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# New Ninth Order J-Halley Method for Solving Nonlinear Equations

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## ABSTRACT

In the paper [1], authors have suggested and analyzed a predictor-corrector Halley method for solving nonlinear equations. In this paper, we modified this method by using the finite difference scheme, which had a quantic convergence. We have compared this modified Halley method with some other iterative methods of ninth order, which shows that this new proposed method is a robust one. Some examples are given to illustrate the efficiency and the performance of this new method.

**Keywords:** Halley Method; Jarratt Method; Iterative Methods; Convergence Order; Numerical Examples

## 1. Introduction

In recent years, several iterative type methods have been developed by using the Taylor series, decomposition and quadrature formulae (see [1-14] and the references therein). Using the technique of updating the solution and Taylor series expansion, Noor and Noor [1] have suggested and analyzed a sixth-order predictor-corrector iterative type Halley method for solving the nonlinear equations. Also, Kou *et al.* [2-4] have also suggested a class of fifth-order iterative methods. In the implementation of these methods, one has to evaluate the second derivative of the function, which is a serious drawback of these methods. To overcome these drawbacks, we modify the predictor-corrector Halley method by replacing the second derivatives of the function by its finite difference scheme. We prove that the new modified predictor-corrector method is of fifth-order convergence. We also present the comparison of the new method with the methods of Kou *et al.* [2-4] and Hu *et al.* [5]. In passing, we would like to point out that the results presented by Kou *et al.* [2-4] are incorrect. We also rectify this error.

Several examples are given to illustrate the efficiency and robustness of the new proposed method.

## 2. Iterative Methods

The Jarratt's fourth-order method [6] which improves the order of convergence is defined by

**Algorithm 1**

where

$$Jf = \frac{3f'(y_n)+f'(x_n)}{6f'(y_n)-2f'(x_n)}.$$

Recently, Kou *et al.* [2] considered the following two-step iteration scheme

**Algorithm 2**

$$y_n = x_n - Jf\frac{f(x_n)}{f'(x_n)},\ f'(x_n)\neq 0$$

$$x_{n+1} = y_n - \frac{f(y_n)}{f'(y_n)},$$

where

$$Jf = \frac{3f'(y_n)+f'(x_n)}{6f'(y_n)-2f'(x_n)}.$$

We now state some fifth-order iterative methods which have been suggested by Noor and Noor [6] and Kou *et al.* [2,3] using quite different techniques.

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**Algorithm 3**

$$y_n = x_n - \frac{2f(x_n)f'(x_n)}{2f'^2(x_n) - f(x_n)f''(x_n)}, \text{where } f'(x_n) \neq 0$$

$$x_{n+1} = x_n - \frac{2[f(x_n) + f(y_n)]f'(x_n)}{2f'^2(y_n) - [f(x_n) + f(y_n)]f''(x_n)},$$

which is a two-step Halley method of fifth-order convergent.

In a recent paper Kou *et al.* [2,3] have suggested following iterative methods.

**Algorithm 4** (SHM [3]). For a given $x_0$, compute the approximate solution xnþ1 by the iterative schemes:

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)} - \frac{f^2(x_n)f''(x_n)}{2f'^3(x_n) - 2f(x_n)f'(x_n)f''(x_n)},$$

where $f'(x_n) \neq 0$

$$x_{n+1} = y_n - \frac{f(y_n)}{f'(x_n) + (y_n - x_n)f''(x_n)},$$

**Algorithm 5** (ISHM [2]). For a given $x_0$, compute the approximate solution xnþ1 by the iterative schemes:

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)} - \frac{f^2(x_n)f''(x_n)}{2f'^3(x_n) - 2f(x_n)f'(x_n)f''(x_n)},$$

where $f'(x_n) \neq 0$

$$x_{n+1} = y_n - \frac{f(y_n)}{f'(x_n)} - \frac{f''(x_n)f(y_n)}{2f'^3(x_n)}.$$

On the basis of the above discussion a new iterative technique is proposed below (named as FAJH):

