Applied Mathematics, 2015, 6, 2248-2256
Published Online December 2015 in SciRes. http://www.scirp.org/journal/am
http://dx.doi.org/10.4236/am.2015.614198

Scientific Research Publishing

Four Nontrivial Solutions for Kirchhoff Problems with Critical Potential, Critical Exponent and a Concave Term

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Received 13 November 2015; accepted 27 December 2015; published 30 December 2015

Abstract

In this paper, we consider the existence of multiple solutions to the Kirchhoff problems with critical potential, critical exponent and a concave term. Our main tools are the Nehari manifold and mountain pass theorem.

Keywords

Kirchhoff Problems, Critical Potential, Concave term, Nehari Manifold, Mountain Pass Theorem

1. Introduction

In this paper, we consider the multiplicity results of nontrivial solutions of the following Kirchhoff problem

$$\begin{cases} L_{\mu,a,b}u = h|u|^4 u + \lambda f |u|^{q-2} u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{1.1}$$

where $L_{\mu,a,b}v := -\left(a + b\left(\int_\Omega |\nabla v|^2 + \mu |x|^{-2} v^2\right)\mathrm{d}x\right)\left(\Delta v + \mu |x|^{-2} v\right)$, Ω is a smooth bounded domain of $\mathbb{R}^3$, $a > 0$, $b > 0$, $\lambda \neq 0$, $1 < q < 2$, $\mu < 1/4$, λ is a real parameter, $f \in \mathcal{H}^{-1} \cap C(\bar{\Omega})$ with $\mathcal{H}^{-1}$ is the topological dual of $\mathcal{H}_0^1(\Omega)$ satisfying suitable conditions, h is a bounded positive function on Ω.

The original one-dimensional Kirchhoff equation was introduced by Kirchhoff [1] in 1883. His model takes into account the changes in length of the strings produced by transverse vibrations.

In recent years, the existence and multiplicity of solutions to the nonlocal problem

How to cite this paper: El Mokhtar, E.O. (2015) Four Nontrivial Solutions for Kirchhoff Problems with Critical Potential, Critical Exponent and a Concave Term. *Applied Mathematics*, **6**, 2248-2256. http://dx.doi.org/10.4236/am.2015.614198

$$\begin{cases} -\left(a+b\int_\Omega |\nabla u|^2\,\mathrm{d}x\right)\Delta u = g(x;u) & \text{in } \Omega, \\ u=0 & \text{on } \partial\Omega \end{cases} \tag{1.2}$$

has been studied by various researchers and many interesting and important results can be found. For instance, positive solutions could be obtained in [2]-[4]. Especially, Chen *et al.* [5] discussed a Kirchhoff type problem when $g(x;u)=f(x)u^{p-2}u+\lambda g(x)|u|^{q-2}u$, where $1<q<2<p<2^*=2N/(N-2)$ if $N\geq 3$, $2^*=\infty$ if $N=1,2$, $f(x)$ and $g(x)$ with some proper conditions are sign-changing weight functions. And they have obtained the existence of two positive solutions if $p>4$, $0<\lambda<\lambda_0(a)$.

Researchers, such as Mao and Zhang [6], Mao and Luan [7], found sign-changing solutions. As for in nitely many solutions, we refer readers to [8] [9]. He and Zou [10] considered the class of Kirchhoff type problem when $g(x;u)=\lambda f(x;u)$ with some conditions and proved a sequence of a.e. positive weak solutions tending to zero in $L^\infty(\Omega)$.

In the case of a bounded domain of $\mathbb{R}^N$ with $N\geq 3$, Tarantello [8] proved, under a suitable condition on *f*, the existence of at least two solutions to (1.2) for $a=0$, $b=1$ and $g(x;u)=|u|^{\frac{4}{N-2}}u+f$.

Before formulating our results, we give some definitions and notation.

The space $\mathcal{H}_0^1(\Omega)$ is equiped with the norm

$$\|u\|=\left(\int_\Omega |\nabla u|^2\,\mathrm{d}x\right)^{1/2}$$

wich equivalent to the norm

$$\|u\|_\mu=\left(\int_\Omega\left(|\nabla u|^2-\mu|x|^{-2}|u|^2\right)\mathrm{d}x\right)^{1/2},$$

with $\mu<1/4$. More explicitly, we have

$$\left(1-4\mu^+\right)^{1/2}\|u\|\leq\|u\|_\mu\leq\left(1-4\mu^-\right)^{1/2}\|u\|,$$

for all $u\in\mathcal{H}_0^1(\Omega)$, with $\mu^+=\max(\mu,0)$ and $\mu^-=\min(\mu,0)$.

