

# Interior and Exterior Differential Systems for Lie Algebroids

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Received March 26, 2011; revised August 9, 2011; accepted August 20, 2011

## Abstract

A theorem of Maurer-Cartan type for Lie algebroids is presented. Suppose that any vector subbundle of a Lie algebroid is called interior differential system (IDS) for that Lie algebroid. A theorem of Frobenius type is obtained. Extending the classical notion of exterior differential system (EDS) to Lie algebroids, a theorem of Cartan type is obtained.

**2000 Mathematics Subject Classification:** 00A69, 58A15, 58B34.

**Keywords:** Vector Bundle, Lie Algebroid, Interior Differential System, Exterior Differential Calculus, Exterior Differential System

## 1. Introduction

Using the exterior differential calculus for Lie algebroids (See [1,2]) the structure equations of Maurer-Cartan type are established. Using the Cartan's moving frame method, there exists the following

**Theorem** (E. Cartan) *If  $N \in |Man_n|$  is a Riemannian manifold and  $X_\alpha = X_\alpha^i \frac{\partial}{\partial x^i}$ ,  $\alpha \in \overline{1, n}$  is  $n$  ortonormal moving frame, then there exists a collection of 1-forms  $\Omega_\beta^\alpha$ ,  $\alpha, \beta \in \overline{1, n}$  uniquely defined by the requirements*

$$\Omega_\beta^\alpha = -\Omega_\alpha^\beta$$

and

$$d^F \Theta^\alpha = \Omega_\beta^\alpha \wedge \Theta^\beta, \alpha \in \overline{1, n}$$

where  $\{\Theta^\alpha, \alpha \in \overline{1, n}\}$  is the coframe. (see [3], p. 151)

We know that an  $r$ -dimensional distribution on a manifold  $N$  is a mapping  $D$  defined on  $N$ , which assigns to each point  $x$  of  $N$  an  $r$ -dimensional linear subspace  $D_x$  of  $T_x N$ . A vector field  $X$  belongs to  $D$  if we have  $X_x \in D_x$  for each  $x \in N$ . When this happens we write  $X \in \Gamma(D)$ .

The distribution  $D$  on a manifold  $N$  is said to be differentiable if for any  $x \in N$  there exists  $r$  differentiable linearly independent vector fields

$X_1, \dots, X_r \in \Gamma(D)$  in a neighborhood of  $x$ . The distri-

bution  $D$  is said to be involutive if for all vector fields  $X, Y \in \Gamma(D)$  we have  $[X, Y] \in \Gamma(D)$ .

In the classical theory we have the following

**Theorem** (Frobenius) *The distribution  $D$  is involutive if and only if for each  $x \in N$  there exists a neighborhood  $U$  and  $n-r$  linearly independent 1-forms  $\Theta^{r+1}, \dots, \Theta^n$  on  $U$  which vanish on  $D$  and satisfy the condition*

$$d^F \Theta^\alpha = \sum_{\beta \in \overline{r+1, n}} \Omega_\beta^\alpha \wedge \Theta^\beta, \alpha \in \overline{r+1, n}$$

for suitable 1-forms  $\Omega_\beta^\alpha$ ,  $\alpha, \beta \in \overline{1, n}$ . (see [4], p. 58)

Extending the notion of distribution we obtain the definition of an IDS of a Lie algebroid. A characterization of the involutivity of an IDS in a result of Frobenius type is presented in Theorem 4.7.

This paper studies the intersection between the geometry of Lie algebroids and some aspects of EDS. In the classical sense, an EDS is a pair  $(M, I)$  consisting of a smooth manifold  $M$  and a homogeneous, differentially closed ideal  $I$  in the algebra of smooth differential forms on  $M$ . (see [5,6]) Using the notion of EDS of an arbitrary Lie algebroid  $((F, \nu, N), [\cdot, \cdot]_F, (\rho, Id_N))$  we obtained a new result of Cartan type in the Theorem 5.1. In the particular case of standard Lie algebroid  $((TM, \tau_M, M), [\cdot, \cdot]_{TM}, (Id_{TM}, Id_M))$  there are obtained similar results those for distributions.

We know that a submanifold  $S$  of  $N$  is said to be

integral manifold for the distribution  $D$  if for every point  $x \in N$ ,  $D_x$  coincides with  $T_x S$ . The distribution  $D$  is said to be integrable if for each point  $x \in N$  there exists an integral manifold of  $D$  containing  $x$ . As a distribution  $D$  is involutive if and only if it is integrable, then the study of the integral manifolds of an  $IDS$  or  $EDS$  is a new direction by research.

### 2. Preliminaries

In general, if  $C$  is a category, then we denote  $|C|$  the class of objects and for any  $A, B \in |C|$ , we denote  $C(A, B)$  the set of morphisms of  $A$  source and  $B$  target. Let  $LieAlg$ ,  $Mod$ , and  $B^v$  be the category of Lie algebras, modules and vector bundles respectively.

We know that if  $(E, \pi, M) \in |B^v|$ ,  $\Gamma(E, \pi, M) = \{u \in Man(M, E) : u \circ \pi = Id_M\}$  and  $F(M) = Man(M, R)$ , then  $(\Gamma(E, \pi, M), +, \cdot)$  is a  $F(M)$ -module.

