

A Family of Generalized Stirling Numbers of the First Kind

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Abstract

A modified approach via differential operator is given to derive a new family of generalized Stirling numbers of the first kind. This approach gives us an extension of the techniques given by El-Desouky [1] and Gould [2]. Some new combinatorial identities and many relations between different types of Stirling numbers are found. Furthermore, some interesting special cases of the generalized Stirling numbers of the first kind are deduced. Also, a connection between these numbers and the generalized harmonic numbers is derived. Finally, some applications in coherent states and matrix representation of some results obtained are given.

Keywords

Stirling Numbers, Comtet Numbers, Creation, Annihilation, Differential Operator, Maple Program

1. Introduction

Gould [2] proved that

$$(e^x D)^n = e^{nx} \sum_{k=1}^n (-1)^{n-k} s(n, k) D^k = e^{nx} \sum_{k=1}^n s_1(n, k) D^k, D = D_x = \frac{d}{dx}, \quad (1)$$

where $s(n, k)$ and $s_1(n, k)$ are the usual Stirling numbers and the singles Stirling numbers of the first kind, respectively, defined by

$$(x)_n = \sum_{k=0}^n s(n, k) x^k, s(n, 0) = \delta_{n,0} \text{ and } s(n, k) = 0 \text{ for } k > n. \quad (2)$$

$$\langle x \rangle_n = \sum_{k=0}^n s_1(n, k) x^k, s_1(n, 0) = \delta_{n,0} \text{ and } s_1(n, k) = 0 \text{ for } k > n, \quad (3)$$

where $(x)_n = x(x-1)\cdots(x-n+1)$ and $\langle x \rangle = x(x+1)\cdots(x+n-1)$.

These numbers satisfy the recurrence relations

$$s(n+1, k) = s(n, k-1) - ns(n, k), \tag{4}$$

$$s_1(n+1, k) = s_1(n, k-1) + ns_1(n, k). \tag{5}$$

EL-Desouky [1] defined the generalized Stirling numbers of the first kind $s(k; \bar{r}, \bar{s})$, called (\bar{r}, \bar{s}) -Stirling numbers of the first kind by

$$e^{r_n x} D^{s_n} \dots e^{r_2 x} D^{s_2} e^{r_1 x} D^{s_1} = e^{\left(\sum_{l=1}^n r_l\right)x} \sum_{k=s_1}^{s_1+s_2+\dots+s_n} s(k; \bar{r}, \bar{s}) D^k, \tag{6}$$

$s(k; \bar{r}, \bar{s}) = 0$ for $k < s_1$ or $k > \sum_{j=1}^n s_j$ and $s(0; \bar{r}, \bar{s}) = 1$, where $\bar{r} := (r_1, r_2, \dots, r_n)$ is a sequence of real numbers and $\bar{s} := (s_1, s_2, \dots, s_n)$ is a sequence of nonnegative integers.

Equation (6) is equivalent to

$$e^{r_n a^+} a^{s_n} \dots e^{r_2 a^+} a^{s_2} e^{r_1 a^+} a^{s_1} = e^{\left(\sum_{l=1}^n r_l\right)a^+} \sum_{k=s_1}^{s_1+s_2+\dots+s_n} s(k; \bar{r}, \bar{s}) a^k, \tag{7}$$

where a^+ and a are boson creation and annihilation operators, respectively, and satisfy the commutation relation $[a^+, a] = 1$.

The numbers $s(k; \bar{r}, \bar{s})$ satisfy the recurrence relation

$$s(k; \bar{r} \oplus r_{n+1}, \bar{s} \oplus s_{n+1}) = \sum_{i=0}^{s_{n+1}} \binom{s_{n+1}}{i} \left(\sum_{j=1}^n r_j\right)^{s_{n+1}-i} s(k-i; \bar{r}, \bar{s}), \tag{8}$$

with the notations $\bar{r} \oplus r_{n+1} = (r_1, r_2, \dots, r_{n+1})$ and $\bar{s} \oplus s_{n+1} = (s_1, s_2, \dots, s_{n+1})$.

The numbers $s(k; \bar{r}, \bar{s})$ have the explicit formula

$$s(k; \bar{r}, \bar{s}) = \sum_{\sigma_{n-1}=\beta_n-k, j \geq 0} \prod_{l=1}^{n-1} \binom{s_{l+1}}{i_l} \left(\sum_{j=1}^l r_j\right)^{i_l}, \tag{9}$$

where $\sigma_n = \sum_{j=0}^n i_j$, with $i_0 = 0$ and $\beta_n = \sum_{j=1}^n s_j$.

Moreover El-Desouky [1] derived many special cases and some applications. For the proofs and more details, see [1].

The generalized falling factorial of x associated with the sequence $\bar{\alpha} := (\alpha_0, \alpha_1, \dots, \alpha_{n-1})$ of order n , where $\alpha_0, \alpha_1, \dots, \alpha_{n-1}$ are real numbers, is defined by $(x, \bar{\alpha})_n = \prod_{i=0}^{n-1} (x - \alpha_i)$, $(x, \bar{\alpha})_0 = 1$.

Comtet [3] [4] and [5] defined $s_{\bar{\alpha}}(n, k)$ the generalized Stirling numbers of the first kind, which are called Comtet numbers, by

$$(x, \bar{\alpha})_n = \sum_{k=0}^n s_{\bar{\alpha}}(n, k) x^k. \tag{10}$$

These numbers satisfy the recurrence relation

$$s_{\bar{\alpha}}(n, k) = s_{\bar{\alpha}}(n-1, k-1) - \alpha_{n-1} s_{\bar{\alpha}}(n-1, k). \tag{11}$$

El-Desouky and Cakic [6] defined $s_{\bar{\alpha}}(n, k; \bar{s})$, the generalized Comtet numbers by

$$\prod_{j=0}^{n-1} (x^{\alpha_j} \delta x^{-\alpha_j})^{s_j} = \prod_{j=0}^{n-1} (\delta - \alpha_j)^{s_j} = \sum_{k=0}^{|s|} s_{\bar{\alpha}}(n, k; \bar{s}) \delta^k, \tag{12}$$

where $|s| = \sum_{j=0}^{n-1} s_j$, $s_{\bar{\alpha}}(n, k; \bar{s}) = 0$ for $k > |s|$, $s_{\bar{\alpha}}(n, 0; \bar{s}) = (-1)^{|s|} \prod_{j=0}^{n-1} (\alpha_j)^{s_j}$ and $\bar{s} = (s_0, s_1, \dots, s_{n-1})$.