**Algorithm 6**

$$y_n = x_n - \frac{2f(x_n)}{3f'(x_n)}, f'(x_n) \neq 0 \tag{2.1}$$

$$z_n = x_n - Jf\frac{f(x_n)}{f'(x_n)}, \tag{2.2}$$

$$x_{n+1} = z_n - \frac{2f(z_n)f'(z_n)}{2f'(z_n) - f(z_n)L}, \tag{2.3}$$

where

$$Jf = \frac{3f'(y_n) + f'(x_n)}{6f'(y_n) - 2f'(x_n)} \tag{2.4}$$

and

$$L = \frac{f'(z_n) - f'(x_n)}{z_n - x_n}. \tag{2.5}$$

## 3. Analysis of Convergence

In this section, we compute the convergence order of the proposed method (FAJH).

**Theorem**: *Let* $\alpha \in I$ *be a simple zero of sufficiently differentiable function* $f : I \subseteq R \to R$ *for an open interval* $I$. *If* $x_0$ *is close to* $\alpha$, *then the three-step algorithm* 6 *has ninth order of convergence.*

**Proof:** The iterative technique is given by

$$y_n = x_n - \frac{2f(x_n)}{3f'(x_n)}, f'(x_n) \neq 0 \tag{3.1}$$

$$z_n = x_n - Jf\frac{f(x_n)}{f'(x_n)}, \tag{3.2}$$

$$x_{n+1} = z_n - \frac{2f(z_n)f'(z_n)}{2f'(z_n) - f(z_n)L}, \tag{3.3}$$

where

$$Jf = \frac{3f'(y_n) + f'(x_n)}{6f'(y_n) - 2f'(x_n)} \tag{3.4}$$

and

$$L = \frac{f'(z_n) - f'(x_n)}{z_n - x_n}. \tag{3.5}$$

Let $\alpha$ be a simple zero of $f$. By Taylor's expansion, we have,

$$\begin{aligned} f(x_n) = f'(\alpha)\big[&e_n + c_2e_n^2 + c_3e_n^3 + c_4e_n^4 + c_5e_n^5 \\ &+ c_6e_n^6 + c_7e_n^7 + c_8e_n^8 + c_9e_n^9 + c_{10}e_n^{10} + O(e_n^{11})\big], \end{aligned} \tag{3.6}$$

$$\begin{aligned} f'(x_n) = f'(\alpha)\big[&1 + 2c_2e_n + 3c_3e_n^2 + 4c_4e_n^3 + 5c_5e_n^4 \\ &+ 6c_6e_n^5 + 7c_7e_n^6 + 8c_8e_n^7 + 9c_9e_n^8 + 10c_{10}e_n^9 + O(e_n^{10})\big], \end{aligned} \tag{3.7}$$

where

$$c_k = \left(\frac{1}{k!}\right)\frac{f^{(k)}(\alpha)}{f'(\alpha)}, k = 2, 3, \cdots,$$

and $e_n = x_n - \alpha.$

Using (3.1), (3.6) and (3.7), we have

$$y_n = \alpha + \frac{1}{3}e_n + \frac{2}{3}c_2e_n^2 + \left(\frac{4}{3}c_3 - \frac{4}{3}c_2^2\right)e_n^3 + O(e_n^4), \tag{3.8}$$

by Taylor's series, we have

$$f(y_n) = f'(\alpha)\left[\frac{1}{3}e_n + \frac{7}{9}c_2e_n^2 + \left(\frac{37}{27}c_3 - \frac{8}{9}c_2^2\right)e_n^3 + O(e_n^4)\right], \tag{3.9}$$

and

$$\begin{aligned} f'(y_n) = f'(\alpha)\Bigg[&\left(1 + \frac{2}{3}c_2e_n\right) + \left(\frac{4}{3}c_2^2 + \frac{1}{3}c_3\right)e_n^2 \\ &+ \left(4c_2c_3 - \frac{8}{3}c_2^3 + \frac{4}{27}7c_4\right)e_n^3 + O(e_n^4)\Bigg]. \end{aligned} \tag{3.10}$$