Let S_μ be the best Sobolev constant, then

$$S_\mu^{12}\leq\frac{\left(\int_\Omega |u|^2\,\mathrm{d}x\right)^2}{\|u\|_\mu^{12}} \tag{2.1}$$

Since our approach is variational, we define the functional J_λ on $\mathcal{H}_0^1(\Omega)$ by

$$J_\lambda(u)=(1/2)a\|u\|_\mu^2+(1/4)b\|u\|_\mu^4-(1/6)\int_\Omega h|u|^6\,\mathrm{d}x-(\lambda/q)\int_\Omega f|u|^q\,\mathrm{d}x, \tag{2.2}$$

A point $u\in\mathcal{H}_0^1(\Omega)$ is a weak solution of the Equation (1.1) if it is the critical point of the functional J_λ. Generally speaking, a function *u* is called a solution of (1.1) if $u\in\mathcal{H}_0^1(\Omega)$ and for all $v\in\mathcal{H}_0^1(\Omega)$ it holds

$$\left(a+b\|u\|_\mu^2\right)\int_\Omega\left(\nabla u\nabla v-\mu|x|^{-2}uv\right)\mathrm{d}x-\int_\Omega h|u|^5uv\mathrm{d}x-\lambda\int_\Omega f|u|^{q-1}uv\mathrm{d}x=0.$$

Throughout this work, we consider the following assumptions:

(F) There exist $\nu_0>0$ and $\delta_0>0$ such that $f(x)\geq\nu_0$, for all *x* in $B(0,2\delta_0)$.

(H) $h\in C(\overline{\Omega}), h(0)=\max_{x\in\overline{\Omega}}h(x)=h_0+o(x^\beta), x\in B(0,2\delta_0), \beta>6\left(\frac{1}{4}-\mu\right)^{\frac{1}{2}}$.

Here, $B(a,r)$ denotes the ball centered at *a* with radius *r*.

In our work, we research the critical points as the minimizers of the energy functional associated to the problem (1.1) on the constraint defined by the Nehari manifold, which are solutions of our system.

Let λ_0 be positive number such that

$$\lambda_0=E(A+BD)^k(A'+B'D),$$

where

$$A=\left(\frac{2-q}{6-q}\right)a,\ B=\left(\frac{4-q}{6-q}\right)b,\ A'=\left(\frac{4}{6-q}\right)a,\ B'=\left(\frac{2}{6-q}\right)b,$$

$$D=\frac{2q(q-2)}{(4-q)(q+2)}\left(\frac{a}{b}\right),\ E=(h_0)^{\frac{q-2}{4}}\|f\|_{H^{-1}}^{-1}(S_\mu)^{\frac{3(2-q)}{2}} \text{ and } k=\frac{q-2}{4}.$$

Now we can state our main results.

Theorem 1. *Assume that* $1<q<2$, $-\infty<\mu<\frac{1}{4}$ *and* (*F*) *satisfied and* λ *verifying* $0<\lambda<\lambda_0$, *then the problem* (1.1) *has at least one positive solution.*

Theorem 2. *In addition to the assumptions of the Theorem* 1, *if* (*H*) *hold and* $b>\frac{3(6-q)}{(4-q)}$ *then there exists* $\lambda_1>0$ *such that for all* λ *verifying* $0<\lambda<\min(\lambda_0,\lambda_1)$ *the problem* (1.1) *has at least two positive solutions.*

Theorem 3. *In addition to the assumptions of the Theorem* 2, *assuming* $\lambda<0$ *then the problem* (1.1) *has at least two positive solutions and two opposite solutions.*

This paper is organized as follows. In Section 2, we give some preliminaries. Section 3 and 4 are devoted to the proofs of Theorems 1 and 2. In the last Section, we prove the Theorem 3.

