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Methods for Lower Approximation Reduction in Inconsistent Decision Table Based on Tolerance Relation

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ABSTRACT

It is well known that most of information systems are based on tolerance relation instead of the classical equivalence relation because of various factors in real-world. To acquire brief decision rules from the information systems, lower approximation reduction is needed. In this paper, the lower approximation reduction is proposed in inconsistent information systems based on tolerance relation. Moreover, the properties are discussed. Furthermore, judgment theorem and discernibility matrix are obtained, from which an approach to lower reductions can be provided in the complicated information systems.

Keywords: Rough Set; Tolerance Relation; Lower Approximation Reduction; Discernibility Matrix

1. Introduction

The rough set theory, proposed by Pawlak in the early 1980s [1], is an extension of the classical set theory for modeling uncertainty or imprecision information. The research has recently roused great interest in the theoretical and application fronts, such as machine learning, pattern recognition, data analysis, and so on.

Attribute reduction is one of the hot research topics of rough set theory. Much study on this area had been reported and many useful results were obtained [2-8]. However, most work was based on consistent information systems, and the main methodology has been developed under equivalence relations (indiscernibility relations). In practice, most of information systems are not only inconsistent, but also based on tolerance relations because of various factors. The tolerance of properties of attributes plays a crucial role in those systems. For this reason, J. Jarinen [9-13] proposed an extension rough sets theory, called the rough sets based on tolerances to take into account the tolerance relation properties of attributes. This innovation is mainly based on substitution of the indiscernibility relation by a tolerance relation. And many studies have been made in DRSA [14-18]. But useful results of attribute reductions are very poor in inconsistent information systems based on tolerance relations until now.

In this paper, the lower approximation reduction is proposed in inconsistent information systems based on tolerance relations. Moreover, some properties are discussed. Furthermore, judgment theorem and discernibility matrix are obtained, from which an approach to lower approximation reductions can be provided in inconsistent information systems based on tolerance relations.

2. Rough Sets and Information Systems Based on Tolerance Relations

The following recalls necessary concepts and preliminaries required in the sequel of our work. Detailed description of the theory can be found in [5,17].

An information system with decisions is an ordered quadruple $I=(U,A\bigcup D,F,G)$, where

$U=\{x_1,x_2,\cdots,x_n\}$ is a non-empty finite set of objects;

$A\bigcup D$ is a non-empty finite attributes set;

$A=\{a_1,a_2,\cdots,a_p\}$ denotes the set of condition attributes;

$D=\{d_1,d_2,\cdots,d_q\}$ denotes the set of decision attributes, and $A\bigcup D=\varnothing$;

$F=\{f_k|U\to V_k,k\le p\}$, $f_k(x)$ is the value of a_k for $x\in U$, v_k is the domain of a_k, where $a_k\in A$;

$G=\{g_{k'}|U\to V_{k'},k'\le q\}$, $g_{k'}(x)$ is the value of $d_{k'}$ for $x\in U$, $v_{k'}$ is the domain of $d_{k'}$, $d_{k'}\in D$.

If a binary relation T on the universe U is reflexive and symmetric, it is called a tolerance relation on U. The set of all tolerance relations on U is denoted by $Tol(U)$. A tolerance relation T can construct a covering of the universe U, not a partition. For any tolerance relation $T\in Tol(U)$ and $x\in U$, denote

$$T(x,y)=\begin{cases}1, & (x,y)\in T\\ 0, & (x,y)\notin T\end{cases};$$

$$T(x)=\{y\in U \mid xTy, x\in U\};$$

where $T(x,y)=1$ means x and y have the tolerance T, or x and y haven't the tolerance T, and the $T(x)$ is called the tolerance neighborhood or tolerance class of the object x.

An information system is called an information system based on tolerance relations, in brief *TIS*, if all relations of condition attributes are tolerance relations.

In general, we call an information system based on tolerance relations with decision to be a decision table based tolerance relations, denoted by TDT, that is $TDT=(U,A\bigcup D,F,G)$. Thus the following definition can be obtained.