Using (3.4), (3.7) and (3.10), we have

$$Jf = +c_2e_n + \left(-c_2^2 + 2c_3\right)e_n^2 + \left(-2c_2c_3 + \frac{26}{9}c_4\right)e_n^3 + O\left(e_n^4\right). \quad (3.11)$$

Using (3.2), (3.6), (3.7) and (3.11), we have

$$z_n = \alpha + \left(\frac{1}{9}c_4 + c_2^3 - c_2c_3\right)e_n^4 + \left(\frac{8}{27}c_5 + 8c_3c_2^2 - \frac{20}{9}c_2c_4 - 2c_3^2 - 4c_2^4\right)e_n^5 + O\left(e_n^6\right), \quad (3.12)$$

by Taylor's series, we have

$$f\left(z_n\right) = \left(\frac{1}{9}c_4 + c_2^3 - c_2c_3\right)e_n^4 + \left(\frac{8}{27}c_5 + 8c_3c_2^2 - \frac{20}{9}c_2c_4 - 2c_3^2 - 4c_2^4\right)e_n^5 + O\left(e_n^6\right), \quad (3.13)$$

and

$$f'\left(z_n\right) = \left(\frac{1}{9}c_4 + c_2^3 - c_2c_3\right)e_n^4 + \left(\frac{8}{27}c_5 + 8c_3c_2^2 - \frac{20}{9}c_2c_4 - 2c_3^2 - 4c_2^4\right)e_n^5 + O\left(e_n^6\right). \quad (3.14)$$

From (3.5), (3.7) and (3.14), we have

$$L = 2c_2 + 3c_3e_n + 4c_4e_n^2 + 5c_5e_n^3 + \left(3c_3c_2^3 - 3c_2c_3^2 + 6c_6 + 1/3\,c_4c_3\right)e_n^4 + O\left(e_n^5\right). \quad (3.15)$$

Using (3.3), (3.7), (3.12), (3.14) and (3.15), we get

$$x_{n+1} = \alpha + \left(\frac{1}{3}c_4c_2c_3^2 - \frac{1}{3}c_4c_3c_2^3 - \frac{3}{2}c_3^3c_2^2 + 3c_3^2c_2^4 - \frac{3}{2}c_3c_2^6 - \frac{1}{54}c_4^2c_3\right)e_n^9 + O\left(e_n^{10}\right), \quad (3.16)$$

implies

$$e_{n+1} = \left(\frac{1}{3}c_4c_2c_3^2 - \frac{1}{3}c_4c_3c_2^3 - \frac{3}{2}c_3^3c_2^2 + 3c_3^2c_2^4 - \frac{3}{2}c_3c_2^6 - \frac{1}{54}c_4^2c_3\right)e_n^9 + O\left(e_n^{10}\right).$$

Thus we observe that the new three-step method (FAJH) has ninth order convergence.

## 4. Numerical Examples

In this section now we consider some numerical examples (see **Table 1**) to demonstrate the performance of the newly developed iterative method. We compare classical Newton method (NW), Kou *et al.* method (see, [2]) (VCM) and (VSHM), Noor *et al.* methods (see [1]) (NR1), (NR2) and also ninth order Zhongyong Hu *et al.* (Z Hu) [5] with the new developed method (FAJH). All the computations for above mentioned methods, are performed using software Maple 9, precision 128 digits and $\varepsilon = 10^{-15}$ as tolerance and also the following criteria is used for estimating the zero:

1) $\delta = \left|x_{n+1} - x_n\right| < \varepsilon$,
2) $\left|f\left(x_n\right)\right| < \varepsilon$,
3) Maximum numbers of iterations $= 500$.