2. Preliminaries

Definition 1. *Let* $c\in\mathbb{R}$, *E a Banach space and* $I\in C^1(E,\mathbb{R})$.

i) $(u_n)_n$ *is a Palais-Smale sequence at level c (in short* $(PS)_c$*) in E for I if*

$$I(u_n)=c+o_n(1) \text{ and } I'(u_n)=o_n(1),$$

where $o_n(1)$ *tends to* 0 *as n goes at infinity.*

ii) *We say that I satisfies the* $(PS)_c$ *condition if any* $(PS)_c$ *sequence in E for I has a convergent subsequence.*

Lemma 1. *Let X Banach space, and* $J\in C^1(X,\mathbb{R})$ *verifying the Palais-Smale condition. Suppose that* $J(0)=0$ *and that*:

i) *there exist* $R>0$, $r>0$ *such that if* $\|u\|=R$, *then* $J(u)\geq r$;

ii) *there exist* $(u_0)\in X$ *such that* $\|u_0\|>R$ *and* $J(u_0)\leq 0$;

let $c=\inf_{\gamma\in\Gamma}\max_{t\in[0,1]}(J(\gamma(t)))$ *where*

$$\Gamma=\{\gamma\in C([0,1];X) \text{ such that } \gamma(0)=0\ et\ \gamma(1)=u_0\},$$

then c is critical value of J such that $c\geq r$.

Nehari Manifold

It is well known that the functional J_λ is of class C^1 in $\mathcal{H}_0^1(\Omega)$ and the solutions of (1.1) are the critical points of J_λ which is not bounded below on $\mathcal{H}_0^1(\Omega)$. Consider the lowing Nehari manifold

$$\mathcal{M}_\lambda=\{u\in\mathcal{H}_0^1(\Omega)\setminus\{0\}:\langle J'_\lambda(u),u\rangle=0\},$$

Thus, $u\in\mathcal{M}_\lambda$ if and only if

$$\left(a+b\|u\|_\mu^2\right)\|u\|_\mu^2-\int_\Omega h|u|^6\,\mathrm{d}x-\lambda\int_\Omega f|u|^q\,\mathrm{d}x=0 \tag{2.3}$$

Define

$$\phi_\lambda(u)=\langle J'_\lambda(u),u\rangle.$$

Then, for $u\in\mathcal{M}_\lambda$

$$\begin{aligned}\langle \phi_\lambda'(u),u\rangle &= \left(2a+4b\|u\|_\mu^2\right)\|u\|_\mu^2-6\int_\Omega h|u|^6\,\mathrm{d}x-\lambda q\int_\Omega f|u|^q\,\mathrm{d}x\\ &=\left[(2-q)a+(4-q)b\|u\|_\mu^2\right]\|u\|_\mu^2-(6-q)\int_\Omega h|u|^6\,\mathrm{d}x\\ &=(6-q)\lambda\int_\Omega f|u|^q\,\mathrm{d}x-\left(4a+2b\|u\|_\mu^2\right)\|u\|_\mu^2.\end{aligned}\tag{2.4}$$

Now, we split $\mathcal{M}_\lambda$ in three parts:

$$\mathcal{M}_\lambda^+=\left\{u\in\mathcal{M}_\lambda:\langle\phi_\lambda'(u),u\rangle>0\right\}$$

$$\mathcal{M}_\lambda^0=\left\{u\in\mathcal{M}_\lambda:\langle\phi_\lambda'(u),u\rangle=0\right\}$$

$$\mathcal{M}_\lambda^-=\left\{u\in\mathcal{M}_\lambda:\langle\phi_\lambda'(u),u\rangle<0\right\}.$$

Note that $\mathcal{M}_\lambda$ contains every nontrivial solution of the problem (1.1). Moreover, we have the following results.

Lemma 2. *J_λ is coercive and bounded from below on $\mathcal{M}_\lambda$.*

Proof. If $u\in\mathcal{M}_\lambda$, then by (2.3) and the Hölder inequality, we deduce that

$$\begin{aligned}J_\lambda(u)&=(1/2)a\|u\|_\mu^2+(1/4)b\|u\|_\mu^4-(1/6)\int_\Omega h|u|^6\,\mathrm{d}x-(\lambda/q)\int_\Omega f|u|^q\,\mathrm{d}x\\ &\ge(1/3)a\|u\|_\mu^2+(1/12)b\|u\|_\mu^4-\lambda\left(\frac{1}{q}-\frac{1}{6}\right)\|f\|_{\mathcal{H}^{-1}}\|u\|_\mu^q.\end{aligned}$$

Thus, J_λ is coercive and bounded from below on $\mathcal{M}_\lambda$.

We have the following results.