For more details on generalized Stirling numbers via differential operators, see [7]-[10] and [11].

The paper is organized as follows:

In Section 2, using the differential operator $(e^{r_n x} D^{s_n} e^{r_n x}) \dots (e^{r_2 x} D^{s_2} e^{r_2 x}) (e^{r_1 x} D^{s_1} e^{r_1 x})$ we define a new family of generalized Stirling numbers of the first kind, denoted by $s(k; \bar{r}, \bar{s})$. A recurrence relation and an explicit formula of these numbers are derived. In Section 3, some interesting special cases are discussed. Moreover some new combinatorial identities and a connection between $s(n, k; r, s)$ and the generalized harmonic numbers $O_n^{(i)}$ are given. In Section 4, some applications in coherent states and matrix representation of some results obtained are given. Section 5 is devoted to the conclusion, which handles the main results derived throughout this work. Finally, a computer program is written using Maple and executed for calculating the generalized Stirling numbers of the first kind and some special cases, see [Appendix](#).

2. Main Results

Let $\bar{r} := (r_1, r_2, \dots, r_n)$ be a sequence of real numbers and $\bar{s} := (s_1, s_2, \dots, s_n)$ be a sequence of nonnegative integers.

Definition 2.1

The generalized Stirling numbers $s(k; \bar{r}, \bar{s})$ are defined by

$$(e^{r_n x} D^{s_n} e^{r_n x}) \dots (e^{r_2 x} D^{s_2} e^{r_2 x}) (e^{r_1 x} D^{s_1} e^{r_1 x}) = e^{(2\sum_{i=1}^n r_i)x} \sum_{k=0}^{\beta_n} s(k; \bar{r}, \bar{s}) D^k, \tag{13}$$

where $\beta_n = \sum_{j=1}^n s_j$, $s(k; \bar{r}, \bar{s}) = 0$ for $k > \beta_n$ and $s(0; \bar{r}, \bar{s}) = 1$.

Equation (13) is equivalent to

$$(e^{r_n a^+} a^{s_n} e^{r_n a^+}) \dots (e^{r_2 a^+} a^{s_2} e^{r_2 a^+}) (e^{r_1 a^+} a^{s_1} e^{r_1 a^+}) = e^{(2\sum_{i=1}^n r_i)a^+} \sum_{k=0}^{\beta_n} s(k; \bar{r}, \bar{s}) a^k. \tag{14}$$

Theorem 2.1

The numbers $s(k; \bar{r}, \bar{s})$ satisfy the recurrence relation

$$s(k; \bar{r} \oplus r_{n+1}, \bar{s} \oplus s_{n+1}) = \sum_{i=0}^{s_{n+1}} \binom{s_{n+1}}{i} \left(2\sum_{j=1}^n r_j + r_{n+1} \right)^{s_{n+1}-i} s(k-i; \bar{r}, \bar{s}), \tag{15}$$

with the notations $\bar{r} \oplus r_{n+1} := (r_1, r_2, \dots, r_{n+1})$ and $\bar{s} \oplus s_{n+1} := (s_1, s_2, \dots, s_{n+1})$.

Proof

$$\begin{aligned} e^{(2\sum_{j=1}^{n+1} r_j)x} \sum_{k=0}^{\beta_{n+1}} s(k; \bar{r}, \bar{s}) D^k &= (e^{r_{n+1}x} D^{s_{n+1}} e^{r_{n+1}x}) \left(e^{(2\sum_{j=1}^n r_j)x} \sum_{k=0}^{\beta_n} s(k; \bar{r}, \bar{s}) D^k \right) \\ &= e^{r_{n+1}x} D^{s_{n+1}} \left(e^{(2\sum_{j=1}^n r_j + r_{n+1})x} \sum_{k=0}^{\beta_n} s(k; \bar{r}, \bar{s}) D^k \right) \\ &= e^{(2\sum_{j=1}^n r_j)x} \sum_{m=0}^{\beta_n} \left(D + \left(2\sum_{j=1}^n r_j + r_{n+1} \right) \right)^{s_{n+1}-m} s(m; \bar{r}, \bar{s}) D^m \\ &= e^{(2\sum_{j=1}^n r_j)x} \sum_{m=0}^{\beta_n} \sum_{i=0}^{s_{n+1}} \binom{s_{n+1}}{i} \left(2\sum_{j=1}^n r_j + r_{n+1} \right)^{s_{n+1}-i} s(m; \bar{r}, \bar{s}) D^{m+i} \\ &= e^{(2\sum_{j=1}^n r_j)x} \sum_{k=0}^{\beta_{n+1}} \sum_{i=0}^{s_{n+1}} \binom{s_{n+1}}{i} \left(2\sum_{j=1}^n r_j + r_{n+1} \right)^{s_{n+1}-i} s(k-i; \bar{r}, \bar{s}) D^k. \end{aligned}$$

Equating the coefficients of D^k on both sides yields (15).

