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Generating Set of the Complete Semigroups of Binary Relations

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Abstract

Difficulties encountered in studying generators of semigroup $B_X(D)$ of binary relations defined by a complete *X*-semilattice of unions *D* arise because of the fact that they are not regular as a rule, which makes their investigation problematic. In this work, for special *D*, it has been seen that the semigroup $B_X(D)$, which are defined by semilattice *D*, can be generated by the set $B = \{\alpha \in B_X(D) \mid V(X^*, \alpha) = D\}$.

Keywords

Semigroups, Binary Relation, Generated Set, Generators

1. Introduction

Theorem 1. *Let* $D = \{\breve{D}, Z_1, Z_2, \cdots, Z_{m-1}\}$ *be some finite X-semilattice of unions and*

$$C(D) = \{P_0, P_1, P_2, \cdots, P_{m-1}\}$$

be the family of sets of pairwise nonintersecting subsets of the set X.

If φ is a mapping of the semilattice D on the family of sets $C(D)$ which satisfies the condition $\varphi(\breve{D}) = P_0$ and $\varphi(Z_i) = P_i$ for any $i = 1, 2, \cdots, m-1$ and $\hat{D}_Z = D \setminus D_T$, then the following equalities are valid:

$$\breve{D} = P_0 \cup P_1 \cup P_2 \cup \cdots \cup P_{m-1},$$
$$Z_i = P_0 \cup \bigcup_{T \in \hat{D}_{Z_i}} \varphi(T). \quad (1)$$

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In the sequel these equalities will be called formal.

It is proved that if the elements of the semilattice D are represented in the form 1, then among the parameters P_i $(i = 0,1,2,\cdots,m-1)$ there exist such parameters that cannot be empty sets for D. Such sets P_i $(0 < i \le m-1)$ are called basis sources, whereas sets P_i $(0 \le j \le m-1)$ which can be empty sets too are called completeness sources.

It is proved that under the mapping φ the number of covering elements of the pre-image of a basis source is always equal to one, while under the mapping φ the number of covering elements of the pre-image of a completeness source either does not exist or is always greater than one (see [1], Chapter 11). Some positive results in this direction can be found in [2]-[6].

Let $P_0, P_1, P_2, \cdots, P_{m-1}$ be parameters in the formal equalities, $\beta \in B_X(D)$ and

$$\overline{\beta} = \bigcup_{i=0}^{m-1}\left(P_i \times \bigcup_{t\in P_i} t\beta\right) \cup \bigcup_{t'\in X\setminus \breve{D}} \left(\{t'\} \times t'\beta\right). \tag{2}$$

$$\tilde{\beta} = \bigcup_{i=0}^{m-1}\left(P_i \times \bigcup_{t\in P_i} t\beta\right) \cup \left(\left(X \setminus \breve{D}\right) \times \breve{D}\right) \tag{3}$$

The representation of the binary relation β of the form $\overline{\beta}$ and $\tilde{\beta}$ will be called subquasinormal and maximal subquasinormal.

If $\overline{\beta}$ and $\tilde{\beta}$ are the subquasinormal and maximal subquasinormal representations of the binary relation β, then for the binary relations $\overline{\beta}$ and $\tilde{\beta}$ the following statements are true:

a) $\overline{\beta}, \tilde{\beta} \in B_X(D)$;

b) $\beta \subseteq \overline{\beta} \subseteq \tilde{\beta}$;

c) the subquasinormal representation of the binary relation β is quasinormal;

d) if

$$\overline{\beta}_1 = \begin{pmatrix} P_0 & P_1 & \cdots & P_{m-1} \\ P_0\overline{\beta} & P_1\overline{\beta} & \cdots & P_{m-1}\overline{\beta} \end{pmatrix},$$

then $\overline{\beta}_1$ is a mapping of the family of sets $C(D) = \{P_0, P_1, P_2, \cdots, P_{m-1}\}$ in the X-semilattice of unions

$$D = \left\{\breve{D}, Z_1, Z_2, \cdots, Z_{m-1}\right\}.$$

e) if $\overline{\beta}_2 : X \setminus \breve{D} \to D$ is a mapping satisfying the condition $\overline{\beta}_2(t') = t'\beta$ for all $t' \in X \setminus \breve{D}$, then

$$\overline{\beta} = \bigcup_{i=0}^{m-1}\left(P_i \times \bigcup_{t\in P_i} t\beta\right) \cup \bigcup_{t'\in X\setminus \breve{D}} \left(\{t'\} \times \overline{\beta}_2(t')\right).$$

2. Results

Proposition 2. *Let* $\alpha, \beta \in B_X(D)$. *Then*

$$\alpha \circ \beta = \alpha \circ \overline{\beta} = \alpha \circ \tilde{\beta}.$$

Proof. It is easy to see the inclusion $\alpha \circ \beta \subseteq \alpha \circ \overline{\beta} \subseteq \alpha \circ \tilde{\beta}$ holds, since $\beta \subseteq \overline{\beta} \subseteq \tilde{\beta}$. If $x(\alpha \circ \tilde{\beta})y$ $(x, y \in X)$, then $x\alpha z\tilde{\beta}y$ for some $z \in X$. So, $z \notin X \setminus \breve{D}$ since $z \in x\alpha \in D$ and $z\left(\bigcup_{i=0}^{m-1}\left(P_i \times \bigcup_{t\in P_i} t\beta\right)\right)y$. Then $z\left(P_k \times \bigcup_{t\in P_k} t\beta\right)y$ for some k $(0 \le k \le m-1)$ *i.e.* $z \in P_k$ and $y \in z\beta \subseteq \bigcup_{t\in P_k} t\beta$. For the last condition follows that $z\beta y$. We have $x\alpha z\beta y$ and $x(\alpha \circ \beta)y$. Therefore, the inclusion $\alpha \circ \tilde{\beta} \subseteq \alpha \circ \beta$ is true. Of this and by inclusion $\alpha \circ \beta \subseteq \alpha \circ \overline{\beta} \subseteq \alpha \circ \tilde{\beta}$ follows that the equality $\alpha \circ \beta = \alpha \circ \overline{\beta} = \alpha \circ \tilde{\beta}$ holds. □

Corollary 1. *If* $\alpha, \delta, \beta \in B_X(D)$ *and* $\beta \subseteq \delta \subseteq \tilde{\beta}$, *then* $\alpha \circ \beta = \alpha \circ \delta = \alpha \circ \tilde{\beta}$.

Proof. We have $\beta \subseteq \delta \subseteq \tilde{\beta}$ and $\alpha \circ \beta \subseteq \alpha \circ \delta \subseteq \alpha \circ \tilde{\beta}$. Of this follows that $\alpha \circ \beta = \alpha \circ \delta = \alpha \circ \tilde{\beta}$ since $\alpha \circ \beta = \alpha \circ \tilde{\beta}$. □

Let the X-semilattice $D = \left\{Z_4, Z_3, Z_2, Z_1, \breve{D}\right\}$ of unions given by the diagram of **Figure 1**. Formal equalities of the given semilattice have a form:

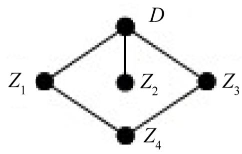

Figure 1. Diagram of D.

$$\begin{aligned} \breve{D} &= P_0 \cup P_1 \cup P_2 \cup P_3 \cup P_4, \\ Z_1 &= P_0 \cup P_2 \cup P_3 \cup P_4, \\ Z_2 &= P_0 \cup P_1 \cup P_3 \cup P_4, \\ Z_3 &= P_0 \cup P_1 \cup P_2 \cup P_4, \\ Z_4 &= P_0 \cup P_2. \end{aligned} \tag{4}$$

The parameters P_1, P_2, P_3 are basis sources and the parameters P_0, P_4 are completeness sources, *i.e.* $|X| \geq 3$.

Example 3. *Let* $X = \{1,2,3,4,5,6,7\}$, $P_1 = \{1,4\}$, $P_2 = \{2,5\}$, $P_3 = \{3,6\}$, $P_0 = P_4 = \varnothing$. *Then for the formal equalities of the semilattice D follows that* $Z_4 = \{2,5\}$, $Z_3 = \{1,2,4,5\}$, $Z_2 = \{1,3,4,6\}$, $Z_1 = \{2,3,5,6\}$, $\breve{D} = \{1,2,3,4,5,6\}$, $D = \{\{2\},\{1,2\},\{1,3\},\{2,3\},\{1,2,3\}\}$, *and*

$\beta = (\{1\} \times \{2,3,5,6\}) \cup (\{2\} \times \{1,3,4,6\}) \cup (\{3\} \times \{1,2,4,5\}) \cup (\{4,5,6,7\} \times \{2,5\})$.