Lemma 3. *Suppose that u_0 is a local minimizer for J_λ on $\mathcal{M}_\lambda$. Then, if $u_0\notin\mathcal{M}_\lambda^0$, u_0 is a critical point of J_λ.*

Proof. If u_0 is a local minimizer for J_λ on $\mathcal{M}_\lambda$, then u_0 is a solution of the optimization problem

$$\min_{\{u/\phi_\lambda(u)=0\}} J_\lambda(u).$$

Hence, there exists a Lagrange multipliers $\theta\in\mathbb{R}$ such that

$$J_\lambda'(u_0)=\theta\phi_\lambda'(u_0)\ \text{ in } \mathcal{H}^{-1}$$

Thus,

$$\langle J_\lambda'(u_0),u_0\rangle=\theta\langle\phi_\lambda'(u_0),u_0\rangle.$$

But $\langle\phi_\lambda'(u_0),u_0\rangle\neq0$, since $u_0\notin\mathcal{M}_\lambda^0$. Hence $\theta=0$. This completes the proof.

Lemma 4. *There exists a positive number λ_0 such that, for all $\lambda\in(0,\lambda_0)$ we have $\mathcal{M}_\lambda^0=\varnothing$.*

Proof. Let us reason by contradiction.

Suppose $\mathcal{M}_\lambda^0\neq\varnothing$ such that $0<\lambda<\lambda_0$. Moreover, by the Hölder inequality and the Sobolev embedding theorem, we obtain

$$\|u\|_\mu^4\ge\left(A'+B'\|u\|_\mu^2\right)^{\frac{4}{q-2}}\|f\|_{\mathcal{H}^{-1}}^{\frac{-4}{q-2}}\lambda^{\frac{-4}{q-2}}$$

and

$$\|u\|_\mu^4\le\left(A+B\|u\|_\mu^2\right)^{-1/(2-q)}h_0\left(S_\mu\right)^{-6},$$

with

$$A=\left(\frac{2-q}{6-q}\right)a,\ B=\left(\frac{4-q}{6-q}\right)b,\ A'=\left(\frac{4}{6-q}\right)a,\ B'=\left(\frac{2}{6-q}\right)b.$$

From (2.5) and (2.6), we obtain $\lambda\ge\lambda_0$, which contradicts an hypothesis.

Thus $\mathcal{M}_\lambda = \mathcal{M}_\lambda^+ \cup \mathcal{M}_\lambda^-$. Define

$$c := \inf_{u\in\mathcal{M}_\lambda} J_\lambda(u),\ c^+ := \inf_{u\in\mathcal{M}_\lambda^+} J_\lambda(u) \text{ and } c^- := \inf_{u\in\mathcal{M}_\lambda^-} J_\lambda(u).$$

For the sequel, we need the following Lemma.

Lemma 5. i) *For all* λ *such that* $0<\lambda<\lambda_0$, *one has* $c \le c^+ < 0$.

ii) *There exists* $\lambda_1 > 0$ *such that for all* $0<\lambda<\lambda_1$, *one has*

$$c^- > C_0 = C_0\left(a,b,q,\|f\|_{\mathcal{H}^{-1}}\right).$$

Proof. i) Let $u \in \mathcal{M}_\lambda^+$. By (2.4), we have

$$\left[\left((2-q)a+(4-q)b\|u\|_\mu^2\right)\big/(6-q)\right]\|u\|_{\mu,a}^2 > \int_\Omega h|u|^6\,\mathrm{d}x$$

and so

$$\begin{aligned} J_\lambda(u) &= \left(\frac{1}{4}-\frac{1}{q}\right)b\|u\|_\mu^4 + \left(\frac{1}{2}-\frac{1}{q}\right)a\|u\|_\mu^2 + \left(\frac{1}{q}-\frac{1}{6}\right)\int_\Omega h|u|^6\,\mathrm{d}x \\ &< -\left[\frac{(4-q)}{12q}b\|u\|_\mu^4 + \frac{(2-q)}{3q}a\|u\|_\mu^2\right]. \end{aligned}$$

We conclude that $c \le c^+ < 0$.

ii) Let $u \in \mathcal{M}_\lambda^-$. By (2.4) and the Hölder inequality we get

$$\begin{aligned} J_\lambda(u) &\ge (1/3)a\|u\|_\mu^2 + (1/12)b\|u\|_\mu^4 - \lambda\left(\frac{1}{q}-\frac{1}{6}\right)\|f\|_{\mathcal{H}^{-1}}\|u\|_\mu^q. \\ &\ge \left[(1/3)a+(1/12)b\right]\min\left(\|u\|_\mu^4,\|u\|_\mu^q\right) - \lambda\left(\frac{1}{q}-\frac{1}{6}\right)\|f\|_{\mathcal{H}^{-1}}\max\left(\|u\|_\mu^4,\|u\|_\mu^q\right). \end{aligned}$$

Thus, for all λ such that $0<\lambda<\lambda_1 = \dfrac{(4a+b)q}{(6-q)\|f\|_{\mathcal{H}^{-1}}}$, we have $J_\lambda(u) \ge C_0$.