Theorem 2.2

The numbers $s(k; \bar{r}, \bar{s})$ have the explicit formula

$$s(k; \bar{r}, \bar{s}) = \sum_{\sigma_n = \beta_n - k, j \geq 0} \prod_{l=1}^n \binom{s_l}{i} \left(2 \left(\sum_{j=0}^{l-1} r_j \right) + r_l \right)^{i_j}, \text{ where } \sigma_n = \sum_{j=1}^n i_j, r_0 = 0. \tag{16}$$

Proof

$$\begin{aligned}
 e^{\eta x} D^{s_1} e^{\eta x} &= e^{\eta x} e^{\eta x} (D + r_1 I)^{s_1} = e^{2\eta x} \sum_{i_1=0}^{s_1} \binom{s_1}{i_1} (r_1)^{i_1} D^{s_1-i_1}, \\
 (e^{r_2 x} D^{s_2} e^{r_2 x})(e^{\eta x} D^{s_1} e^{\eta x}) &= e^{r_2 x} D^{s_2} e^{r_2 x} \left(e^{2\eta x} \sum_{i_1=0}^{s_1} \binom{s_1}{i_1} (r_1)^{i_1} D^{s_1-i_1} \right) = e^{r_2 x} D^{s_2} e^{(2\eta+r_2)x} \sum_{i_1=0}^{s_1} \binom{s_1}{i_1} (r_1)^{i_1} D^{s_1-i_1} \\
 &= e^{r_2 x} e^{(2\eta+r_2)x} \sum_{i_1=0}^{s_1} \binom{s_1}{i_1} (r_1)^{i_1} (D + (2r_1 + r_2) I)^{s_2} D^{s_1-i_1} \\
 &= e^{2(\eta+r_2)x} \sum_{i_1=0}^{s_1} \sum_{i_2=0}^{s_2} \binom{s_1}{i_1} \binom{s_2}{i_2} (r_1)^{i_1} (2r_1 + r_2)^{i_2} D^{s_1+s_2-i_1-i_2},
 \end{aligned}$$

thus, by iteration, we get

$$\begin{aligned}
 &(e^{r_n x} D^{s_n} e^{r_n x}) \dots (e^{r_2 x} D^{s_2} e^{r_2 x})(e^{\eta x} D^{s_1} e^{\eta x}) \\
 &= e^{2(\sum_{l=1}^n r_l)x} \sum_{i_1=0}^{s_1} \sum_{i_2=0}^{s_2} \dots \sum_{i_n=0}^{s_n} \prod_{j=1}^n \binom{s_j}{i_j} \left(2 \left(\sum_{l=1}^{j-1} r_l \right) + r_j \right)^{i_j} D^{\sum_{j=1}^n s_j - \sum_{j=1}^n i_j}, r_0 = 0.
 \end{aligned} \tag{17}$$

Setting $\sum_{j=1}^n s_j - \sum_{j=1}^n i_j = \beta_n - \sigma_n = k$, we obtain

$$(e^{r_n x} D^{s_n} e^{r_n x}) \dots (e^{r_2 x} D^{s_2} e^{r_2 x})(e^{\eta x} D^{s_1} e^{\eta x}) = e^{(2\sum_{l=1}^n r_l)x} \sum_{k=0}^{\beta_n} \sum_{\sigma_n=\beta_n-k, j \geq 0} \prod_{j=1}^n \binom{s_j}{i_j} \left(2 \left(\sum_{l=1}^{j-1} r_l \right) + r_j \right)^{i_j} D^k. \tag{18}$$

Comparing (13) and (18) yields (16).

3. Special Cases

Setting $r_i = r$ and $s_i = s, i = 1, \dots, n$ in (13), we have the following definition.

Definition 3.1

For any real number r and nonnegative integer s , let the numbers $\mathbf{s}(n, k; r, s)$, be defined by

$$(e^{rx} D^s e^{rx})^n = e^{(2nr)x} \sum_{k=0}^{ns} \mathbf{s}(n, k; r, s) D^k, \tag{19}$$

where $\mathbf{s}(0, 0; r, s) = 1$ and $\mathbf{s}(n, k; r, s) = 0$ for $k > ns$.

Equation (19) is equivalent to

$$(e^{ra^+} a^s e^{ra^+})^n = e^{(2nr)a^+} \sum_{k=0}^{ns} \mathbf{s}(n, k; r, s) a^k. \tag{20}$$

Corollary 3.1

The numbers $\mathbf{s}(n, k; r, s)$ satisfy the recurrence relation

$$\mathbf{s}(n+1, k; r, s) = \sum_{i=0}^s \binom{s}{i} ((2n+1)r)^{s-i} \mathbf{s}(n, k-i; r, s). \tag{21}$$

Proof

The proof follows directly from Equation (15) by setting $r_i = r$ and $s_i = s, i = 1, 2, \dots, n$.

Corollary 3.2

The numbers $\mathbf{s}(n, k; r, s)$ have the explicit formula

$$\mathbf{s}(n, k; r, s) = r^{ns-k} \sum_{\sigma_n=ns-k, i_j \geq 0} \left(\prod_{j=1}^n \binom{s}{i_j} (2j-1)^{i_j} \right). \tag{22}$$

Proof

By substituting $r_i = r$ and $s_i = s, i = 1, 2, \dots, n$ in Equation (17), yields

$$\left(e^{rx} D^s e^{rx} \right)^n = e^{(2nr)x} \sum_{0 \leq i_1, \dots, i_n \leq s} \left(\prod_{j=1}^n \binom{s}{i_j} \right) (2j-1)^{i_j} r^{\sigma_n} D^{ns-\sigma_n},$$

then setting $ns - \sigma_n = k$ we have

$$\left(e^{rx} D^s e^{rx} \right)^n = e^{(2nr)x} \sum_{k=0}^{ns} r^{ns-k} \sum_{\sigma_n=ns-k, i_j \geq 0} \left(\prod_{j=1}^n \binom{s}{i_j} \right) (2j-1)^{i_j} D^k, \tag{23}$$

hence comparing Equations (19) and (23) we obtain Equation (22).

Furthermore we handle the following special cases.

i) If $r = 1$, then we have

Definition 3.2

$$\left(e^x D^s e^x \right)^n = e^{(2n)x} \sum_{k=0}^{ns} \mathbf{s}(n, k; 1, s) D^k, \tag{24}$$

where $\mathbf{s}(0, 0; 1, s) = 1$ and $\mathbf{s}(n, k; 1, s) = 0$ for $k > ns$.

Corollary 3.3

The numbers $\mathbf{s}(n, k; 1, s)$ satisfy the recurrence relation

$$\mathbf{s}(n+1, k; 1, s) = \sum_{i=0}^s \binom{s}{i} (2n+1)^{s-i} \mathbf{s}(n, k-i; 1, s). \tag{25}$$

Proof:

The proof follows directly from Equation (21) by setting $r = 1$.