Then we have:

$$\bigcup_{t \in P_1} t\beta = \{2,3,5,6\}, \bigcup_{t \in P_2} t\beta = \{1,2,3,4,5,6\}, \bigcup_{t \in P_3} t\beta = \{1,2,4,5\},$$

$$\bigcup_{t \in P_0} t\beta = \bigcup_{t \in P_4} t\beta = \varnothing;$$

$$\overline{\beta} = (\{1,4\} \times \{2,3,5,6\}) \cup (\{2,5\} \times \{1,2,3,4,5,6\}) \cup (\{3,6\} \times \{1,2,4,5\}) \cup (\{7\} \times \{2,5\});$$

$$\tilde{\beta} = (\{1,4\} \times \{2,3,5,6\}) \cup (\{2,5\} \times \{1,2,3,4,5,6\}) \cup (\{3,6\} \times \{1,2,4,5\}) \cup (\{7\} \times \{1,2,3,4,5,6,7\}).$$

Theorem 4. *Let the X-semilattice* $D = \{Z_4, Z_3, Z_2, Z_1, \breve{D}\}$ *of unions given by the diagram of* **Figure 1**, $B = \{\alpha \in B_X(D) \mid V(X^*, \alpha) = D\}$ *and* $|X \setminus \breve{D}| \geq 3$. *Then the set B is generating set of the semigroup* $B_X(D)$.

Proof. It is easy to see that $|X| \geq 6$ since $P_1, P_2, P_3 \notin \{\varnothing\}$ and $|X \setminus \breve{D}| \geq 3$. Now, let α be any binary relation of the semigroup $B_X(D)$; $\alpha_1, \alpha_2, \cdots, \alpha_n \in B$, $\alpha = \alpha_1 \circ \alpha_2 \circ \cdots \circ \alpha_n$ and $\beta = \alpha_2 \circ \alpha_3 \circ \cdots \circ \alpha_n$. Then the equality $\alpha = \alpha_1 \circ \beta = \alpha_1 \circ \overline{\beta}$ ($\overline{\beta}$ is subquasinormal representation of a binary relation β) is true. By assumption $\alpha_1 \in B_X(D)$, *i.e.* the quasinormal representation of a binary relation α_1 have a form

$$\alpha_1 = (Y_4^{\alpha_1} \times Z_4) \cup (Y_3^{\alpha_1} \times Z_3) \cup (Y_2^{\alpha_1} \times Z_2) \cup (Y_1^{\alpha_1} \times Z_1) \cup (Y_0^{\alpha_1} \times \breve{D}).$$

Of this follows that

$$\begin{aligned} \alpha = \alpha_1 \circ \overline{\beta} &= (Y_4^{\alpha_1} \times Z_4) \cup (Y_3^{\alpha_1} \times Z_3) \cup (Y_2^{\alpha_1} \times Z_2) \cup (Y_1^{\alpha_1} \times Z_1) \cup (Y_0^{\alpha_1} \times \breve{D}) \circ \overline{\beta} \\ &= (Y_4^{\alpha_1} \times Z_4\overline{\beta}) \cup (Y_3^{\alpha_1} \times Z_3\overline{\beta}) \cup (Y_2^{\alpha_1} \times Z_2\overline{\beta}) \cup (Y_1^{\alpha_1} \times Z_1\overline{\beta}) \cup (Y_0^{\alpha_1} \times \breve{D}\overline{\beta}). \end{aligned} \tag{5}$$

For the binary relation α we consider the following case.

a) Let $|V(X^*, \alpha)| = 1$. Then $\alpha = X \times T$, where $T \in D$. By element T we consider the following cases:

1. $T \neq \breve{D}$. In this case suppose that

$$\overline{\beta}_1 = \begin{pmatrix} P_0 & P_1 & P_2 & P_3 & P_4 \\ \varnothing & T & T & T & \varnothing \end{pmatrix}$$

and $\overline{\beta}_2$ are mapping of the set $X \setminus \breve{D}$ on the set $\{Z_4, Z_3, Z_2, Z_1\} \setminus \{T\}$. Then

$$\overline{\beta} = (P_0 \times \varnothing) \cup (P_1 \times T) \cup (P_2 \times T) \cup (P_3 \times T) \cup (P_4 \times \varnothing) \cup \bigcup_{t' \in X \setminus \breve{D}} (\{t'\} \times \overline{\beta}_2(t')),$$
$$\alpha_1 = (Y_4^{\alpha_1} \times Z_4) \cup (Y_3^{\alpha_1} \times Z_3) \cup (Y_2^{\alpha_1} \times Z_2) \cup (Y_1^{\alpha_1} \times Z_1) \cup (Y_0^{\alpha_1} \times \breve{D}), \tag{6}$$

where $Y_4^{\alpha_1}, Y_3^{\alpha_1}, Y_2^{\alpha_1}, Y_1^{\alpha_1} \notin \{\varnothing\}$, then it is easy to see, that $\alpha_1, \overline{\beta} \in B$ since $V(D, \alpha_1) = V(D, \overline{\beta}) = D$. From the formal equality and equalities (6) and (5) we have:

$$Z_4\overline{\beta} = (P_0 \cup P_2)\overline{\beta} = P_0\overline{\beta} \cup P_2\overline{\beta} = \varnothing \cup T = T;$$
$$Z_3\overline{\beta} = (P_0 \cup P_1 \cup P_2 \cup P_4)\overline{\beta} = P_0\overline{\beta} \cup P_1\overline{\beta} \cup P_2\overline{\beta} \cup P_4\overline{\beta} = \varnothing \cup T \cup T \cup \varnothing = T;$$
$$Z_2\overline{\beta} = (P_0 \cup P_1 \cup P_3 \cup P_4)\overline{\beta} = P_0\overline{\beta} \cup P_1\overline{\beta} \cup P_3\overline{\beta} \cup P_4\overline{\beta} = \varnothing \cup T \cup T \cup \varnothing = T;$$
$$Z_1\overline{\beta} = (P_0 \cup P_2 \cup P_3 \cup P_4)\overline{\beta} = P_0\overline{\beta} \cup P_2\overline{\beta} \cup P_3\overline{\beta} \cup P_4\overline{\beta} = \varnothing \cup T \cup T \cup \varnothing = T;$$
$$\breve{D}\overline{\beta} = (P_0 \cup P_1 \cup P_2 \cup P_3 \cup P_4)\overline{\beta} = P_0\overline{\beta} \cup P_1\overline{\beta} \cup P_2\overline{\beta} \cup P_3\overline{\beta} \cup P_4\overline{\beta}$$
$$= \varnothing \cup T \cup T \cup T \cup \varnothing = T;$$
$$\alpha = \alpha_1 \circ \overline{\beta}$$
$$= (Y_4^{\alpha_1} \times Z_4\overline{\beta}) \cup (Y_3^{\alpha_1} \times Z_3\overline{\beta}) \cup (Y_2^{\alpha_1} \times Z_2\overline{\beta}) \cup (Y_1^{\alpha_1} \times Z_1\overline{\beta}) \cup (Y_0^{\alpha_1} \times \breve{D}\overline{\beta})$$
$$= (Y_4^{\alpha_1} \times T) \cup (Y_3^{\alpha_1} \times T) \cup (Y_2^{\alpha_1} \times T) \cup (Y_1^{\alpha_1} \times T) \cup (Y_0^{\alpha_1} \times T) = X \times T$$

since $Y_4^{\alpha_1} \cup Y_3^{\alpha_1} \cup Y_2^{\alpha_1} \cup Y_1^{\alpha_1} \cup Y_0^{\alpha_1} = X$.