We used the following examples for comparison:

## 5. Conclusion

In **Tables 2-11**, we observe that our iterative method

**Table 1. (Table of functions).**

| TABLE # 1 OF FUNCTIONS | |
|---|---|
| Functions | Roots |
| $f_1 = 4x^4 - 4x^2$ | 1 |
| $f_2 = (x-2)^{23} - 1$ | 3 |
| $f_3 = \exp(x)\cdot\sin(x) + \ln(x^2+1)$ | 3.237562984023 |
| $f_4 = (x+2)\exp(x) - 1$ | −0.442854401002 |
| $f_5 = x^3 + 4x^2 - 15$ | 1.631980805566 |
| $f_6 = p(x^2 + 7x - 30) - 1$ | 3 |
| $f_7 = \exp(1-x) - 1$ | 1 |
| $f_8 = x^3 - 2x^2 - 5$ | 2.690647448028 |
| $f_9 = (x-1)\exp(-x)$ | 1 |
| $f_{10} = (1/x) - 1$ | 1 |

**Table 2. Comparison of Methods for Example 1.**

| $f_1, x_0 = 0.75$ | | | |
|---|---|---|---|
| | Numbers of iteration | $f(x_n)$ | $\delta$ |
| NW | 10 | 7.1e-40 | 5.9e-21 |
| VCM | 33 | 0 | 1.9e-42 |
| VSHM | 8 | −1.0e-127 | 3.6e-25 |
| NR1 | 5 | 1.8e-37 | 9.5e-20 |
| NR2 | 11 | 3.4e-36 | 4.1e-19 |
| Z Hu | 6 | 1.5e-99 | 6.5e-20 |
| FAJH | 5 | 1.4e-103 | 6.4e-27 |

**Table 3. Comparison of Methods for Example 2.**

| $f_2, x_0 = 2.9$ | | | |
|---|---|---|---|
| | Numbers of iteration | $f(x_n)$ | $\delta$ |
| NW | 13 | 7.0e-44 | 1.6e-23 |
| VCM | DIVERGE | --- | --- |
| VSHM | DIVERGE | ---- | ---- |
| NR1 | 6 | 1.9e-31 | 8.8e-19 |
| NR2 | 20 | 3.1e-29 | 3.5e-16 |
| Z Hu | 6 | 4.5e-65 | 1.5e-15 |
| FAJH | 5 | 4.2e-60 | 1.3e-16 |

**Table 4. Comparison of Methods for Example 3.**

| $f_3,\ x_0 = 2.9$ | | | |
|---|---|---|---|
| | Numbers of iteration | $f(x_n)$ | $\delta$ |
| NW | 7 | −1.1e-51 | 6.6e-27 |
| VCM | DIVERGE | --- | ---- |
| VSHM | 4 | 5.0e-127 | 1.9e-67 |
| NR1 | 4 | −1.0e-9 | 1.2e-20 |
| NR2 | DIVERGE | ---- | ---- |
| Z Hu | 5 | 4.5e-65 | 1.5e-15 |
| FAJH | 4 | 5.0e-127 | 8.2e-34 |

**Table 5. Comparison of Methods for Example 4.**

| $f_4, x_0 = -9$ | | | |
|---|---|---|---|
| | Numbers of iteration | $f(x_n)$ | $\delta$ |
| NW | 6 | 3.4e-29 | 5.5e-15 |
| VCM | 4 | 1.0e-127 | 4.8e-26 |
| VSHM | 4 | −7.0e-128 | 5.2e-73 |
| NR1 | 4 | 3.8e-38 | 1.8e-19 |
| NR2 | 45 | 9.6e-50 | 2.8e-25 |
| Z Hu | 4 | 4.5e-65 | 1.5e-15 |
| FAJH | 4 | 7.0e-129 | 1.2e-42 |