Corollary 3.4

The numbers $\mathbf{s}(n, k; 1, s)$ have the explicit formula

$$\mathbf{s}(n, k; 1, s) = \sum_{\sigma_n=ns-k, i_j \geq 0} \left[\prod_{j=1}^n \binom{s}{i_j} (2j-1)^{i_j} \right]. \tag{26}$$

Proof

The proof follows directly from Equation (22) by setting $r = 1$.

ii) If $s = 1$, then we have

Definition 3.3

The numbers $\mathbf{s}(n, k; r, 1)$ are defined by

$$\left(e^{rx} D e^{rx} \right)^n = e^{(2nr)x} \sum_{k=0}^n \mathbf{s}(n, k; r, 1) D^k, \tag{27}$$

where $\mathbf{s}(n, 0; r, 1) = \delta_{n,0}$ and $\mathbf{s}(n, k; r, 1) = 0$ for $k > n$.

Corollary 3.5

The numbers $\mathbf{s}(n, k; r, 1)$ satisfy the triangular recurrence relation

$$\mathbf{s}(n, k; r, 1) = \mathbf{s}(n-1, k-1; r, 1) + (2n-1)r \mathbf{s}(n-1, k; r, 1). \tag{28}$$

Proof

The proof follows easily from (22) by setting $s = 1$.

Corollary 3.6

The numbers $\mathbf{s}(n, k; r, 1)$ have the following explicit formula

$$\mathbf{s}(n, k; r, 1) = r^{n-k} \sum_{\sigma_n=n-k, i_j \in \{0,1\}} \prod_{j=1}^n (2j-1)^{i_j}. \tag{29}$$

Proof

The proof follows from (22) by setting $s = 1$.

Also, using the recurrence relation (28) we can find the following explicit formula.

Theorem 3.1

The numbers $\mathbf{s}(n, k; r, 1)$ have the following explicit expression

$$\mathbf{s}(n, k; r, 1) = \sum_{i_1 + \dots + i_n = k, i_j \in \{0, 1\}} \binom{i_1 + r}{1 - i_1} \binom{i_2 + 3r}{1 - i_2} \dots \binom{i_n + (2n-1)r}{1 - i_n}. \tag{30}$$

Proof

For $k = 0$, $\mathbf{s}(n, 0; r, 1) = (r)(3r)(5r) \dots ((2n-1)r) = r^n (1.2.3 \dots (2n-1))$.

For $i_n \in \{0, 1\}$, we get

$$\begin{aligned} &\mathbf{s}(n, k; r, 1) \\ &= \sum_{i_1 + \dots + i_{n-1} = k-0, i_j \in \{0, 1\}} \binom{i_1 + r}{1 - i_1} \binom{i_2 + 3r}{1 - i_2} \dots \binom{i_{n-1} + (2n-3)r}{1 - i_{n-1}} \cdot (2n-1)r \\ &\quad + \sum_{i_1 + \dots + i_{n-1} = k-1, i_j \in \{0, 1\}} \binom{i_1 + r}{1 - i_1} \binom{i_2 + 3r}{1 - i_2} \dots \binom{i_{n-1} + (2n-3)r}{1 - i_{n-1}} \cdot 1 \\ &= \mathbf{s}(n-1, k-1; r, 1) + (2n-1)r\mathbf{s}(n-1, k; r, 1). \end{aligned}$$

That is the same recurrence relation (28) for the numbers $\mathbf{s}(n, k; r, 1)$. This completes the proof.

iii) If $r = 1$ and $s = 1$, then we have

Definition 3.4

The numbers $\mathbf{s}(n, k) := \mathbf{s}(n, k; 1, 1)$ are defined by

$$(e^x De^x)^n = e^{2nx} \sum_{k=0}^n \mathbf{s}(n, k) D^k, \tag{31}$$

where $\mathbf{s}(n, 0) = \delta_{n,0}$ and $\mathbf{s}(n, k) = 0$ for $k > n$.

Equation (31) is equivalent to

$$(e^{a^+} De^{a^+})^n = e^{2na^+} \sum_{k=0}^n \mathbf{s}(n, k) a^k. \tag{32}$$

Corollary 3.7

The numbers $\mathbf{s}(n, k)$ satisfy the triangular recurrence relation

$$\mathbf{s}(n+1, k) = \mathbf{s}(n, k-1) + (2n+1)\mathbf{s}(n, k). \tag{33}$$

Proof

The proof follows by setting $r = 1$ in Equation (28).

Corollary 3.8

The numbers $\mathbf{s}(n, k)$ have the explicit formula

$$\mathbf{s}(n, k) = \sum_{\sigma_n = n-k, i_j \in \{0, 1\}} \prod_{j=1}^n (2j-1)^{i_j}. \tag{34}$$

Proof

The proof follows by setting $r = 1$ in Equation (29).

Moreover $\mathbf{s}(n, k)$ have the following explicit formula.

Corollary 3.9

The numbers $\mathbf{s}(n, k)$ have the following explicit expression

$$\mathbf{s}(n, k) = \sum_{i_1 + \dots + i_n = k, i_j \in \{0, 1\}} \binom{i_1 + 1}{1 - i_1} \binom{i_2 + 3}{1 - i_2} \binom{i_2 + 5}{1 - i_3} \dots \binom{i_n + (2n-1)}{1 - i_n}. \tag{35}$$

Proof

The proof follows by setting $r = 1$ in (30).