For each $u \in \mathcal{H}_0^1(\Omega)$ with $P = \int_\Omega h|u|^6\,\mathrm{d}x > a\|u\|_\mu^2$, we write

$$t_m := t_{\max}(u) = \left[\frac{3\left(P-a\|u\|_\mu^2\right)\left[1+\sqrt{1+\dfrac{20aP\|u\|_\mu^2}{9\left(P-a\|u\|_\mu^2\right)}}\right]}{3P}\right]^{1/2} > 0.$$

Lemma 6. *Let* λ *real parameters such that* $0<\lambda<\lambda_0$. *For each* $u\in\mathcal{H}_0^1(\Omega)$ *with* $\int_\Omega h|u|^6\,\mathrm{d}x > a\|u\|_\mu^2$, *there exist unique* t^+ *and* t^- *such that* $0<t^+<t_m<t^-$, $(t^+u)\in\mathcal{M}_\lambda^+$, $(t^-u)\in\mathcal{M}_\lambda^-$,

$$J_\lambda(t^+u) = \inf_{0\le t\le t_m} J_\lambda(tu) \text{ and } J_\lambda(t^-u) = \sup_{t\ge 0} J_\lambda(tu).$$

Proof. With minor modifications, we refer to [11].

Proposition 1. (*see* [11])

i) For all λ such that $0<\lambda<\lambda_0$, there exists a $(PS)_{c^+}$ sequence in $\mathcal{M}_\lambda^+$.

ii) For all λ such that $0<\lambda<\lambda_1$, there exists a a $(PS)_{c^-}$ sequence in $\mathcal{M}_\lambda^-$.

3. Proof of Theorem 1

Now, taking as a starting point the work of Tarantello [8], we establish the existence of a local minimum for J_λ

on $\mathcal{M}_\lambda^+$.

Proposition 2. *For all* λ *such that* $0<\lambda<\lambda_0$, *the functional* J_λ *has a minimizer* $u_0^+\in\mathcal{M}_\lambda^+$ *and it satisfies*:

i) $J_\lambda\left(u_0^+\right)=c=c^+$,

ii) $\left(u_0^+\right)$ is a nontrivial solution of (1.1).

Proof. If $0<\lambda<\lambda_0$, then by Proposition 1. i) there exists a $\left(u_n\right)_n$ $\left(PS\right)_{c^+}$ sequence in $\mathcal{M}_\lambda^+$, thus it bounded by Lemma 2. Then, there exists $u_0^+\in\mathcal{H}_0^1\left(\Omega\right)$ and we can extract a subsequence which will denoted by $\left(u_n\right)_n$ such that

$$\begin{aligned} &u_n\rightharpoonup u_0^+ \text{ weakly in } \mathcal{H}_0^1\left(\Omega\right)\\ &u_n\rightharpoonup u_0^+ \text{ weakly in } L^6\left(\Omega\right)\\ &u_n\rightarrow u_0^+ \text{ strongly in } L^q\left(\Omega\right)\\ &u_n\rightarrow u_0^+ \text{ a.e in } \Omega \end{aligned} \tag{3.1}$$

Thus, by (3.1), u_0^+ is a weak nontrivial solution of (1.1). Now, we show that u_n converges to u_0^+ strongly in $\mathcal{H}_0^1\left(\Omega\right)$. Suppose otherwise. By the lower semi-continuity of the norm, then either $\left\|u_0^+\right\|_{\mu,a}<\liminf_{n\to\infty}\left\|u_n\right\|_{\mu,a}$ and we obtain

$$c\le J_\lambda\left(u_0^+\right)=\left(a/3\right)\left\|u_0^+\right\|_\mu^2+\left(b/12\right)\left\|u_0^+\right\|_\mu^4-\lambda\left(\frac{1}{q}-\frac{1}{6}\right)\int_\Omega f\left|u_0^+\right|^q\mathrm{d}x<\liminf_{n\to\infty} J_\lambda\left(u_n\right)=c.$$