Definition 2.1. Let $TDT=(U,A\bigcup D,F,G)$ be a decision table based on tolerance relations, for any $B\subseteq A$, denote R_B^T and R_D^T are tolerance relations of information system TDT.

If we denote

$$[x_i]_B^T=\{x_j\in U \mid (x_i,x_j)\in R_B^T\};$$

$$[x_i]_D^T=\{x_j\in U \mid (x_i,x_j)\in R_D^T\},$$

then the following properties of a tolerance relation are trivial.

Proposition 2.1. Let R_A^T be a tolerance relation. The following hold.

(1) R_A^T is reflexive, symmetric, but not transitive, so it is not an equivalence relation.

(2) If $B\subseteq A$, then $R_A^T\subseteq R_B^T$.

(3) If $B\subseteq A$, then $[x_i]_A^T\subseteq[x_i]_B^T$

(4) $J=\bigcup\{[x]_A^T \mid x\in U\}$ constitutes a covering of U.

For any subset X of U, and A of TDT define

$$\underline{R}_A^T(X)=\{x\in U \mid [x]_A^T\subseteq X\};$$

$$\overline{R}_A^T(X)=\{x\in U \mid [x]_A^T\cap X\neq\varnothing\},$$

$\underline{R}_A^T(X)$ and $\overline{R}_A^T(X)$ are said to be the lower and upper approximation of X with respect to a tolerance relation R_A^T. And the approximations have also some properties which are similar to those of Pawlak approximation spaces.

Proposition 2.2. Let $TDT=(U,A\bigcup D,F,G)$ be an information systems based on tolerance relation and $X,Y\subseteq U$, then its lower and upper approximations satisfy the following properties.

(1) $\underline{R}_A^T(X)\subseteq X\subseteq\overline{R}_A^T(X)$.

(2) $\overline{R}_A^T(X\cup Y)=\overline{R}_A^T(X)\cup\overline{R}_A^T(Y)$; $\underline{R}_A^T(X\cap Y)=\underline{R}_A^T(X)\cap\underline{R}_A^T(Y)$.

(3) $\overline{R}_A^T(X\cap Y)\subseteq\overline{R}_A^T(X)\cap\overline{R}_A^T(Y)$; $\underline{R}_A^T(X)\cup\underline{R}_A^T(Y)\subseteq\underline{R}_A^T(X\cup Y)$.

(4) $\overline{R}_A^T(\sim X)=\sim\overline{R}_A^T(X)$; $\underline{R}_A^T(\sim X)=\sim\underline{R}_A^T(X)$.

(5) $\overline{R}_A^T(\varnothing)=\varnothing$; $\underline{R}_A^T(U)=U$.

(6) If $X\subseteq Y$, then $\underline{R}_A^T(X)\subseteq\underline{R}_A^T(Y)$ and $\overline{R}_A^T(X)\subseteq\overline{R}_A^T(Y)$;

where $\sim X$ is the complement of X.

Definition 2.2. For an information system based on tolerance relations with decisions $TDT=(U,A\bigcup D,F,G)$, if $R_A^T\subseteq R_D^T$, then this information system is consistent, otherwise, this system is inconsistent.

Example 2.1. Given an information system based on tolerance relations in **Table 1**.

Define the tolerance relation R_A^T as following:

$$R_A^T=\{(x_i,x_j) \mid |f(x_i,a_i)-f(x_j,a_i)|\le 1, \forall a_i\in A\}$$

From the table, we have

$$[x_1]_A^T=\{x_1,x_3\};$$
$$[x_2]_A^T=\{x_2,x_4,x_5,x_6\};$$
$$[x_3]_A^T=\{x_1,x_3,x_4\};$$
$$[x_4]_A^T=\{x_2,x_3,x_4,x_6\};$$
$$[x_5]_A^T=\{x_2,x_5,x_6\};$$
$$[x_6]_A^T=\{x_2,x_4,x_5,x_6\};$$

and

$$[x_1]_d^T=[x_5]_d^T=\{x_1,x_2,x_4,x_5\};$$
$$[x_2]_d^T=[x_4]_d^T=\{x_1,x_2,x_3,x_4,x_5,x_6\};$$
$$[x_3]_d^T=[x_6]_d^T=\{x_2,x_3,x_4,x_6\}.$$

Obviously, by the above, we have $R_A^T\subsetneq R_D^T$, so the system in **Table 1** is inconsistent.