From Equations (29) and (30) (also from Equations (34) and (35)) we have the combinatorial identities

$$\sum_{i_1+\dots+i_n=k, i_j \in \{0,1\}} \binom{i_1+r}{1-i_1} \binom{i_2+3r}{1-i_2} \dots \binom{i_n+(2n-1)r}{1-i_n} = r^{n-k} \sum_{\sigma_n=n-k, i_j \in \{0,1\}} \prod_{j=1}^n (2j-1)^{i_j}. \tag{36}$$

$$\sum_{i_1+\dots+i_n=k, i_j \in \{0,1\}} \binom{i_1+1}{1-i_1} \binom{i_2+3}{1-i_2} \dots \binom{i_n+(2n-1)}{1-i_n} = \sum_{\sigma_n=n-k, i_j \in \{0,1\}} \prod_{j=1}^n (2j-1)^{i_j}. \tag{37}$$

From Equations (29) and (34) we obtain that

$$\mathbf{s}(n, k; r, 1) = r^{n-k} \mathbf{s}(n, k). \tag{38}$$

Remark 3.1

Operating with both sides of Equation (13) on the exponential function e^{lx} , we get

$$(l+r_1)^{s_1} (l+2r_1+r_2)^{s_2} + \dots + (l+2r_1+\dots+2r_{n-1}+r_n)^{s_n} = \sum_{k=s}^{\beta_n} \mathbf{s}(k; \bar{r}, \bar{s}) l^k.$$

Therefore, since a nonzero polynomial can have only a finite set of zeros, we have

$$\prod_{j=0}^{n-1} \left(2 \sum_{m=0}^j r_m + r_{j+1} + x \right)^{s_{j+1}} = \sum_{k=0}^{\beta_n} \mathbf{s}(k; \bar{r}, \bar{s}) x^k, r_0 = 0. \tag{39}$$

If $x = 1$, we obtain

$$\sum_{k=0}^{\beta_n} \mathbf{s}(k; \bar{r}, \bar{s}) = \prod_{j=0}^{n-1} \left(1 + 2 \sum_{m=0}^j r_m + r_{j+1} \right)^{s_{j+1}}. \tag{40}$$

Remark 3.2

From relation (39), by replacing s_j with s_{j-1} , and relation (18) we conclude that

$$\mathbf{s}(k; \bar{r}, \bar{s}) = s_{\bar{\alpha}}(n, k; \tilde{s}), \text{ where } \alpha_i = -\left(2 \sum_{m=0}^i r_m + r_{i+1} \right), i = 0, 1, \dots, n-1. \tag{41}$$

This gives us a connection between $\mathbf{s}(k; \bar{r}, \bar{s})$ and $s_{\bar{\alpha}}(n, k; \tilde{s})$, the generalized Comtet numbers, see [6].

Setting $r_i = r$ and $s_i = s, i = 1, \dots, n$, in (39), we get

$$\prod_{j=0}^{n-1} ((2j+1)r+x)^s = \sum_{k=0}^{ns} \mathbf{s}(n, k; r, s) x^k, \tag{42}$$

hence, we have $\mathbf{s}(n, k; r, s) = s_{\bar{\alpha}}(n, k; s)$, where $\alpha_i = -(2i+1)r, i = 0, 1, \dots, n-1$, see [6].

If $x = 1$, then

$$\sum_{k=0}^{ns} \mathbf{s}(n, k; r, s) = \prod_{j=0}^{n-1} ((2j+1)r+1)^s, n \geq 1. \tag{43}$$

Next we discuss the following special cases of (42) and (43):

i) If $r = 1$, then

$$\left. \begin{aligned} \prod_{j=0}^{n-1} ((2j+1)+x)^s &= \sum_{k=0}^{ns} \mathbf{s}(n, k; 1, s) x^k, \\ \sum_{k=0}^{ns} \mathbf{s}(n, k; 1, s) &= \prod_{j=0}^{n-1} ((2j+1)+1)^s, n \geq 1, \end{aligned} \right\} \tag{44}$$

hence we have $\mathbf{s}(n, k; 1, s) = s_{\bar{\alpha}}(n, k)$, the generalized Comtet numbers, where $\alpha_i = -(2i+1)$,

$i = 0, 1, \dots, n-1$, see [6].

ii) If $s = 1$, then we have

$$\left. \begin{aligned} \prod_{j=0}^{n-1} ((2j+1)r+x) &= \sum_{k=0}^n \mathbf{s}(n, k; r, 1) x^k, \\ \sum_{k=0}^n \mathbf{s}(n, k; r, 1) &= \prod_{j=0}^{n-1} ((2j+1)r+1), n \geq 1. \end{aligned} \right\} \quad (45)$$

hence we obtain $\mathbf{s}(n, k; r, s) = s_{\bar{\alpha}}(n, k)$, Comtet numbers, where $\alpha_i = -(2i+1)r, i = 0, 1, \dots, n-1$, see [3] and [4].

For example if $n = 3, r = 2$ and $s = 2$ in (43) we have

$$\sum_{k=0}^6 \mathbf{s}(3, k; 2, 2) = \prod_{j=0}^2 (4j+3)^2. \quad (46)$$

Using **Table 2**,

L.H.S. of (46) = $s(3,0;2,2) + s(3,1;2,2) + s(3,2;2,2) + s(3,3;2,2) + s(3,4;2,2) + s(3,5;2,2) + s(3,6;2,2) = 14400 + 22080 + 12784 + 3552 + 508 + 36 + 1 = 53361$.

R.H.S. of (46) = $\prod_{j=0}^2 (4j+3)^2 = (3^2)(7^2)(11^2) = 53361$.

This confirms (46) and hence (43).

Another example if $n = 2, r = 2$ and $s = 3$ in (43) we have

$$\sum_{k=0}^6 \mathbf{s}(2, k; 2, 3) = \prod_{j=0}^1 (4j+3)^3. \quad (47)$$

Using **Table 3**,

L.H.S. of (47) = $s(2,0;2,3) + s(2,1;2,3) + s(2,2;2,3) + s(2,3;2,3) + s(2,4;2,3) + s(2,5;2,3) + s(2,6;2,3) = 1728 + 3456 + 2736 + 1088 + 228 + 24 + 1 = 9261$.

R.H.S. of (46) = $\prod_{j=0}^1 (4j+3)^3 = (3^3)(7^3) = 9261$.