**Table 6. Comparison of Methods for Example 5.**

| $f_5, x_0 = 0.9$ | | | |
|---|---|---|---|
| | Numbers of iteration | $f(x_n)$ | $\delta$ |
| NW | 7 | 6.1e-51 | 2.6e-26 |
| VCM | 6 | 1.0e-126 | 7.7e-43 |
| VSHM | 4 | 0 | 1.5e-67 |
| NR1 | 4 | 5.6e-40 | 8.0e-21 |
| NR2 | 14 | 1.6e-30 | 4.3e-18 |
| Z Hu | 6 | 4.5e-65 | 1.5e-15 |
| FAJH | 4 | 0 | 1.9e-39 |

**Table 7. Comparison of Methods for Example 6.**

| $f_6, x_0 = 2.8$ | | | |
|---|---|---|---|
| | Numbers of iteration | $f(x_n)$ | $\delta$ |
| NW | 17 | 8.2e-33 | 9.8e-18 |
| VCM | DIVERGE | --- | ---- |
| VSHM | DIVERGE | 5.1e-37 | 1.0e-18 |
| NR1 | 8 | 6.9e-52 | 2.8e-27 |
| NR2 | 42 | 1.9e-33 | 4.7e-18 |
| Z Hu | 7 | 2.5e-65 | 5.5e-15 |
| FAJH | 6 | 1.0e-110 | 5.2e-27 |

**Table 8. Comparison of Methods for Example 7.**

| $f_7, x_0 = 1.1$ | | | |
|---|---|---|---|
| | Numbers of iteration | $f(x_n)$ | $\delta$ |
| NW | 5 | 7.8e-42 | 3.9e-21 |
| VCM | 3 | 0 | 4.3e-39 |
| VSHM | 3 | 0 | 2.2e-42 |
| NR1 | 3 | 2.4e-33 | 7.0e-17 |
| NR2 | 4 | 4.9e-37 | 9.9e-19 |
| Z Hu | 4 | 4.5e-65 | 1.5e-15 |
| FAJH | 3 | 0 | 3.5e-23 |

**Table 9. Comparison of Methods for Example 8.**

| $f_8, x_0 = 2$ | | | |
|---|---|---|---|
| | Numbers of iteration | $f(x_n)$ | $\delta$ |
| NW | 7 | 1.0e-37 | 1.3e-19 |
| VCM | 53 | 0 | 3.7e-29 |
| VSHM | 4 | −1.0e-126 | 2.8e-36 |
| NR1 | 4 | 7.2e-38 | 1.0e-19 |
| NR2 | 9 | 5.8e-51 | 3.1e-26 |
| Z Hu | 5 | 4.5e-65 | 1.5e-30 |
| FAJH | 4 | −1.0e-126 | 9.8e-33 |

**Table 10. Comparison of Methods for Example 9.**

| $f_9, x_0 = 1$ | | | |
|---|---|---|---|
| | Numbers of iteration | $f(x_n)$ | $\delta$ |
| NW | 1 | 0 | 0 |
| VCM | DIVERGE | --- | ---- |
| VSHM | DIVERGE | --- | ---- |
| NR1 | 1 | 0 | 0 |
| NR2 | DIVERGE | --- | ---- |
| Z Hu | 2 | 4.5e-65 | 1.5e-15 |
| FAJH | 1 | 0 | 0 |

**Table 11. Comparison of Methods for Example 10.**

| $f_{10}, x_0 = 1.5$ | | | |
|---|---|---|---|
| | Numbers of iteration | $f(x_n)$ | $\delta$ |
| NW | 7 | 2.9e-39 | 5.4e-20 |
| VCM | 4 | 4.6e-105 | 3.0e-18 |
| VSHM | 4 | 0 | 8.3e-41 |
| NR1 | 3 | 1.2e-38 | 1.1e-19 |
| NR2 | DIVERGE | --- | ---- |
| Z Hu | 5 | 4.5e-6 | 7.5e-35 |
| FAJH | 4 | 0 | 6.5e-39 |

(FAJH) is comparable with all the methods cited in the above mentioned tables and gives better results even than ninth orders method of Hu *et al.* [5]. With the help of the technique and idea of this paper, one can develop higher-order multi-step iterative methods for solving nonlinear equations, as well as a system of nonlinear equations.

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