2. $T = \breve{D}$. In this case suppose that

$$\overline{\beta}_1 = \begin{pmatrix} P_0 & P_1 & P_2 & P_3 & P_4 \\ \breve{D} & Z_1 & Z_2 & Z_3 & Z_4 \end{pmatrix}$$

and $\overline{\beta}_2$ are mapping of the set $X \setminus \breve{D}$ in the set *D*. Then

$$\overline{\beta} = (P_0 \times \breve{D}) \cup (P_1 \times Z_1) \cup (P_2 \times Z_2) \cup (P_3 \times Z_3) \cup (P_4 \times Z_4) \cup \bigcup_{t' \in X \setminus \breve{D}} (\{t'\} \times \overline{\beta}_2(t')),$$
$$\alpha_1 = (Y_4^{\alpha_1} \times Z_4) \cup (Y_3^{\alpha_1} \times Z_3) \cup (Y_2^{\alpha_1} \times Z_2) \cup (Y_1^{\alpha_1} \times Z_1) \cup (Y_0^{\alpha_1} \times \breve{D}), \tag{7}$$

where $Y_4^{\alpha_1}, Y_3^{\alpha_1}, Y_2^{\alpha_1}, Y_1^{\alpha_1} \notin \{\varnothing\}$, then it is easy to see, that $\alpha_1, \overline{\beta} \in B$ since $V(D, \alpha_1) = V(D, \overline{\beta}) = D$. From the formal equality and equalities (7) and (5) we have:

$$Z_4\overline{\beta} = (P_0 \cup P_2)\overline{\beta} = P_0\overline{\beta} \cup P_2\overline{\beta} = \breve{D};$$
$$Z_3\overline{\beta} = (P_0 \cup P_1 \cup P_2 \cup P_4)\overline{\beta} = P_0\overline{\beta} \cup P_1\overline{\beta} \cup P_2\overline{\beta} \cup P_4\overline{\beta} = \breve{D};$$
$$Z_2\overline{\beta} = (P_0 \cup P_1 \cup P_3 \cup P_4)\overline{\beta} = P_0\overline{\beta} \cup P_1\overline{\beta} \cup P_3\overline{\beta} \cup P_4\overline{\beta} = \breve{D};$$
$$Z_1\overline{\beta} = (P_0 \cup P_2 \cup P_3 \cup P_4)\overline{\beta} = P_0\overline{\beta} \cup P_2\overline{\beta} \cup P_3\overline{\beta} \cup P_4\overline{\beta} = \breve{D};$$
$$\breve{D}\overline{\beta} = (P_0 \cup P_1 \cup P_2 \cup P_3 \cup P_4)\overline{\beta} = P_0\overline{\beta} \cup P_1\overline{\beta} \cup P_2\overline{\beta} \cup P_3\overline{\beta} \cup P_4\overline{\beta} = \breve{D};$$
$$\alpha = \alpha_1 \circ \overline{\beta}$$
$$= (Y_4^{\alpha_1} \times Z_4\overline{\beta}) \cup (Y_3^{\alpha_1} \times Z_3\overline{\beta}) \cup (Y_2^{\alpha_1} \times Z_2\overline{\beta}) \cup (Y_1^{\alpha_1} \times Z_1\overline{\beta}) \cup (Y_0^{\alpha_1} \times \breve{D}\overline{\beta})$$
$$= (Y_4^{\alpha_1} \times \breve{D}) \cup (Y_3^{\alpha_1} \times \breve{D}) \cup (Y_2^{\alpha_1} \times \breve{D}) \cup (Y_1^{\alpha_1} \times \breve{D}) \cup (Y_0^{\alpha_1} \times \breve{D})$$
$$= X \times \breve{D}.$$

b) $|V(X^*, \alpha)| = 2$. Then

$$V(X^*, \alpha) \in \{\{Z_4, Z_1\}, \{Z_4, Z_3\}, \{Z_4, \breve{D}\}, \{Z_3, \breve{D}\}, \{Z_2, \breve{D}\}, \{Z_1, \breve{D}\}\}$$

since $V(X^*, \alpha)$ is *X*-semilattice of unions. For the semilattice of unions $V(X^*, \alpha)$ consider the following cases.

1. Let $V(X^*, \alpha) = \{Z_4, T\}$, where, $T \in \{Z_1, Z_3\}$. Then binary relation α has representation of the form

$\alpha = \left(Y_4^\alpha \times Z_4\right) \cup \left(Y_T^\alpha \times T\right)$. In this case suppose that

$$\overline{\beta}_1 = \begin{pmatrix} P_0 & P_1 & P_2 & P_3 & P_4 \\ \varnothing & T & Z_4 & T & \varnothing \end{pmatrix}$$

and $\overline{\beta}_2$ are mapping of the set $X \setminus \breve{D}$ on the set $D \setminus \{Z_4, T\}$. Then

$$\begin{aligned} &\overline{\beta} = (P_0 \times \varnothing) \cup (P_1 \times T) \cup (P_2 \times Z_4) \cup (P_3 \times T) \cup (P_4 \times \varnothing) \cup \bigcup_{t' \in X \setminus \breve{D}} \left(\{t'\} \times \overline{\beta}_2(t')\right), \\ &\alpha_1 = \left(Y_4^{\alpha_1} \times Z_4\right) \cup \left(Y_3^{\alpha_1} \times Z_3\right) \cup \left(Y_2^{\alpha_1} \times Z_2\right) \cup \left(Y_1^{\alpha_1} \times Z_1\right) \cup \left(Y_0^{\alpha_1} \times \breve{D}\right), \end{aligned} \tag{8}$$

where $Y_4^{\alpha_1}, Y_3^{\alpha_1}, Y_2^{\alpha_1}, Y_1^{\alpha_1} \notin \{\varnothing\}$, $Y_4^{\alpha_1} = Y_4^\alpha$ and $Y_3^{\alpha_1} \cup Y_2^{\alpha_1} \cup Y_1^{\alpha_1} \cup Y_0^{\alpha_1} = Y_0^\alpha$, then it is easy to see, that $\alpha_1, \overline{\beta} \in B$ since $V(D, \alpha_1) = V\left(D, \overline{\beta}\right) = D$. From the formal equality and equalities (8) and (5) we have:

$$\begin{aligned}
Z_4\overline{\beta} &= (P_0 \cup P_2)\overline{\beta} = P_0\overline{\beta} \cup P_2\overline{\beta} = \varnothing \cup Z_4 = Z_4; \\
Z_3\overline{\beta} &= (P_0 \cup P_1 \cup P_2 \cup P_4)\overline{\beta} = P_0\overline{\beta} \cup P_1\overline{\beta} \cup P_2\overline{\beta} \cup P_4\overline{\beta} \\
&= \varnothing \cup T \cup Z_4 \cup \varnothing = T; \\
Z_2\overline{\beta} &= (P_0 \cup P_1 \cup P_3 \cup P_4)\overline{\beta} = P_0\overline{\beta} \cup P_1\overline{\beta} \cup P_3\overline{\beta} \cup P_4\overline{\beta} \\
&= \varnothing \cup T \cup T \cup \varnothing = T; \\
Z_1\overline{\beta} &= (P_0 \cup P_2 \cup P_3 \cup P_4)\overline{\beta} = P_0\overline{\beta} \cup P_2\overline{\beta} \cup P_3\overline{\beta} \cup P_4\overline{\beta} \\
&= \varnothing \cup Z_4 \cup T \cup \varnothing = T; \\
\breve{D}\overline{\beta} &= (P_0 \cup P_1 \cup P_2 \cup P_3 \cup P_4)\overline{\beta} = P_0\overline{\beta} \cup P_1\overline{\beta} \cup P_2\overline{\beta} \cup P_3\overline{\beta} \cup P_4\overline{\beta} \\
&= \varnothing \cup T \cup Z_4 \cup T \cup \varnothing = T; \\
\alpha &= \alpha_1 \circ \overline{\beta} \\
&= \left(Y_4^{\alpha_1} \times Z_4\overline{\beta}\right) \cup \left(Y_3^{\alpha_1} \times Z_3\overline{\beta}\right) \cup \left(Y_2^{\alpha_1} \times Z_2\overline{\beta}\right) \cup \left(Y_1^{\alpha_1} \times Z_1\overline{\beta}\right) \cup \left(Y_0^{\alpha_1} \times \breve{D}\overline{\beta}\right) \\
&= \left(Y_4^{\alpha_1} \times Z_4\right) \cup \left(Y_3^{\alpha_1} \times T\right) \cup \left(Y_2^{\alpha_1} \times T\right) \cup \left(Y_1^{\alpha_1} \times T\right) \cup \left(Y_0^{\alpha_1} \times T\right) \\
&= \left(Y_4^{\alpha_1} \times Z_4\right) \cup \left(\left(Y_3^{\alpha_1} \cup Y_2^{\alpha_1} \cup Y_1^{\alpha_1} \cup Y_0^{\alpha_1}\right) \times T\right) = \left(Y_4^\alpha \times Z_4\right) \cup \left(Y_0^\alpha \times T\right).
\end{aligned}$$