For simple description, the following information system with decisions is based on tolerance relations, *i.e.* information systems based on tolerance relations.

3. Theories of Lower Approximation Reduction in Inconsistent Information Systems Based on Tolerance Relation

Let $TDT=(U,A\bigcup D,F,G)$ be an information system based on tolerance relations with decisions, and R_B^T,

Table 1. An information system based on tolerance relations.

U	a_1	a_2	a_3	d
x_1	1	2	1	3
x_2	3	2	2	2
x_3	1	1	2	1
x_4	2	1	3	2
x_5	3	3	2	3
x_6	3	2	3	1

R_D^T be tolerance relations derived from condition attributes set A and decision attributes set D respectively. For $B \subseteq A$, denote

$$U/R_B^T = \left\{[x_i]_B^T \middle| x_i \in U\right\};$$

$$U/R_D^T = \{D_1, D_2, \cdots, D_r\};$$

$$L_B = \left(\underline{R}_B^T(D_1), \underline{R}_B^T(D_2), \cdots, \underline{R}_B^T(D_r)\right);$$

where $[x]_B^T = \{y \in U | (x, y) \in R_B^T\}$. Furthermore, we said L_B is the lower approximation function about attributions sets *B*.

Definition 3.1. Let $\alpha = (a_1, a_2, \cdots, a_n)^{\mathrm{T}}$ and $\beta = (b_1, b_2, \cdots, b_n)^{\mathrm{T}}$ be two vectors with n dimensions. If $a_i = b_i$ $(i = 1, \cdots, n)$, we said that α is equal to β, denoted by $\alpha = \beta$. If $a_i \le b_i$ $(i = 1, \cdots, n)$, we said that α is less than β, denoted by $\alpha \le \beta$. Otherwise, If it exists i_0, $(i_0 \in \{1, 2, \cdots, n\})$ such that $a_{i_0} > b_{i_0}$, we said α is not less than β, denoted by $\alpha \not\le \beta$.

Such as $(1,2,3) \not\le (1,1,4)$, and $(1,1,4) \not\le (1,2,3)$.

From the above, we can have the following propositions immediately.

Proposition 3.1. Let $TDT = (U, A \cup D, F, G)$ be an information system based on tolerance relations with decisions. If $B \subseteq A$, then $L_A \le L_B$.

Definition 3.2. Let $TDT = (U, A \cup D, F, G)$ be an Inconsistent information system. If $L_B = L_A$, for all, we say that B is a lower approximation consistent set of TDT. If B is a lower approximation consistent set, and no proper subset of B is lower approximation consistent set, then B is called an lower approximation consistent reduction of TDT.

Example 3.1. Consider the system in **Table 1**.

For the system in **Table 1**, we denote

$$D_1 = [x_1]_d^T = [x_5]_d^T;$$

$$D_2 = [x_2]_d^T = [x_4]_d^T;$$

$$D_3 = [x_3]_d^T = [x_6]_d^T,$$

We can have

$$\underline{R}_A(D_1) = \varnothing$$

$$\underline{R}_A(D_2) = \{x_1, x_2, x_3, x_4, x_5, x_6\}$$

$$\underline{R}_A(D_3) = \{x_4\}$$

When $B = \{a_2, a_3\}$, it can be easily checked that $\underline{R}_A(D_i) = \underline{R}_B(D_i)$, for all $\forall D_i \in U/R_d^T$. So that $L_A = L_B$, and $B = \{a_2, a_3\}$ is a lower approximation consistent set of TDT. Furthermore, we can examine that $\{a_2\}$ and $\{a_3\}$ are not lower approximation consistent set of TDT. That is to say $B = \{a_2, a_3\}$ is a lower approximation reduction of TDT.