This confirms (43).

iii) If $r = s = 1$, then we get

$$\left. \begin{aligned} \prod_{j=0}^{n-1} ((2j+1)+x) &= \sum_{k=0}^n \mathbf{s}(n, k) x^k, \\ \sum_{k=0}^n \mathbf{s}(n, k) &= \prod_{j=0}^{n-1} 2(j+1) = 2^n n!, n \geq 1, \end{aligned} \right\} \quad (48)$$

hence we have $\mathbf{s}(n, k) = s_{\bar{\alpha}}(n, k)$, which is a special case of Comtet numbers, where $\alpha_i = -(2i+1), i = 0, 1, \dots, n-1$, see [3] and [4] and **Table 1**.

Setting $e^x = t$, we have $D := d/dx = t(d/dt) = tD_t = \delta_t$, then substituting in (2.1) it becomes

$$(t^{r_n} \delta_t^{s_n} t^{r_n}) \cdots (t^{r_2} \delta_t^{s_2} t^{r_2}) (t^{r_1} \delta_t^{s_1} t^{r_1}) = e^{(2\sum_{i=1}^n r_i)x} \sum_{k=0}^{\beta_n} \mathbf{s}(k; \bar{r}, \bar{s}) \delta_t^k. \quad (49)$$

Using, see [12],

$$F(\delta)(x^\alpha f(x)) = x^\alpha F(\delta + \alpha) f(x),$$

then Equation (49) yields

$$(\delta_t + 2r_1 + \cdots + 2r_{n-1} + r_n)^{s_n} \cdots (\delta_t + 2r_2 + r_1)^{s_2} (\delta_t + r_1)^{s_1} = \sum_{k=0}^{\beta_n} \mathbf{s}(k; \bar{r}, \bar{s}) \delta_t^k. \quad (50)$$

Comparing this equation with Equation (4.1) in [6], we get

$$\mathbf{s}(k; \bar{r}, \bar{s}) = s_{\bar{\alpha}}(n, k; \tilde{s}), \quad (51)$$

where $\alpha_i = -(2\sum_{m=0}^i r_m + r_{i+1}), i = 0, 1, \dots, n-1$ and $s_{\bar{\alpha}}(n, k; \tilde{s})$, are the generalized Comtet numbers of the first

kind.

Furthermore, using our notations, it is easy from Equation (4.4) in [6] and (41) to show that

$$S(n, i; \bar{\alpha}, \tilde{s}) = \sum_{k=i}^{|s|} \mathbf{s}(n, k; \bar{r}, \tilde{s}) S(k, i), |s| = s_0 + s_1 + \dots + s_{n-1}, \tag{52}$$

where $\alpha_i = -\left(2\sum_{m=0}^i r_m + r_{i+1}\right), i = 0, 1, \dots, n-1$ and $S(n, k)$ are the Stirling numbers of the second kind.

Next, we find a connection between $s(n, k; r, s)$ and the generalized harmonic numbers $O_n^{(i)}$ which are defined by, see [13] and [14],

$$O_n^{(i)} = \sum_{j=1}^n \frac{1}{(2j-1)^i}.$$

From (42), we have

$$\begin{aligned} & \sum_{k=0}^{ns} \mathbf{s}(n, k; r, s) x^k \\ &= \prod_{j=0}^{n-1} ((2j+1)r)^s \prod_{j=0}^{n-1} (1+x/(2j+1)r)^s = \prod_{j=0}^{n-1} ((2j+1)r)^s \exp\left(\log\left(\prod_{j=0}^{n-1} (1+x/(2j+1)r)^s\right)\right) \\ &= \prod_{j=0}^{n-1} ((2j+1)r)^s \exp\left(\sum_{j=0}^{n-1} s \log(1+x/(2j+1)r)\right) = \prod_{j=0}^{n-1} ((2j+1)r)^s \exp\left(\sum_{i=1}^{\infty} s \sum_{j=0}^{n-1} \frac{(-1)^{i+1}}{((2j+1)r)^i} \frac{x^i}{i}\right) \\ &= \prod_{j=0}^{n-1} ((2j+1)r)^s \exp\left(s \sum_{i=1}^{\infty} O_n^{(i)} \frac{(-1)^{i+1}}{r^i i} x^i\right) = \prod_{j=0}^{n-1} ((2j+1)r)^s \sum_{l=0}^{\infty} \left(s \sum_{i=1}^{\infty} O_n^{(i)} \frac{(-1)^{i+1}}{r^i i} x^i\right)^l / l! \\ &= \prod_{j=0}^{n-1} ((2j+1)r)^s \sum_{l=0}^{\infty} \frac{s^l}{l!} \sum_{k=l}^{\infty} \sum_{i_1+\dots+i_l=k} (-1)^{k+l} \frac{O_n^{(i_1)} \dots O_n^{(i_l)}}{i_1 \dots i_l r^k} x^k \\ &= \prod_{j=0}^{n-1} ((2j+1)r)^s \sum_{l=0}^{\infty} \sum_{k=l}^{\infty} \frac{s^l}{l!} \sum_{i_1+\dots+i_l=k} (-1)^{k+l} \frac{O_n^{(i_1)} \dots O_n^{(i_l)}}{i_1 \dots i_l r^k} x^k. \end{aligned}$$

Equating the coefficients of x^k on both sides, we obtain

$$\mathbf{s}(n, k; r, s) = r^{ns-k} \prod_{j=0}^{n-1} (2j+1)^s \sum_{l=0}^{\infty} \frac{s^l}{l!} \sum_{k=l}^{\infty} \sum_{i_1+\dots+i_l=k} (-1)^{k+l} \frac{O_n^{(i_1)} \dots O_n^{(i_l)}}{i_1 \dots i_l r^k}. \tag{53}$$

From (22) and (53), we have the combinatorial identity

$$\sum_{\substack{\sigma_n = ns-k, \\ i_j \geq 0}} \left[\prod_{j=1}^n \binom{s}{i_j} \right] (2j-1)^{i_j} = \prod_{j=0}^{n-1} (2j+1)^s \sum_{l=0}^{\infty} \frac{s^l}{l!} \sum_{k=l}^{\infty} \sum_{i_1+\dots+i_l=k} (-1)^{k+l} \frac{O_n^{(i_1)} \dots O_n^{(i_l)}}{i_1 \dots i_l r^k}. \tag{54}$$

hence, setting $s = 1$, we get the identity

$$\sum_{\substack{\sigma_n = n-k, \\ i_j \geq 0}} \left[\prod_{j=1}^n (2j-1)^{i_j} \right] = \prod_{j=0}^{n-1} (2j+1) \sum_{l=0}^k \sum_{i_1+\dots+i_l=k} (-1)^{k+l} \frac{O_n^{(i_1)} \dots O_n^{(i_l)}}{i_1 \dots i_l l!}. \tag{55}$$

4. Some Applications

4.1. Coherent State and Normal Ordering

Coherent states play an important role in quantum mechanics especially in optics. The normally ordered form of the boson operator in which all the creation operators a^+ stand to the left of the annihilation operators a . Using the properties of coherent states we can define and represent the generalized polynomial $\bar{P}_{r,s}(x)$ and generalized

number $\bar{P}_{r,s}$ as follows.