2. Let $V\left(X^*, \alpha\right) = \left\{T, \breve{D}\right\}$, where, $T \in \{Z_4, Z_3, Z_2, Z_1\}$. Then binary relation α has representation of the form $\alpha = \left(Y_T^\alpha \times T\right) \cup \left(Y_0^\alpha \times \breve{D}\right)$. In this case suppose that

$$\overline{\beta}_1 = \begin{pmatrix} P_0 & P_1 & P_2 & P_3 & P_4 \\ \varnothing & \breve{D} & T & \breve{D} & \varnothing \end{pmatrix}$$

and $\overline{\beta}_2$ are mapping of the set $X \setminus \breve{D}$ on the set $D \setminus \left\{T, \breve{D}\right\}$. Then

$$\begin{aligned} &\overline{\beta} = (P_0 \times \varnothing) \cup \left(P_1 \times \breve{D}\right) \cup (P_2 \times T) \cup \left(P_3 \times \breve{D}\right) \cup (P_4 \times \varnothing) \cup \bigcup_{t' \in X \setminus \breve{D}} \left(\{t'\} \times f(t')\right), \\ &\alpha_1 = \left(Y_4^{\alpha_1} \times Z_4\right) \cup \left(Y_3^{\alpha_1} \times Z_3\right) \cup \left(Y_2^{\alpha_1} \times Z_2\right) \cup \left(Y_1^{\alpha_1} \times Z_1\right) \cup \left(Y_0^{\alpha_1} \times \breve{D}\right), \end{aligned} \tag{9}$$

where $Y_4^{\alpha_1}, Y_3^{\alpha_1}, Y_2^{\alpha_1}, Y_1^{\alpha_1} \notin \{\varnothing\}$, $Y_4^{\alpha_1} = Y_T^\alpha$ and $Y_3^{\alpha_1} \cup Y_2^{\alpha_1} \cup Y_1^{\alpha_1} \cup Y_0^{\alpha_1} = Y_0^\alpha$, then it is easy to see, that $\alpha_1, \overline{\beta} \in B$ since $V(D, \alpha_1) = V\left(D, \overline{\beta}\right) = D$. From the formal equality and equalities (9) and (5) we have:

$$Z_4\overline{\beta} = (P_0 \cup P_2)\overline{\beta} = P_0\overline{\beta} \cup P_2\overline{\beta} = \varnothing \cup T = T;$$
$$Z_3\overline{\beta} = (P_0 \cup P_1 \cup P_2 \cup P_4)\overline{\beta} = P_0\overline{\beta} \cup P_1\overline{\beta} \cup P_2\overline{\beta} \cup P_4\overline{\beta} = \varnothing \cup \breve{D} \cup T \cup \varnothing = \breve{D};$$
$$Z_2\overline{\beta} = (P_0 \cup P_1 \cup P_3 \cup P_4)\overline{\beta} = P_0\overline{\beta} \cup P_1\overline{\beta} \cup P_3\overline{\beta} \cup P_4\overline{\beta} = \varnothing \cup \breve{D} \cup \breve{D} \cup \varnothing = \breve{D};$$
$$Z_1\overline{\beta} = (P_0 \cup P_2 \cup P_3 \cup P_4)\overline{\beta} = P_0\overline{\beta} \cup P_2\overline{\beta} \cup P_3\overline{\beta} \cup P_4\overline{\beta} = \varnothing \cup T \cup \breve{D} \cup \varnothing = \breve{D};$$
$$\breve{D}\overline{\beta} = (P_0 \cup P_1 \cup P_2 \cup P_3 \cup P_4)\overline{\beta} = P_0\overline{\beta} \cup P_1\overline{\beta} \cup P_2\overline{\beta} \cup P_3\overline{\beta} \cup P_4\overline{\beta}$$
$$= \varnothing \cup \breve{D} \cup T \cup \breve{D} \cup \varnothing = \breve{D};$$
$$\alpha = \alpha_1 \circ \overline{\beta}$$
$$= \left(Y_4^{\alpha_1} \times Z_4\overline{\beta}\right) \cup \left(Y_3^{\alpha_1} \times Z_3\overline{\beta}\right) \cup \left(Y_2^{\alpha_1} \times Z_2\overline{\beta}\right) \cup \left(Y_1^{\alpha_1} \times Z_1\overline{\beta}\right) \cup \left(Y_0^{\alpha_1} \times \breve{D}\overline{\beta}\right)$$
$$= \left(Y_4^{\alpha_1} \times T\right) \cup \left(Y_3^{\alpha_1} \times \breve{D}\right) \cup \left(Y_2^{\alpha_1} \times \breve{D}\right) \cup \left(Y_1^{\alpha_1} \times \breve{D}\right) \cup \left(Y_0^{\alpha_1} \times \breve{D}\right)$$
$$= \left(Y_4^{\alpha_1} \times T\right) \cup \left(\left(Y_3^{\alpha_1} \cup Y_2^{\alpha_1} \cup Y_1^{\alpha_1} \cup Y_0^{\alpha_1}\right) \times \breve{D}\right) = \left(Y_T^{\alpha} \times T\right) \cup \left(Y_0^{\alpha} \times \breve{D}\right).$$

c) $\left|V\left(X^*,\alpha\right)\right| = 3$. Then

$$V\left(X^*,\alpha\right) \in \left\{\left\{Z_4,Z_1,\breve{D}\right\},\left\{Z_4,Z_3,\breve{D}\right\},\left\{Z_4,Z_2,\breve{D}\right\},\left\{Z_1,Z_2,\breve{D}\right\},\left\{Z_1,Z_3,\breve{D}\right\},\left\{Z_2,Z_3,\breve{D}\right\}\right\}$$

since $V\left(X^*,\alpha\right)$ is *X*-semilattice of unions. For the semilattice of unions $V\left(X^*,\alpha\right)$ consider the following cases.

1. Let $V\left(X^*,\alpha\right) = \left\{Z_4,Z_1,\breve{D}\right\}$. Then binary relation α has representation of the form $\alpha = \left(Y_4^{\alpha} \times Z_4\right) \cup \left(Y_1^{\alpha} \times Z_1\right) \cup \left(Y_0^{\alpha} \times \breve{D}\right)$. In this case suppose that

$$\overline{\beta}_1 = \begin{pmatrix} P_0 & P_1 & P_2 & P_3 & P_4 \\ \varnothing & Z_2 & Z_4 & Z_1 & \varnothing \end{pmatrix}$$

and $\overline{\beta}_2$ are mapping of the set $X \setminus \breve{D}$ on the set $D \setminus \left\{Z_4,Z_2,Z_1\right\}$. Then

$$\overline{\beta} = \left(P_0 \times \varnothing\right) \cup \left(P_1 \times Z_2\right) \cup \left(P_2 \times Z_4\right) \cup \left(P_3 \times Z_1\right) \cup \left(P_4 \times \varnothing\right) \cup \bigcup_{t' \in X \setminus \breve{D}} \left(\left\{t'\right\} \times f\left(t'\right)\right),$$
$$\alpha_1 = \left(Y_4^{\alpha_1} \times Z_4\right) \cup \left(Y_3^{\alpha_1} \times Z_3\right) \cup \left(Y_2^{\alpha_1} \times Z_2\right) \cup \left(Y_1^{\alpha_1} \times Z_1\right) \cup \left(Y_0^{\alpha_1} \times \breve{D}\right), \tag{10}$$

where $Y_4^{\alpha_1}, Y_3^{\alpha_1}, Y_2^{\alpha_1}, Y_1^{\alpha_1} \notin \{\varnothing\}$, $Y_4^{\alpha_1} = Y_4^{\alpha}$, $Y_1^{\alpha_1} = Y_1^{\alpha}$ and $Y_3^{\alpha_1} \cup Y_2^{\alpha_1} \cup Y_0^{\alpha_1} = Y_0^{\alpha}$, then it is easy to see, that $\alpha_1, \overline{\beta} \in B$ since $V\left(D,\alpha_1\right) = V\left(D,\overline{\beta}\right) = D$. From the formal equality and equalities (10) and (5) we have:

$$Z_4\overline{\beta} = (P_0 \cup P_2)\overline{\beta} = P_0\overline{\beta} \cup P_2\overline{\beta} = \varnothing \cup Z_4 = Z_4;$$
$$Z_3\overline{\beta} = (P_0 \cup P_1 \cup P_2 \cup P_4)\overline{\beta} = P_0\overline{\beta} \cup P_1\overline{\beta} \cup P_2\overline{\beta} \cup P_4\overline{\beta} = \varnothing \cup Z_2 \cup Z_4 \cup \varnothing = \breve{D};$$
$$Z_2\overline{\beta} = (P_0 \cup P_1 \cup P_3 \cup P_4)\overline{\beta} = P_0\overline{\beta} \cup P_1\overline{\beta} \cup P_3\overline{\beta} \cup P_4\overline{\beta} = \varnothing \cup Z_2 \cup Z_1 \cup \varnothing = \breve{D};$$
$$Z_1\overline{\beta} = (P_0 \cup P_2 \cup P_3 \cup P_4)\overline{\beta} = P_0\overline{\beta} \cup P_2\overline{\beta} \cup P_3\overline{\beta} \cup P_4\overline{\beta} = \varnothing \cup Z_4 \cup Z_1 \cup \varnothing = Z_1;$$
$$\breve{D}\overline{\beta} = (P_0 \cup P_1 \cup P_2 \cup P_3 \cup P_4)\overline{\beta} = P_0\overline{\beta} \cup P_1\overline{\beta} \cup P_2\overline{\beta} \cup P_3\overline{\beta} \cup P_4\overline{\beta}$$
$$= \varnothing \cup Z_2 \cup Z_4 \cup Z_1 \cup \varnothing = \breve{D};$$
$$\alpha = \alpha_1 \circ \overline{\beta}$$
$$= \left(Y_4^{\alpha_1} \times Z_4\overline{\beta}\right) \cup \left(Y_3^{\alpha_1} \times Z_3\overline{\beta}\right) \cup \left(Y_2^{\alpha_1} \times Z_2\overline{\beta}\right) \cup \left(Y_1^{\alpha_1} \times Z_1\overline{\beta}\right) \cup \left(Y_0^{\alpha_1} \times \breve{D}\overline{\beta}\right)$$
$$= \left(Y_4^{\alpha_1} \times Z_4\right) \cup \left(Y_3^{\alpha_1} \times \breve{D}\right) \cup \left(Y_2^{\alpha_1} \times \breve{D}\right) \cup \left(Y_1^{\alpha_1} \times Z_1\right) \cup \left(Y_0^{\alpha_1} \times \breve{D}\right)$$
$$= \left(Y_4^{\alpha_1} \times Z_4\right) \cup \left(Y_1^{\alpha_1} \times Z_1\right) \cup \left(\left(Y_3^{\alpha_1} \cup Y_2^{\alpha_1} \cup Y_0^{\alpha_1}\right) \times \breve{D}\right)$$
$$= \left(Y_4^{\alpha} \times Z_4\right) \cup \left(Y_1^{\alpha} \times Z_1\right) \cup \left(Y_0^{\alpha} \times \breve{D}\right).$$

2. Let $V\left(X^*,\alpha\right) = \left\{Z_4,Z_3,\breve{D}\right\}$. Then binary relation α has representation of the form

$\alpha = \left(Y_4^\alpha \times Z_4\right) \cup \left(Y_3^\alpha \times Z_3\right) \cup \left(Y_0^\alpha \times \breve{D}\right)$. In this case suppose that

$$\overline{\beta}_1 = \begin{pmatrix} P_0 & P_1 & P_2 & P_3 & P_4 \\ \varnothing & Z_3 & Z_4 & Z_2 & \varnothing \end{pmatrix}$$

and $\overline{\beta}_2$ are mapping of the set $X \setminus \breve{D}$ on the set $D \setminus \{Z_4, Z_3, Z_2\}$. Then

$$\begin{aligned} \overline{\beta} &= (P_0 \times \varnothing) \cup (P_1 \times Z_3) \cup (P_2 \times Z_4) \cup (P_3 \times Z_2) \cup (P_4 \times \varnothing) \cup \bigcup_{t' \in X \setminus \breve{D}} \left(\{t'\} \times f(t')\right), \\ \alpha_1 &= \left(Y_4^{\alpha_1} \times Z_4\right) \cup \left(Y_3^{\alpha_1} \times Z_3\right) \cup \left(Y_2^{\alpha_1} \times Z_2\right) \cup \left(Y_1^{\alpha_1} \times Z_1\right) \cup \left(Y_0^{\alpha_1} \times \breve{D}\right), \end{aligned} \tag{11}$$

where $Y_4^{\alpha_1}, Y_3^{\alpha_1}, Y_2^{\alpha_1}, Y_1^{\alpha_1} \notin \{\varnothing\}$, $Y_4^{\alpha_1} = Y_4^\alpha$, $Y_3^{\alpha_1} = Y_3^\alpha$ and $Y_2^{\alpha_1} \cup Y_1^{\alpha_1} \cup Y_0^{\alpha_1} = Y_0^\alpha$, then it is easy to see, that $\alpha_1, \overline{\beta} \in B$ since $V(D, \alpha_1) = V(D, \overline{\beta}) = D$. From the formal equality and equalities (11) and (5) we have:

$$\begin{aligned} Z_4\overline{\beta} &= (P_0 \cup P_2)\overline{\beta} = P_0\overline{\beta} \cup P_2\overline{\beta} = \varnothing \cup Z_4 = Z_4; \\ Z_3\overline{\beta} &= (P_0 \cup P_1 \cup P_2 \cup P_4)\overline{\beta} = P_0\overline{\beta} \cup P_1\overline{\beta} \cup P_2\overline{\beta} \cup P_4\overline{\beta} \\ &= \varnothing \cup Z_3 \cup Z_4 \cup \varnothing = Z_3; \\ Z_2\overline{\beta} &= (P_0 \cup P_1 \cup P_3 \cup P_4)\overline{\beta} = P_0\overline{\beta} \cup P_1\overline{\beta} \cup P_3\overline{\beta} \cup P_4\overline{\beta} \\ &= \varnothing \cup Z_3 \cup Z_2 \cup \varnothing = \breve{D}; \\ Z_1\overline{\beta} &= (P_0 \cup P_2 \cup P_3 \cup P_4)\overline{\beta} = P_0\overline{\beta} \cup P_2\overline{\beta} \cup P_3\overline{\beta} \cup P_4\overline{\beta} \\ &= \varnothing \cup Z_4 \cup Z_2 \cup \varnothing = \breve{D}; \\ \breve{D}\overline{\beta} &= (P_0 \cup P_1 \cup P_2 \cup P_3 \cup P_4)\overline{\beta} = P_0\overline{\beta} \cup P_1\overline{\beta} \cup P_2\overline{\beta} \cup P_3\overline{\beta} \cup P_4\overline{\beta} \\ &= \varnothing \cup Z_3 \cup Z_4 \cup Z_2 \cup \varnothing = \breve{D}; \\ \alpha &= \alpha_1 \circ \overline{\beta} \\ &= \left(Y_4^{\alpha_1} \times Z_4\overline{\beta}\right) \cup \left(Y_3^{\alpha_1} \times Z_3\overline{\beta}\right) \cup \left(Y_2^{\alpha_1} \times Z_2\overline{\beta}\right) \cup \left(Y_1^{\alpha_1} \times Z_1\overline{\beta}\right) \cup \left(Y_0^{\alpha_1} \times \breve{D}\overline{\beta}\right) \\ &= \left(Y_4^{\alpha_1} \times Z_4\right) \cup \left(Y_3^{\alpha_1} \times Z_3\right) \cup \left(Y_2^{\alpha_1} \times \breve{D}\right) \cup \left(Y_1^{\alpha_1} \times \breve{D}\right) \cup \left(Y_0^{\alpha_1} \times \breve{D}\right) \\ &= \left(Y_4^{\alpha_1} \times Z_4\right) \cup \left(Y_3^{\alpha_1} \times Z_3\right) \cup \left(\left(Y_2^{\alpha_1} \cup Y_1^{\alpha_1} \cup Y_0^{\alpha_1}\right) \times \breve{D}\right) \\ &= \left(Y_4^\alpha \times Z_4\right) \cup \left(Y_3^\alpha \times Z_3\right) \cup \left(Y_0^\alpha \times \breve{D}\right). \end{aligned}$$