Moreover, it can be easily calculated that $B' = \{a_1, a_3\}$ and $B'' = \{a_1, a_2\}$ are not lower approximation consistent sets of TDT. Thus there exist only one lower approximation reduction of TDT in the system of **Table 1**, which is $\{a_2, a_3\}$.

In the following, detailed judgment theorems of lower approximation reduction are obtained.

4. Methods for Attribute Reduction in Inconsistent Information Systems Based on Tolerance Relations

This section provides approaches to lower approximation reduction in inconsistent information systems.

Definition 4.1. Let $TDT = (U, A \cup D, F, G)$ be an information system, Denote

$$D_L^* = \left\{(x_i, x_j) \middle| x_i \in \underline{R}_A^T(D_i), x_j \notin \underline{R}_A^T(D_i)\right\};\ \exists D_i \in U/R_D^T$$

$$D_L(x_i, x_j) = \begin{cases} \left\{a_k \in A \middle| R_{a_k}^T(x_i, x_j) = 0\right\}, & (x_i, x_j) \in D_L^* \\ \varnothing, & (x_i, x_j) \notin D_L^* \end{cases}$$

$D_L(x_i, x_j)$ is called lower approximation discernibility attribute set, and matrixes $M_L = \left(D_L(x_i, x_j) \middle| x_i, x_j \in U\right)$ is referred to lower approximation discernibility matrix of TDT respectively.

Theorem 4.1. Let $TDT = (U, A \cup D, F, G)$ be an information system, $B \subseteq A$, then B is a lower approximation consistent set $\Leftrightarrow$ if $x \in R_A^T(D_i), y \notin R_A^T(D_i)$, then exists $b \in B$ such that $R_b(x, y) = 0$, for any $D_i \in U/R_D^T$.

Proof. $\Rightarrow$ Assume that exists D_i, for any $b \in B$, and $R_b(x, y) = 1$, if $x \in R_A^T(D_i), y \notin R_A^T(D_i)$, then $y \in [x]_B^T$.

Since *B* is a lower approximation consistent set, therefore, for any $D_i \in U/R_D^T$, we can have $\underline{R}_A^T(D_i) = \underline{R}_B^T(D_i)$. According to $x \in R_A^T(D_i)$, so we can obtain $x \in R_A^T(D_i)$ and $[x]_B^T \subseteq D_i$. Moreover for $y \in [x]_B^T$, then $[y]_B^T \subseteq [x]_B^T$, therefore $[y]_B^T \subseteq D_i$ and

$y \in R_B^T(D_i)$. Hence we have $y \in R_A^T(D_i)$, which is contradiction.

$\Leftarrow$ Supposed that B isn't a lower approximation consistent set, then exists $D_i \in U/R_D^T$, such that $\underline{R}_A^T(D_i) \neq \underline{R}_B^T(D_i)$, therefore exists $x_0 \in \underline{R}_A^T(D_i)$ and $x_0 \notin \underline{R}_B^T(D_i)$. So we can have $[x_0]_A^T \subseteq D_i$ and $[x_0]_B^T \not\subset D_i$.

Moreover, $[x_0]_A^T \subseteq [x_0]_B^T$, so there exists $y_0 \in [x_0]_B^T$ and $y_0 \notin D_i$, *i.e.* $y_0 \notin \underline{R}_A^T(D_i)$. In addition, we can have that exists $b \in B$ such that $R_b(x_0, y_0) = 0$, which is contradict with $[x_0]$.

Therefore B is a lower approximation consistent set.