Definition 4.1

The generalized polynomial $\bar{P}_{r,s}(x)$ is defined by

$$\bar{P}_{r,s}(x) = \sum_{k=0}^{\beta_n} \mathbf{s}(k; \bar{\mathbf{r}}, \bar{\mathbf{s}}) x^k, \tag{56}$$

and the generalized number $\bar{P}_{r,s}$

$$\bar{P}_{r,s} = \bar{P}_{r,s}(1) = \sum_{k=0}^{\beta_n} \mathbf{s}(k; \bar{\mathbf{r}}, \bar{\mathbf{s}}). \tag{57}$$

For convenience we apply the convention

$$\mathbf{s}(k; \bar{\mathbf{r}}, \bar{\mathbf{s}}) = 0 \text{ for } k < 0 \text{ or } k > \beta_n. \tag{58}$$

Now we come back to normal ordering. Using the properties of coherent states, see [7], the coherent state matrix element of the boson string yields the generalized polynomial $\bar{P}_{r,s}(x)$

$$\begin{aligned} & \left\langle z \left| \left(e^{r_1 a^+} a^{s_1} e^{r_1 a^+} \right) \dots \left(e^{r_2 a^+} a^{s_2} e^{r_2 a^+} \right) \left(e^{\eta a^+} a^{s_1} e^{\eta a^+} \right) \right| z \right\rangle \\ &= \left\langle z \left| e^{(2\sum_{i=1}^n r_i) a^+} \sum_{k=0}^{\beta_n} \mathbf{s}(k; \bar{\mathbf{r}}, \bar{\mathbf{s}}) a^k \right| z \right\rangle = e^{(2\sum_{i=1}^n r_i) z} \sum_{k=0}^{\beta_n} \mathbf{s}(k; \bar{\mathbf{r}}, \bar{\mathbf{s}}) z^k \langle z | z \rangle = e^{(2\sum_{i=1}^n r_i) z} \sum_{k=0}^{\beta_n} \mathbf{s}(k; \bar{\mathbf{r}}, \bar{\mathbf{s}}) z^k \tag{59} \\ &= e^{(2\sum_{i=1}^n r_i) z} \bar{P}_{r,s}(z) = e^{(2\sum_{i=1}^n r_i) z} \bar{P}_{r,s} \left(\frac{|z|^2}{z} \right), \text{ where } \beta_n = \sum_{j=1}^n s_j. \end{aligned}$$

Definition 4.2

We define the polynomial $\bar{P}(n, x)$ as

$$\bar{P}(n, x) = \sum_{k=1}^n \mathbf{s}(n, k) x^k, \tag{60}$$

and the numbers

$$\bar{P}(n) = \bar{P}(n, 1) = \sum_{k=1}^n \mathbf{s}(n, k). \tag{61}$$

For convenience we apply the conventions

$$\mathbf{s}(n, 0) = \delta_{n,0} \text{ and } \mathbf{s}(n, k) = 0 \text{ for } k > n \text{ and } \bar{P}(0) = \bar{P}(0, x) = 1. \tag{62}$$

Similarly, using the properties of coherent states and (32) we have

$$\begin{aligned} \left\langle z \left| \left(e^{a^+} a e^{a^+} \right) \right| z \right\rangle &= \left\langle z \left| \left(e^{2na^+} \sum_{k=0}^n \mathbf{s}(n, k) \right) a^k \right| z \right\rangle \\ &= e^{2nz} \sum_{k=0}^n \mathbf{s}(n, k) z^k \langle z | z \rangle = e^{2nz} \sum_{k=0}^n \mathbf{s}(n, k) z^k \tag{63} \\ &= e^{2nz} \bar{P}(n, z) = e^{2nz} \bar{P} \left(n, \frac{|z|^2}{z} \right). \end{aligned}$$

4.2. Matrix Representation

In this subsection we derive a matrix representation of some results obtained.

Let s_r be $n \times n$ lower triangle matrix, where s_r is the matrix whose entries are the numbers $\mathbf{s}(n, k; r, 1)$, i.e. $\mathbf{s}_r = [\mathbf{s}(i, j; r, 1)]_{i,j \geq 0}$. Furthermore let N_r be an $n \times n$ lower triangle matrix defined by

$N_r = [e^{2irx}(i, j; r, 1)]_{i, j \geq 0}$, M_r is a diagonal matrix whose entries of the main diagonal are $e^{2irx}, i = 0, 1, \dots, n$,
 i.e. $M_r = \text{diag}(e^{0rx}, e^{2rx}, e^{4rx}, \dots, e^{2nrx})$, $R_r = ((e^{rx}De^{rx})^0, (e^{rx}De^{rx})^1, \dots, (e^{rx}De^{rx})^n)^T$ and
 $D = (D^0, D^1, D^2, \dots, D^n)^T$.