3. Let $V(X^*, \alpha) = \{Z_4, Z_2, \breve{D}\}$. Then binary relation α has representation of the form $\alpha = \left(Y_4^\alpha \times Z_4\right) \cup \left(Y_2^\alpha \times Z_2\right) \cup \left(Y_0^\alpha \times \breve{D}\right)$. In this case suppose that

$$\overline{\beta}_1 = \begin{pmatrix} P_0 & P_1 & P_2 & P_3 & P_4 \\ \varnothing & Z_2 & Z_4 & Z_2 & \varnothing \end{pmatrix}$$

and $\overline{\beta}_2$ are mapping of the set $X \setminus \breve{D}$ on the set $D \setminus \{Z_4, Z_2\}$. Then

$$\begin{aligned} \overline{\beta} &= (P_0 \times \varnothing) \cup (P_1 \times Z_2) \cup (P_2 \times Z_4) \cup (P_3 \times Z_2) \cup (P_4 \times \varnothing) \cup \bigcup_{t' \in X \setminus \breve{D}} \left(\{t'\} \times f(t')\right), \\ \alpha_1 &= \left(Y_4^{\alpha_1} \times Z_4\right) \cup \left(Y_3^{\alpha_1} \times Z_3\right) \cup \left(Y_2^{\alpha_1} \times Z_2\right) \cup \left(Y_1^{\alpha_1} \times Z_1\right) \cup \left(Y_0^{\alpha_1} \times \breve{D}\right), \end{aligned} \tag{12}$$

where $Y_4^{\alpha_1}, Y_3^{\alpha_1}, Y_2^{\alpha_1}, Y_1^{\alpha_1} \notin \{\varnothing\}$, $Y_4^{\alpha_1} = Y_4^\alpha$, $Y_2^{\alpha_1} = Y_2^\alpha$ and $Y_3^{\alpha_1} \cup Y_1^{\alpha_1} \cup Y_0^{\alpha_1} = Y_0^\alpha$, then it is easy to see, that $\alpha_1, \overline{\beta} \in B$ since $V(D, \alpha_1) = V(D, \overline{\beta}) = D$. From the formal equality and equalities (12) and (5) we have:

$$Z_4\overline{\beta} = (P_0 \cup P_2)\overline{\beta} = P_0\overline{\beta} \cup P_2\overline{\beta} = \varnothing \cup Z_4 = Z_4;$$
$$Z_3\overline{\beta} = (P_0 \cup P_1 \cup P_2 \cup P_4)\overline{\beta} = P_0\overline{\beta} \cup P_1\overline{\beta} \cup P_2\overline{\beta} \cup P_4\overline{\beta} = \varnothing \cup Z_2 \cup Z_4 \cup \varnothing = \breve{D};$$
$$Z_2\overline{\beta} = (P_0 \cup P_1 \cup P_3 \cup P_4)\overline{\beta} = P_0\overline{\beta} \cup P_1\overline{\beta} \cup P_3\overline{\beta} \cup P_4\overline{\beta} = \varnothing \cup Z_2 \cup Z_2 \cup \varnothing = Z_2;$$
$$Z_1\overline{\beta} = (P_0 \cup P_2 \cup P_3 \cup P_4)\overline{\beta} = P_0\overline{\beta} \cup P_2\overline{\beta} \cup P_3\overline{\beta} \cup P_4\overline{\beta} = \varnothing \cup Z_4 \cup Z_2 \cup \varnothing = \breve{D};$$
$$\breve{D}\overline{\beta} = (P_0 \cup P_1 \cup P_2 \cup P_3 \cup P_4)\overline{\beta} = P_0\overline{\beta} \cup P_1\overline{\beta} \cup P_2\overline{\beta} \cup P_3\overline{\beta} \cup P_4\overline{\beta}$$
$$= \varnothing \cup Z_2 \cup Z_4 \cup Z_2 \cup \varnothing = \breve{D};$$
$$\alpha = \alpha_1 \circ \overline{\beta}$$
$$= \left(Y_4^{\alpha_1} \times Z_4\overline{\beta}\right) \cup \left(Y_3^{\alpha_1} \times Z_3\overline{\beta}\right) \cup \left(Y_2^{\alpha_1} \times Z_2\overline{\beta}\right) \cup \left(Y_1^{\alpha_1} \times Z_1\overline{\beta}\right) \cup \left(Y_0^{\alpha_1} \times \breve{D}\overline{\beta}\right)$$
$$= \left(Y_4^{\alpha_1} \times Z_4\right) \cup \left(Y_3^{\alpha_1} \times \breve{D}\right) \cup \left(Y_2^{\alpha_1} \times Z_2\right) \cup \left(Y_1^{\alpha_1} \times \breve{D}\right) \cup \left(Y_0^{\alpha_1} \times \breve{D}\right)$$
$$= \left(Y_4^{\alpha_1} \times Z_4\right) \cup \left(Y_2^{\alpha_1} \times Z_2\right) \cup \left(\left(Y_3^{\alpha_1} \cup Y_1^{\alpha_1} \cup Y_0^{\alpha_1}\right) \times \breve{D}\right)$$
$$= \left(Y_4^{\alpha} \times Z_4\right) \cup \left(Y_2^{\alpha} \times Z_2\right) \cup \left(Y_0^{\alpha} \times \breve{D}\right).$$

4. Let $V(X^*, \alpha) = \{Z_1, Z_2, \breve{D}\}$. Then binary relation α has representation of the form $\alpha = \left(Y_1^{\alpha} \times Z_1\right) \cup \left(Y_2^{\alpha} \times Z_2\right) \cup \left(Y_0^{\alpha} \times \breve{D}\right)$. In this case suppose that

$$\overline{\beta}_1 = \begin{pmatrix} P_0 & P_1 & P_2 & P_3 & P_4 \\ \varnothing & Z_1 & Z_2 & Z_1 & \varnothing \end{pmatrix}$$

and $\overline{\beta}_2$ are mapping of the set $X \setminus \breve{D}$ on the set $D \setminus \{Z_2, Z_1\}$. Then

$$\overline{\beta} = (P_0 \times \varnothing) \cup (P_1 \times Z_1) \cup (P_2 \times Z_2) \cup (P_3 \times Z_1) \cup (P_4 \times \varnothing) \cup \bigcup_{t' \in X \setminus \breve{D}} \left(\{t'\} \times f(t')\right),$$
$$\alpha_1 = \left(Y_4^{\alpha_1} \times Z_4\right) \cup \left(Y_3^{\alpha_1} \times Z_3\right) \cup \left(Y_2^{\alpha_1} \times Z_2\right) \cup \left(Y_1^{\alpha_1} \times Z_1\right) \cup \left(Y_0^{\alpha_1} \times \breve{D}\right), \tag{13}$$

where $Y_4^{\alpha_1}, Y_3^{\alpha_1}, Y_2^{\alpha_1}, Y_1^{\alpha_1} \notin \{\varnothing\}$, $Y_2^{\alpha_1} = Y_1^{\alpha}$, $Y_4^{\alpha_1} = Y_2^{\alpha}$ and $Y_3^{\alpha_1} \cup Y_1^{\alpha_1} \cup Y_0^{\alpha_1} = Y_0^{\alpha}$, then it is easy to see, that $\alpha_1, \overline{\beta} \in B$ since $V(D, \alpha_1) = V(D, \overline{\beta}) = D$. From the formal equality and equalities (13) and (5) we have:

$$Z_4\overline{\beta} = (P_0 \cup P_2)\overline{\beta} = P_0\overline{\beta} \cup P_2\overline{\beta} = \varnothing \cup Z_2 = Z_2;$$
$$Z_3\overline{\beta} = (P_0 \cup P_1 \cup P_2 \cup P_4)\overline{\beta} = P_0\overline{\beta} \cup P_1\overline{\beta} \cup P_2\overline{\beta} \cup P_4\overline{\beta} = \varnothing \cup Z_1 \cup Z_2 \cup \varnothing = \breve{D};$$
$$Z_2\overline{\beta} = (P_0 \cup P_1 \cup P_3 \cup P_4)\overline{\beta} = P_0\overline{\beta} \cup P_1\overline{\beta} \cup P_3\overline{\beta} \cup P_4\overline{\beta} = \varnothing \cup Z_1 \cup Z_1 \cup \varnothing = Z_1;$$
$$Z_1\overline{\beta} = (P_0 \cup P_2 \cup P_3 \cup P_4)\overline{\beta} = P_0\overline{\beta} \cup P_2\overline{\beta} \cup P_3\overline{\beta} \cup P_4\overline{\beta} = \varnothing \cup Z_2 \cup Z_1 \cup \varnothing = \breve{D};$$
$$\breve{D}\overline{\beta} = (P_0 \cup P_1 \cup P_2 \cup P_3 \cup P_4)\overline{\beta} = P_0\overline{\beta} \cup P_1\overline{\beta} \cup P_2\overline{\beta} \cup P_3\overline{\beta} \cup P_4\overline{\beta}$$
$$= \varnothing \cup Z_1 \cup Z_2 \cup Z_1 \cup \varnothing = \breve{D};$$
$$\alpha = \alpha_1 \circ \overline{\beta}$$
$$= \left(Y_4^{\alpha_1} \times Z_4\overline{\beta}\right) \cup \left(Y_3^{\alpha_1} \times Z_3\overline{\beta}\right) \cup \left(Y_2^{\alpha_1} \times Z_2\overline{\beta}\right) \cup \left(Y_1^{\alpha_1} \times Z_1\overline{\beta}\right) \cup \left(Y_0^{\alpha_1} \times \breve{D}\overline{\beta}\right)$$
$$= \left(Y_4^{\alpha_1} \times Z_2\right) \cup \left(Y_3^{\alpha_1} \times \breve{D}\right) \cup \left(Y_2^{\alpha_1} \times Z_1\right) \cup \left(Y_1^{\alpha_1} \times \breve{D}\right) \cup \left(Y_0^{\alpha_1} \times \breve{D}\right)$$
$$= \left(Y_2^{\alpha_1} \times Z_1\right) \cup \left(Y_4^{\alpha_1} \times Z_2\right) \cup \left(\left(Y_3^{\alpha_1} \cup Y_1^{\alpha_1} \cup Y_0^{\alpha_1}\right) \times \breve{D}\right)$$
$$= \left(Y_1^{\alpha} \times Z_1\right) \cup \left(Y_2^{\alpha} \times Z_2\right) \cup \left(Y_0^{\alpha} \times \breve{D}\right).$$

5. Let $V(X^*, \alpha) = \{Z_1, Z_3, \breve{D}\}$. Then binary relation α has representation of the form $\alpha = \left(Y_1^{\alpha} \times Z_1\right) \cup \left(Y_3^{\alpha} \times Z_3\right) \cup \left(Y_0^{\alpha} \times \breve{D}\right)$. In this case suppose that

$$\overline{\beta}_1 = \begin{pmatrix} P_0 & P_1 & P_2 & P_3 & P_4 \\ \varnothing & Z_3 & Z_1 & Z_3 & \varnothing \end{pmatrix}$$

and $\overline{\beta}_2$ are mapping of the set $X \setminus \breve{D}$ on the set $D \setminus \{Z_3, Z_1\}$. Then

$$\overline{\beta} = (P_0 \times \varnothing) \cup (P_1 \times Z_3) \cup (P_2 \times Z_1) \cup (P_3 \times Z_3) \cup (P_4 \times \varnothing) \cup \bigcup_{t' \in X \setminus \breve{D}} (\{t'\} \times f(t')),$$
$$\alpha_1 = (Y_4^{\alpha_1} \times Z_4) \cup (Y_3^{\alpha_1} \times Z_3) \cup (Y_2^{\alpha_1} \times Z_2) \cup (Y_1^{\alpha_1} \times Z_1) \cup (Y_0^{\alpha_1} \times \breve{D}), \tag{14}$$

where $Y_4^{\alpha_1}, Y_3^{\alpha_1}, Y_2^{\alpha_1}, Y_1^{\alpha_1} \notin \{\varnothing\}$, $Y_4^{\alpha_1} = Y_1^{\alpha}$, $Y_2^{\alpha_1} = Y_3^{\alpha}$ and $Y_3^{\alpha_1} \cup Y_1^{\alpha_1} \cup Y_0^{\alpha_1} = Y_0^{\alpha}$, then it is easy to see, that $\alpha_1, \overline{\beta} \in B$ since $V(D, \alpha_1) = V(D, \overline{\beta}) = D$. From the formal equality and equalities (14) and (5) we have:

$$Z_4\overline{\beta} = (P_0 \cup P_2)\overline{\beta} = P_0\overline{\beta} \cup P_2\overline{\beta} = \varnothing \cap Z_1 = Z_1;$$
$$Z_3\overline{\beta} = (P_0 \cup P_1 \cup P_2 \cup P_4)\overline{\beta} = P_0\overline{\beta} \cup P_1\overline{\beta} \cup P_2\overline{\beta} \cup P_4\overline{\beta} = \varnothing \cup Z_3 \cup Z_1 \cup \varnothing = \breve{D};$$
$$Z_2\overline{\beta} = (P_0 \cup P_1 \cup P_3 \cup P_4)\overline{\beta} = P_0\overline{\beta} \cup P_1\overline{\beta} \cup P_3\overline{\beta} \cup P_4\overline{\beta} = \varnothing \cup Z_3 \cup Z_3 \cup \varnothing = Z_3;$$
$$Z_1\overline{\beta} = (P_0 \cup P_2 \cup P_3 \cup P_4)\overline{\beta} = P_0\overline{\beta} \cup P_2\overline{\beta} \cup P_3\overline{\beta} \cup P_4\overline{\beta} = \varnothing \cup Z_1 \cup Z_3 \cup \varnothing = \breve{D};$$
$$\breve{D}\overline{\beta} = (P_0 \cup P_1 \cup P_2 \cup P_3 \cup P_4)\overline{\beta} = P_0\overline{\beta} \cup P_1\overline{\beta} \cup P_2\overline{\beta} \cup P_3\overline{\beta} \cup P_4\overline{\beta}$$
$$= \varnothing \cup Z_3 \cup Z_1 \cup Z_3 \cup \varnothing = \breve{D};$$
$$\alpha = \alpha_1 \circ \overline{\beta}$$
$$= (Y_4^{\alpha_1} \times Z_4\overline{\beta}) \cup (Y_3^{\alpha_1} \times Z_3\overline{\beta}) \cup (Y_2^{\alpha_1} \times Z_2\overline{\beta}) \cup (Y_1^{\alpha_1} \times Z_1\overline{\beta}) \cup (Y_0^{\alpha_1} \times \breve{D}\overline{\beta})$$
$$= (Y_4^{\alpha_1} \times Z_1) \cup (Y_3^{\alpha_1} \times \breve{D}) \cup (Y_2^{\alpha_1} \times Z_3) \cup (Y_1^{\alpha_1} \times \breve{D}) \cup (Y_0^{\alpha_1} \times \breve{D})$$
$$= (Y_4^{\alpha_1} \times Z_1) \cup (Y_2^{\alpha_1} \times Z_3) \cup ((Y_3^{\alpha_1} \cup Y_1^{\alpha_1} \cup Y_0^{\alpha_1}) \times \breve{D})$$
$$= (Y_1^{\alpha} \times Z_1) \cup (Y_3^{\alpha} \times Z_3) \cup (Y_0^{\alpha} \times \breve{D}).$$

6. Let $V(X^*, \alpha) = \{Z_2, Z_3, \breve{D}\}$. Then binary relation α has representation of the form $\alpha = (Y_2^{\alpha} \times Z_2) \cup (Y_3^{\alpha} \times Z_3) \cup (Y_0^{\alpha} \times \breve{D})$. In this case suppose that

$$\overline{\beta}_1 = \begin{pmatrix} P_0 & P_1 & P_2 & P_3 & P_4 \\ \varnothing & Z_3 & Z_2 & Z_3 & \varnothing \end{pmatrix}$$

and $\overline{\beta}_2$ are mapping of the set $X \setminus \breve{D}$ on the set $D \setminus \{Z_3, Z_2\}$. Then

$$\overline{\beta} = (P_0 \times \varnothing) \cup (P_1 \times Z_3) \cup (P_2 \times Z_2) \cup (P_3 \times Z_3) \cup (P_4 \times \varnothing) \cup \bigcup_{t' \in X \setminus \breve{D}} (\{t'\} \times f(t'))$$
$$\alpha_1 = (Y_4^{\alpha_1} \times Z_4) \cup (Y_3^{\alpha_1} \times Z_3) \cup (Y_2^{\alpha_1} \times Z_2) \cup (Y_1^{\alpha_1} \times Z_1) \cup (Y_0^{\alpha_1} \times \breve{D}), \tag{15}$$