We get a contradiction. Therefore, u_n converge to u_0^+ strongly in $\mathcal{H}_0^1\left(\Omega\right)$. Moreover, we have $u_0^+\in\mathcal{M}_\lambda^+$. If not, then by Lemma 6, there are two numbers t_0^+ and t_0^-, uniquely defined so that $\left(t_0^+u_0^+\right)\in\mathcal{M}_\lambda^+$ and $\left(t^-u_0^+\right)\in\mathcal{M}_\lambda^-$. In particular, we have $t_0^+<t_0^-=1$. Since

$$\frac{\mathrm{d}}{\mathrm{d}t}J_\lambda\left(tu_0^+\right)_{|t=t_0^+}=0 \text{ and } \frac{\mathrm{d}^2}{\mathrm{d}t^2}J_\lambda\left(tu_0^+\right)_{|t=t_0^+}>0,$$

there exists $t_0^+<t^-\le t_0^-$ such that $J_\lambda\left(t_0^+u_0^+\right)<J_\lambda\left(t^-u_0^+\right)$. By Lemma 6, we get

$$J_\lambda\left(t_0^+u_0^+\right)<J_\lambda\left(t^-u_0^+\right)<J_\lambda\left(t_0^-u_0^+\right)=J_\lambda\left(u_0^+\right),$$

which contradicts the fact that $J_\lambda\left(u_0^+\right)=c^+$. Since $J_\lambda\left(u_0^+\right)=J_\lambda\left(\left|u_0^+\right|\right)$ and $\left|u_0^+\right|\in\mathcal{M}_\lambda^+$, then by Lemma 3, we may assume that u_0^+ is a nontrivial nonnegative solution of (1.1). By the Harnack inequality, we conclude that $u_0^+>0$ and $v_0^+>0$, see for exanmple [12].

4. Proof of Theorem 2

Next, we establish the existence of a local minimum for J_λ on $\mathcal{M}_\lambda^-$. For this, we require the following Lemma.

Lemma 7. *Assume that* $b>\dfrac{3\left(6-q\right)}{\left(4-q\right)}$ *then for all* λ *such that* $0<\lambda<\lambda_1$, *the functional* J_λ *has a minimizer* u_0^- *in* $\mathcal{M}_\lambda^-$ *and it satisfies*:

i) $J_\lambda\left(u_0^-\right)=c^->0$,

ii) u_0^- *is a nontrivial solution of* (1.1) *in* $\mathcal{H}_0^1\left(\Omega\right)$.

Proof. If $0<\lambda<\lambda_1$, then by Proposition 1. ii) there exists a $\left(u_n\right)_n$, $\left(PS\right)_{c^-}$ sequence in $\mathcal{M}_\lambda^-$, thus it bounded by Lemma 2. Then, there exists $u_0^-\in\mathcal{H}_0^1\left(\Omega\right)$ and we can extract a subsequence which will denoted by $\left(u_n\right)_n$ such that

$$\begin{aligned} &u_n\rightharpoonup u_0^- \text{ weakly in } \mathcal{H}_0^1\left(\Omega\right)\\ &u_n\rightharpoonup u_0^- \text{ weakly in } L^6\left(\Omega\right)\\ &u_n\rightarrow u_0^- \text{ strongly in } L^q\left(\Omega\right)\\ &u_n\rightarrow u_0^- \text{ a.e in } \Omega \end{aligned}$$

This implies that

$$\int_\Omega h|u_n|^6 \,dx \to \int_\Omega h|u_0^-|^6 \,dx, \text{ as } n \text{ goes to } \infty.$$