Equation (27), may be represented in a matrix form as

$$R_r = N_r D = M_r s_r D, \tag{64}$$

for example if $n = 3$ then

$$\begin{aligned} \begin{bmatrix} (e^{rx}De^{rx})^0 \\ (e^{rx}De^{rx})^1 \\ (e^{rx}De^{rx})^2 \\ (e^{rx}De^{rx})^3 \end{bmatrix} &= \begin{bmatrix} e^{0rx} & 0 & 0 & 0 \\ re^{2rx} & e^{2rx} & 0 & 0 \\ 3r^2e^{4rx} & 4re^{4rx} & e^{4rx} & 0 \\ 15r^3e^{6rx} & 23r^2e^{6rx} & 9re^{6rx} & e^{6rx} \end{bmatrix} \begin{bmatrix} D^0 \\ D^1 \\ D^2 \\ D^3 \end{bmatrix} \\ &= \begin{bmatrix} e^{0rx} & 0 & 0 & 0 \\ 0 & e^{2rx} & 0 & 0 \\ 0 & 0 & e^{4rx} & 0 \\ 0 & 0 & 0 & e^{6rx} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ r & 1 & 0 & 0 \\ 3r^2 & 4r & 1 & 0 \\ 15r^3 & 23r^2 & 9r & 1 \end{bmatrix} \begin{bmatrix} D^0 \\ D^1 \\ D^2 \\ D^3 \end{bmatrix}, \end{aligned} \tag{65}$$

its inverse is given by

$$D = N_r^{-1} R_r = s_r^{-1} M_r^{-1} R_r. \tag{66}$$

Setting $r = 1$ in (64), we get

$$R_1 = N_1 D = M_1 s_1 D, \tag{67}$$

$$\begin{aligned} \begin{bmatrix} (e^xDe^x)^0 \\ (e^xDe^x)^1 \\ (e^xDe^x)^2 \\ (e^xDe^x)^3 \end{bmatrix} &= \begin{bmatrix} e^{0x} & 0 & 0 & 0 \\ e^{2x} & e^{2x} & 0 & 0 \\ 3e^{4x} & 4e^{4x} & e^{4x} & 0 \\ 15e^{6x} & 23e^{6x} & 9e^{6x} & e^{6x} \end{bmatrix} \begin{bmatrix} D^0 \\ D^1 \\ D^2 \\ D^3 \end{bmatrix} = \begin{bmatrix} e^{0x} & 0 & 0 & 0 \\ 0 & e^{2x} & 0 & 0 \\ 0 & 0 & e^{4x} & 0 \\ 0 & 0 & 0 & e^{6x} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 3 & 4 & 1 & 0 \\ 15 & 23 & 9 & 1 \end{bmatrix} \begin{bmatrix} D^0 \\ D^1 \\ D^2 \\ D^3 \end{bmatrix}, \end{aligned} \tag{68}$$

hence

$$D = N_1^{-1} R_1 = s_1^{-1} M_1^{-1} R_1.$$

For $n = 3$, we have

$$\begin{aligned} \begin{bmatrix} D^0 \\ D^1 \\ D^2 \\ D^3 \end{bmatrix} &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & e^{-2x} & 0 & 0 \\ 1 & -4e^{-2x} & e^{-4x} & 0 \\ -1 & 13e^{-2x} & -9e^{-4x} & e^{-6x} \end{bmatrix} \begin{bmatrix} (e^xDe^x)^0 \\ (e^xDe^x)^1 \\ (e^xDe^x)^2 \\ (e^xDe^x)^3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 1 & -4 & 1 & 0 \\ -1 & 13 & -9 & 1 \end{bmatrix} \begin{bmatrix} e^{0x} & 0 & 0 & 0 \\ 0 & e^{-2x} & 0 & 0 \\ 0 & 0 & e^{-4x} & 0 \\ 0 & 0 & 0 & e^{-6x} \end{bmatrix} \begin{bmatrix} (e^xDe^x)^0 \\ (e^xDe^x)^1 \\ (e^xDe^x)^2 \\ (e^xDe^x)^3 \end{bmatrix} \end{aligned} \tag{69}$$

5. Conclusion

In this article we investigated a new family of generalized Stirling numbers of the first kind. Recurrence relations and an explicit formula of these numbers are derived. Moreover some interesting special cases and new

combinatorial identities are obtained. A connection between this family and the generalized harmonic numbers is given. Finally, some applications in coherent states and matrix representation of some results are obtained.

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Appendix

Tables of $\mathbf{s}(n, k; r, s)$ calculated using Maple, for some values of n, k, r and s :

Table 1. $0 \leq n, k \leq 4, r = s = 1.$

| | | | | | | |
|-----|-----|----|----|---|---|-----|
| 1 | 0 | 0 | 0 | 0 | 0 | 1 |
| 1 | 1 | 0 | 0 | 0 | 0 | 2 |
| 3 | 4 | 1 | 0 | 0 | 0 | 8 |
| 15 | 23 | 9 | 1 | 0 | 0 | 48 |
| 105 | 176 | 86 | 16 | 1 | 0 | 384 |

Table 2. $0 \leq n, k \leq 4, r = s = 2.$

| | | | | | | | | | | |
|---------|---------|---------|---------|--------|-------|------|----|---|---|----------|
| 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 |
| 4 | 4 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 9 |
| 144 | 192 | 88 | 16 | 1 | 0 | 0 | 0 | 0 | 0 | 441 |
| 14400 | 22080 | 12784 | 3552 | 508 | 36 | 1 | 0 | 0 | 0 | 53361 |
| 2822400 | 4730880 | 3138304 | 1076224 | 211808 | 24832 | 1712 | 64 | 1 | 0 | 12006225 |

Table 3. $0 \leq n, k \leq 4, r = 2, \text{ and } s = 3.$

| | | | | | | | | | | | | | |
|------------|-------------|-------------|------------|------------|-----------|-----------|----------|---------|--------|------|-----|---|-------------|
| 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 |
| 8 | 12 | 6 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 27 |
| 1728 | 3456 | 2736 | 1088 | 228 | 24 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 9261 |
| 1728000 | 3974400 | 3824640 | 2014208 | 639936 | 127776 | 16128 | 1248 | 54 | 1 | 0 | 0 | 0 | 12326391 |
| 4741632000 | 11921817600 | 12904335360 | 7944527872 | 3104947968 | 515321088 | 148279040 | 18914304 | 1687152 | 103040 | 4104 | 961 | 0 | 41601569625 |

Notice that the last column in all tables is just the sum of the entries of the corresponding row.