where $Y_4^{\alpha_1}, Y_3^{\alpha_1}, Y_2^{\alpha_1}, Y_1^{\alpha_1} \notin \{\varnothing\}$, $Y_4^{\alpha_1} = Y_2^{\alpha}$, $Y_2^{\alpha_1} = Y_3^{\alpha}$ and $Y_3^{\alpha_1} \cup Y_1^{\alpha_1} \cup Y_0^{\alpha_1} = Y_0^{\alpha}$, then it is easy to see, that $\alpha_1, \overline{\beta} \in B$ since $V(D, \alpha_1) = V(D, \overline{\beta}) = D$. From the formal equality and equalities (15) and (5) we have:

$$Z_4\overline{\beta} = (P_0 \cup P_2)\overline{\beta} = P_0\overline{\beta} \cup P_2\overline{\beta} = \varnothing \cup Z_2 = Z_2;$$
$$Z_3\overline{\beta} = (P_0 \cup P_1 \cup P_2 \cup P_4)\overline{\beta} = P_0\overline{\beta} \cup P_1\overline{\beta} \cup P_2\overline{\beta} \cup P_4\overline{\beta} = \varnothing \cup Z_3 \cup Z_2 \cup \varnothing = \breve{D};$$
$$Z_2\overline{\beta} = (P_0 \cup P_1 \cup P_3 \cup P_4)\overline{\beta} = P_0\overline{\beta} \cup P_1\overline{\beta} \cup P_3\overline{\beta} \cup P_4\overline{\beta} = \varnothing \cup Z_3 \cup Z_3 \cup \varnothing = Z_3;$$
$$Z_1\overline{\beta} = (P_0 \cup P_2 \cup P_3 \cup P_4)\overline{\beta} = P_0\overline{\beta} \cup P_2\overline{\beta} \cup P_3\overline{\beta} \cup P_4\overline{\beta} = \varnothing \cup Z_2 \cup Z_3 \cup \varnothing = \breve{D};$$
$$\breve{D}\overline{\beta} = (P_0 \cup P_1 \cup P_2 \cup P_3 \cup P_4)\overline{\beta} = P_0\overline{\beta} \cup P_1\overline{\beta} \cup P_2\overline{\beta} \cup P_3\overline{\beta} \cup P_4\overline{\beta}$$
$$= \varnothing \cup Z_3 \cup Z_2 \cup Z_3 \cup \varnothing = \breve{D};$$
$$\alpha = \alpha_1 \circ \overline{\beta}$$
$$= (Y_4^{\alpha_1} \times Z_4\overline{\beta}) \cup (Y_3^{\alpha_1} \times Z_3\overline{\beta}) \cup (Y_2^{\alpha_1} \times Z_2\overline{\beta}) \cup (Y_1^{\alpha_1} \times Z_1\overline{\beta}) \cup (Y_0^{\alpha_1} \times \breve{D}\overline{\beta})$$
$$= (Y_4^{\alpha_1} \times Z_2) \cup (Y_3^{\alpha_1} \times \breve{D}) \cup (Y_2^{\alpha_1} \times Z_3) \cup (Y_1^{\alpha_1} \times \breve{D}) \cup (Y_0^{\alpha_1} \times \breve{D})$$
$$= (Y_4^{\alpha_1} \times Z_2) \cup (Y_2^{\alpha_1} \times Z_3) \cup ((Y_3^{\alpha_1} \cup Y_1^{\alpha_1} \cup Y_0^{\alpha_1}) \times \breve{D})$$
$$= (Y_2^{\alpha} \times Z_2) \cup (Y_3^{\alpha} \times Z_3) \cup (Y_0^{\alpha} \times \breve{D}).$$

7. Let $V(X^*,\alpha)=\{Z_4,Z_3,Z_1,\breve{D}\}$. Then binary relation α has representation of the form $\alpha=(Y_4^\alpha\times Z_4)\cup(Y_3^\alpha\times Z_3)\cup(Y_1^\alpha\times Z_1)\cup(Y_0^\alpha\times\breve{D})$. In this case suppose that

$$\overline{\beta}_1=\begin{pmatrix}P_0 & P_1 & P_2 & P_3 & P_4\\ \varnothing & Z_3 & Z_4 & Z_1 & \varnothing\end{pmatrix}$$

and $\overline{\beta}_2$ are mapping of the set $X\setminus\breve{D}$ on the set $D\setminus\{Z_4,Z_3,Z_1\}$. Then

$$\begin{aligned}\overline{\beta}&=(P_0\times\varnothing)\cup(P_1\times Z_3)\cup(P_2\times Z_4)\cup(P_3\times Z_1)\cup(P_4\times\varnothing)\cup\bigcup_{t'\in X\setminus\breve{D}}(\{t'\}\times f(t')),\\ \alpha_1&=(Y_4^{\alpha_1}\times Z_4)\cup(Y_3^{\alpha_1}\times Z_3)\cup(Y_2^{\alpha_1}\times Z_2)\cup(Y_1^{\alpha_1}\times Z_1)\cup(Y_0^{\alpha_1}\times\breve{D}),\end{aligned}\tag{16}$$

where $Y_4^{\alpha_1},Y_3^{\alpha_1},Y_2^{\alpha_1},Y_1^{\alpha_1}\notin\{\varnothing\}$, $Y_4^{\alpha_1}=Y_4^\alpha$, $Y_3^{\alpha_1}=Y_3^\alpha$, $Y_1^{\alpha_1}=Y_1^\alpha$ and $Y_2^{\alpha_1}\cup Y_0^{\alpha_1}=Y_0^\alpha$, then it is easy to see, that $\alpha_1,\overline{\beta}\in B$ since $V(D,\alpha_1)=V(D,\overline{\beta})=D$. From the formal equality and equalities (16) and (5) we have:

$$\begin{aligned}&Z_4\overline{\beta}=(P_0\cup P_2)\overline{\beta}=P_0\overline{\beta}\cup P_2\overline{\beta}=\varnothing\cup Z_4=Z_4;\\ &Z_3\overline{\beta}=(P_0\cup P_1\cup P_2\cup P_4)\overline{\beta}=P_0\overline{\beta}\cup P_1\overline{\beta}\cup P_2\overline{\beta}\cup P_4\overline{\beta}=\varnothing\cup Z_3\cup Z_4\cup\varnothing=Z_3;\\ &Z_2\overline{\beta}=(P_0\cup P_1\cup P_3\cup P_4)\overline{\beta}=P_0\overline{\beta}\cup P_1\overline{\beta}\cup P_3\overline{\beta}\cup P_4\overline{\beta}=\varnothing\cup Z_3\cup Z_1\cup\varnothing=\breve{D};\\ &Z_1\overline{\beta}=(P_0\cup P_2\cup P_3\cup P_4)\overline{\beta}=P_0\overline{\beta}\cup P_2\overline{\beta}\cup P_3\overline{\beta}\cup P_4\overline{\beta}=\varnothing\cup Z_4\cup Z_1\cup\varnothing=Z_1;\\ &\breve{D}\overline{\beta}=(P_0\cup P_1\cup P_2\cup P_3\cup P_4)\overline{\beta}=P_0\overline{\beta}\cup P_1\overline{\beta}\cup P_2\overline{\beta}\cup P_3\overline{\beta}\cup P_4\overline{\beta}\\ &\quad=\varnothing\cup Z_3\cup Z_4\cup Z_1\cup\varnothing=\breve{D};\\ &\alpha=\alpha_1\circ\overline{\beta}\\ &\;=(Y_4^{\alpha_1}\times Z_4\overline{\beta})\cup(Y_3^{\alpha_1}\times Z_3\overline{\beta})\cup(Y_2^{\alpha_1}\times Z_2\overline{\beta})\cup(Y_1^{\alpha_1}\times Z_1\overline{\beta})\cup(Y_0^{\alpha_1}\times\breve{D}\overline{\beta})\\ &\;=(Y_4^{\alpha_1}\times Z_4)\cup(Y_3^{\alpha_1}\times Z_3)\cup(Y_2^{\alpha_1}\times\breve{D})\cup(Y_1^{\alpha_1}\times Z_1)\cup(Y_0^{\alpha_1}\times\breve{D})\\ &\;=(Y_4^{\alpha_1}\times Z_4)\cup(Y_3^{\alpha_1}\times Z_3)\cup(Y_1^{\alpha_1}\times Z_1)\cup((Y_2^{\alpha_1}\cup Y_0^{\alpha_1})\times\breve{D})\\ &\;=(Y_4^\alpha\times Z_4)\cup(Y_3^\alpha\times Z_3)\cup(Y_1^\alpha\times Z_1)\cup(Y_0^\alpha\times\breve{D}).\end{aligned}$$

□

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