=1}^p \mathrm{Fix}(S_i,C)\cap\mathrm{Sol}(f,C)}\left(x^0\right).$$

Indeed, as in Step 4 and Step 5 of the proof of Theorem 4.5, we can claim that for every weakly cluster point $\bar{x}$ of the sequence $\{x^k\}$ satisfies

$$\bar{x}\in\bigcap_{i=1}^p\mathrm{Fix}\left(S_i,C\right)\cap\mathrm{Sol}\left(f,C\right).$$

On the other hand, using the definition of Q_k, we have

$$x^k=\mathrm{Pr}_{Q_k}\left(x^0\right).$$

Combining this with Step 1, we obtain

$$\left\|x^0 - x^k\right\| \le \left\|x^0 - x\right\| \quad \forall x \in \bigcap_{i=1}^{p} \mathrm{Fix}(S_i, C) \cap \mathrm{Sol}(f, C).$$

With $x = x^*$, we have

$$\left\|x^0 - x^k\right\| \le \left\|x^0 - x^*\right\|.$$

By Lemma 4.3, we claim that the sequence $\{x^k\}$ converges strongly to x^* as $k \to \infty$, where

$$x^* = Pr_{\bigcap_{i=1}^{p} \mathrm{Fix}(S_i, C) \cap \mathrm{Sol}(f, C)}\left(x^0\right).$$

Hence, we also have $y_k \to x^*$ as $k \to \infty$. □

5. Acknowledgements

The work of the first author was supported by the Vietnam National Foundation for Science Technology Development (NAFOSTED), code 101.02-2011.07.

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