Moreover, by (H) and (2.4) we obtain

$$\begin{aligned}\int_\Omega h|u_n|^6 \,dx &> \left[\frac{(4-q)}{(6-q)}b\|u_n\|_\mu^4 + \frac{(2-q)}{(6-q)}a\|u_n\|_\mu^2\right] \\ &> \left[\frac{(4-q)}{(6-q)}b\|u_n\|_\mu^4 + \frac{(2-q)}{(6-q)}a\|u_n\|_\mu^2\right] - \|u_n\|_\mu^2 \\ &> C_1 = \left[\frac{(2-q)}{(6-q)}a\right]^2\left[\frac{(4-q)}{(6-q)}b-3\right]\left[\frac{2(4-q)}{(6-q)}b-2\right]^{-2}\end{aligned}$$

if $b > \frac{3(6-q)}{(4-q)}$ we get

$$\int_\Omega h|u_n|^6 \,dx > C_1 > 0. \tag{4.1}$$

This implies that

$$\int_\Omega h|u_0^-|^6 \,dx \geq C_1.$$

Now, we prove that $(u_n)_n$ converges to u_0^- strongly in $\mathcal{H}_0^1(\Omega)$. Suppose otherwise. Then, either $\|u_0^-\|_\mu < \liminf_{n\to\infty} \|u_n\|_\mu$. By Lemma 6 there is a unique t_0^- such that $(t_0^- u_0^-) \in \mathcal{M}_\lambda^-$. Since

$$u_n \in \mathcal{M}_\lambda^-, J_\lambda(u_n) \geq J_\lambda(tu_n), \text{ for all } t \geq 0,$$

we have

$$J_\lambda(t_0^- u_0^-) < \lim_{n\to\infty} J_\lambda(t_0^- u_n) \leq \lim_{n\to\infty} J_\lambda(u_n) = c^-,$$

and this is a contradiction. Hence,

$$(u_n)_n \to u_0^- \text{ strongly in } \mathcal{H}_0^1(\Omega).$$

Thus,

$$J_\lambda(u_n) \text{ converges to } J_\lambda(u_0^-) = c^- \text{ as } n \text{ tends to } +\infty.$$

Since $J_\lambda(u_0^-) = J_\lambda(|u_0^-|)$ and $u_0^- \in \mathcal{M}_\lambda^-$, then by (4.1) and Lemma 3, we may assume that u_0^- is a nontrivial nonnegative solution of (1.1). By the maximum principle, we conclude that $u_0^- > 0$.

Now, we complete the proof of Theorem 2. By Propositions 2 and Lemma 7, we obtain that (1.1) has two positive solutions $u_0^+ \in \mathcal{M}_\lambda^+$ and $u_0^- \in \mathcal{M}_\lambda^-$. Since $\mathcal{M}_\lambda^+ \cap \mathcal{M}_\lambda^- = \varnothing$, this implies that u_0^+ and u_0^- are distinct.

5. Proof of Theorem 3

In this section, we consider the following Nehari submanifold of $\mathcal{M}_\lambda$

$$\mathcal{M}_{\lambda,\varrho} = \left\{u \in \mathcal{H}_0^1(\Omega)\setminus\{0\} : \langle J'_\lambda(u), u\rangle = 0 \text{ and } \|u\|_\mu \geq \varrho > 0\right\}.$$

Thus, $u \in \mathcal{M}_{\lambda,\varrho}$ if and only if

$$\left(a+b\|u\|_\mu^2\right)\|u\|_\mu^2 - \int_\Omega h|u|^6 \,dx - \lambda\int_\Omega f|u|^q \,dx = 0 \text{ and } \|u\|_\mu \geq \varrho > 0.$$

Firsly, we need the following Lemmas.

Lemma 8. *Under the hypothesis of theorem* 3, *there exist* ϱ_0 *such that* $\mathcal{M}_{\lambda,\varrho}$ *is nonempty for any* $\lambda < 0$ *and* $\varrho \in (0, \varrho_0)$.

Proof. Fix $u_0 \in \mathcal{H}_0^1(\Omega)\setminus\{0\}$ and let

$$g(t)=\left\langle J_\lambda'(tu_0),tu_0\right\rangle = at^2\|u_0\|_\mu^2 + bt^4\|u_0\|_\mu^4 - t^6\int_\Omega h|u_0|^6\,\mathrm{d}x - \lambda t^q\int_\Omega f|u_0|^q\,\mathrm{d}x.$$

Clearly $g(0)=0$ and $g(t)\to-\infty$ as $n\to+\infty$. Moreover, we have

$$\begin{aligned} g(1) &= a\|u_0\|_\mu^2 + b\|u_0\|_\mu^4 - \int_\Omega h|u_0|^6\,\mathrm{d}x - \lambda\int_\Omega f|u_0|^q\,\mathrm{d}x \\ &\geq \left[a\|u_0\|_\mu^2 - (S_\mu)^{-3}h_0\|u_0\|_\mu^6\right]. \end{aligned}$$

If $\|u_0\|_\mu \geq \varrho > 0$ for $0<\varrho<\varrho_0=\left(ah_0^{-1}\right)^{1/4}(S_\mu)^{3/4}$, then there exists $t_0>0$ such that $g(t_0)=0$. Thus, $(t_0u_0)\in\mathcal{M}_{\lambda,\varrho}$ and $\mathcal{M}_{\lambda,\varrho}$ is nonempty for any $\lambda<0$.