Theorem 4.2. Let $TDT = (U, A \cup D, F, G)$ be an information system. $B \subseteq A$, then B is a lower approximation consistent set if and only if for any $(x, y) \in D_L^*$, we can have $B \cap D_L(x, y) \neq \varnothing$.

Proof. $\Rightarrow$ For any $(x, y) \in D_L^*$, there exists $D_i \in U/R_D^T$ such that $x \in R_A^T(D_i), y \notin R_A^T(D_i)$, so according to Theorem 4.1, we can have that exists $b \in B$ such that $R_b(x, y) = 0$, so $b \in D_L(x, y)$.

Therefore, if B is a lower approximation consistent set then for any $(x, y) \in D_L^*$, we can have $B \cap D_L(x, y) \neq \varnothing$.

$\Leftarrow$ If for any $(x, y) \in D_L^*$, $B \cap D_L(x, y) \neq \varnothing$, then exists $a_k \in B$ such that $a_k \in D_L(x, y)$, so we have $R_{a_k}(x, y) = 0$, and $x \in \underline{R}_A^T(D_i), y \notin \underline{R}_A^T(D_i)$.

Therefore B is a lower approximation consistent set according to Theorem 4.1.

Definition 4.2. Let $TDT = (U, A \cup D, F, G)$ be an information system, M_L is referred to lower approximation discernibility matrix of TDT, denote

$$F_L = \wedge\left\{\vee\left\{a_k \middle| a_k \in D_L(x_i, x_j)\right\}, x_i, x_j \in U\right\}$$
$$= \wedge\left\{\vee\left\{a_k \middle| a_k \in D_L(x_i, x_j)\right\}, (x_i, x_j) \in D_L^*\right\}$$

F_L is called discernibility formula of lower approximation.

Theorem 4.3. Let $TDT = (U, A \cup D, F, G)$ be an information system. The minimal disjunctive normal form of discernibility formula of lower approximation is

$$F_L = \bigvee_{k=1}^{p}\left(\bigwedge_{s=1}^{q_k} a_s\right)$$

Denote $B_k = \left\{a_s \middle| s = 1, 2, \cdots, q_k\right\}$, then $\left\{B_k \middle| k = 1, 2, \cdots, p\right\}$ is just set of all distribution re- ductions of TDT.

Proof. It follows directly from Theorem 4.1 and the definition of minimal disjunctive normal of the discernibility formula of lower approximation.

Theorem 4.3 provides a practical approach to lower approximation reductions of information systems with decisions based on tolerance relation. The following we will consider the system in **Table 1** using this approach.

Table 2. Lower approximation discernibility matrix M_L.

M_L	x_1	x_2	x_3	x_4	x_5	x_6
x_1	$\varnothing$	$\varnothing$	$\varnothing$	a_3	$\varnothing$	$\varnothing$
x_2	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$
x_3	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$
x_4	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$
x_5	$\varnothing$	$\varnothing$	$\varnothing$	a_2	$\varnothing$	$\varnothing$
x_6	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$

Example 4.1. For the system in **Table 1**, the function of distribution and maximum distribution have been obtained in Example 3.1. In additional, we can have

$$D_\mu^* = \left\{(x_1, x_4), (x_2, x_4), (x_3, x_4), (x_5, x_4), (x_6, x_4)\right\};$$

The above table (**Table 2**) is the lower approximation discernibility matrix of system in **Table 1**.

Consequently, we have

$$F_L = a_2 \wedge a_3$$

Therefore, we obtain that $\{a_2, a_3\}$ is all lower approximation reduction of information system in **Table 1**, which accords with the result of Example 3.1.

5. Conclusion

It is well known that most of information systems are not only inconsistent, but also based on tolerance relations because of various factors in practice. Therefore, it is important to study the lower approximation reduction in inconsistent information systems. In this paper, we are concerned with approaches to the problem. The lower approximation reduction is introduced in inconsistent information systems based on tolerance relations. The judgment theorem and discernibility matrix are obtained, from which we can provide the approach to attribute reductions in inconsistent information systems based on tolerance relations.

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