Lemma 9. *There exist M positive real such that*

$$\left\langle \phi_\lambda'(u),u\right\rangle < -M < 0,$$

for $u\in\mathcal{M}_{\lambda,\varrho}$ *and any* $\lambda<0$.

Proof. Let $u\in\mathcal{M}_{\lambda,\varrho}$, then by (2.3), (2.4) and the Holder inequality, allows us to write

$$\left\langle \phi_\lambda'(u),u\right\rangle \leq \|u\|_\mu^2\left[(6-q)\lambda\|f\|_{\mathcal{H}^{-1}} - (4a+2b)\right].$$

Thus, if $\lambda<0$ then we obtain that

$$\left\langle \phi_\lambda'(u),u\right\rangle < 0, \text{ for any } u\in\mathcal{M}_{\lambda,\varrho}. \tag{5.1}$$

Lemma 10. *There exist r and* η *positive constants such that*

i) *we have*

$$J_\lambda(u)\geq\eta>0 \text{ for } \|u\|_{\mu,a}=r.$$

ii) *there exists* $\sigma\in\mathcal{M}_{\lambda,\varrho}$ *when* $\|\sigma\|_\mu>r$, *with* $r=\|u\|_\mu$, *such that* $J_\lambda(\sigma)\leq 0$.

Proof. We can suppose that the minima of J_λ are realized by (u_0^+) and u_0^-. The geometric conditions of the mountain pass theorem are satisfied. Indeed, we have

i) By (2.4), (5.1), the Holder inequality and the fact that $\int_\Omega h|u|^6\,\mathrm{d}x \leq (S_\mu)^{-3}h_0\|u\|_\mu^6$, we get

$$J_\lambda(u)\geq\frac{a}{2}\|u\|_\mu^2+\frac{b}{4}\|u\|_\mu^4 - S_\mu^{-3}\left(\frac{h_0}{2}\right)\|u\|_\mu^6 - \frac{\lambda}{q}\|f\|_{\mathcal{H}^{-1}}\|u\|_\mu^q.$$

Thus, for $\lambda<0$ there exist $\eta,r>0$ such that

$$J_\lambda(u)\geq\eta>0 \text{ when } r=\|u\|_{\mu,a} \text{ small.}$$

ii) Let $t>0$, then we have for all $\theta\in\mathcal{M}_{\lambda,\varrho}$

$$J_\lambda(t\theta)=\frac{a}{2}t^2\|\theta\|_\mu^2+\frac{b}{4}t^4\|\theta\|_\mu^4-\frac{1}{6}t^6\int_\Omega h|\theta|^6\,\mathrm{d}x-\frac{\lambda}{q}t^q\int_\Omega f|\theta|^q\,\mathrm{d}x.$$

Letting $\sigma=t\theta$ for *t* large enough, we obtain $J_\lambda(\sigma)\leq 0$. For *t* large enough we can ensure $\|\sigma\|_{\mu,a}>r$.

Let Γ and *c* defined by

$$\Gamma:=\left\{\gamma:[0,1]\to\mathcal{M}_{\lambda,\varrho}:\gamma(0)=u_0^- \text{ and } \gamma(1)=u_0^+\right\}$$

and

$$c:=\inf_{\gamma\in\Pi}\max_{t\in[0,1]}\left(J_\lambda(\gamma(t))\right).$$

Proof of Theorem 3.

If $\lambda<0$ then, by the Lemmas 2 and Proposition 1. ii), J_λ verifying the Palais-Smale condition in $\mathcal{M}_{\lambda,\varrho}$. Moreover, from the Lemmas 3, 9 and 10, there exists u_c such that

$$J_\lambda(u_c) = c \text{ and } u_c \in \mathcal{M}_{\lambda,\varrho}.$$

Thus u_c is the third solution of our system such that $u_c \neq u_0^+$ and $u_c \neq u_0^-$. Since (1.1) is odd with respect u, we obtain that $-u_c$ is also a solution of (1.1).

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