We know that a Lie algebroid is a vector bundle  $(F, \nu, N) \in |B^v|$  so that there exists

$$(\rho, Id_N) \in B^v((F, \nu, N), (TN, \tau_N, N))$$

and also an operation

$$\begin{aligned} \Gamma(F, \nu, N) \times \Gamma(F, \nu, N) &\xrightarrow{[\cdot]_F} \Gamma(F, \nu, N) \\ (u, v) &\mapsto [u, v]_F \end{aligned}$$

with the following properties:

$LA_1$ . the equality holds good

$$[u, f \cdot v]_F = f[u, v]_F + \Gamma(\rho, Id_N)(u)f \cdot v$$

for all  $u, v \in \Gamma(F, \nu, N)$  and  $f \in F(N)$ ,

$LA_2$ . the 4-tuple  $(\Gamma(F, \nu, N), +, \cdot, [\cdot]_F)$  is a Lie  $F(N)$ -algebra,

$LA_3$ . the  $Mod$ -morphism  $\Gamma(\rho, Id_N)$  is a  $LieAlg$ -morphism of  $(\Gamma(F, \nu, N), +, \cdot, [\cdot]_F)$  source and  $(\Gamma(TN, \tau_N, N), +, \cdot, [\cdot]_{TN})$  target.

Let  $((F, \nu, N), [\cdot]_F, (\rho, Id_N))$  be a Lie algebroid.

Locally, for any  $\alpha, \beta \in \overline{1, p}$ , we set  $[t_\alpha, t_\beta]_F = L'_{\alpha\beta} t_\gamma$ . We easily obtain that  $L'_{\alpha\beta} = -L'_{\beta\alpha}$ , for any  $\alpha, \beta, \gamma \in \overline{1, p}$ .

The real local functions  $L'_{\alpha\beta}, \alpha, \beta, \gamma \in \overline{1, p}$  are called the *structure functions*.

We assume that  $(F, \nu, N)$  is a vector bundle with type fibre the real vector space  $(R^p, +, \cdot)$  and structure group a Lie subgroup of  $(GL(p, R), \cdot)$ . We denote  $(x^i, z^\alpha)$  the canonical local coordinates on  $(F, \nu, N)$ , where  $i \in \overline{1, n}$  and  $\alpha \in \overline{1, p}$ .

Consider  $(x^i, z^\alpha) \rightarrow (x^{i'}, z^{\alpha'})$  a change of coordinates on  $(F, \nu, N)$ . Then the coordinates  $z^\alpha$  change to  $z^{\alpha'}$  according to the rule:

$$z^{\alpha'} = \Lambda_{\alpha'}^\alpha z^\alpha \tag{2.1}$$

If  $z^\alpha t_\alpha \in \Gamma(F, \nu, N)$  is arbitrary, then

$$[\Gamma(\rho, Id_N)(z^\alpha t_\alpha) f](x) = \left( \rho_\alpha^i z^\alpha \frac{\partial f}{\partial x^i} \right)(x) \tag{2.2}$$

for any  $f \in F(N)$  and  $x \in N$ .

The coefficients  $\rho_\alpha^i$  change to  $\rho_{\alpha'}^{i'}$  according to the rule:

$$\rho_{\alpha'}^{i'} = \Lambda_{\alpha'}^\alpha \rho_\alpha^i \frac{\partial x^i}{\partial x^{i'}}, \tag{2.3}$$

where  $\|\Lambda_{\alpha'}^\alpha\| = \|\Lambda_{\alpha'}^\alpha\|^{-1}$ .

The following equalities hold good:

$$\left( \rho_\alpha^i \frac{\partial}{\partial x^i} \right)(f) = \left( \rho_{\alpha'}^{i'} \frac{\partial f}{\partial x^{i'}} \right), \forall f \in F(N) \tag{2.4}$$

and

$$L'_{\alpha\beta} \cdot \rho_\gamma^k = \rho_{\alpha'}^{i'} \frac{\partial \rho_{\beta'}^k}{\partial x^i} - \rho_{\beta'}^j \frac{\partial \rho_{\alpha'}^k}{\partial x^j}. \tag{2.5}$$

### 3. Interior Differential Systems

Let  $((F, \nu, N), [\cdot]_F, (\rho, Id_N))$  be a Lie algebroid.

**Definition 3.1** Any vector subbundle  $(E, \pi, M)$  of the vector bundle  $(F, \nu, N)$  will be called *interior differential system (IDS) of the Lie algebroid*

$((F, \nu, N), [\cdot]_F, (\rho, Id_N))$ .

*Remark 3.1* If  $(E, \pi, N)$  is an  $IDS$  of the Lie algebroid  $((F, \nu, N), [\cdot]_F, (\rho, Id_N))$  then we obtain a vector subbundle  $(E^0, \pi^0, N)$  of the dual vector bundle

$(F^*, \nu^*, N)$  so that

$$\begin{aligned} &\Gamma(E^0, \pi^0, N) \\ &= \left\{ \Omega \in \Gamma(F^*, \nu^*, N) : \Omega(S) = 0, \forall S \in \Gamma(E, \pi, N) \right\}. \end{aligned}$$

The vector subbundle  $(E^0, \pi^0, N)$  will be called the *annihilator vector subbundle of the IDS*  $(E, \pi, M)$ .

**Proposition 3.1** If  $(E, \pi, N)$  is an  $IDS$  of the Lie algebroid  $((F, \nu, N), [\cdot]_F, (\rho, Id_N))$  so that  $\Gamma(E, \pi, N) =$

$\langle S_1, \dots, S_r \rangle$ , then it exists  $\Theta^{r+1}, \dots, \Theta^p \in \Gamma(F^*, \nu^*, N)$

linearly independent so that

$$\Gamma(E^0, \pi^0, N) = \langle \Theta^{r+1}, \dots, \Theta^p \rangle.$$

**Definition 3.2** The  $IDS$   $(E, \pi, N)$  of the Lie algebroid  $((F, \nu, N), [\cdot]_F, (\rho, Id_N))$  will be called *involutive* if  $[S, T]_F \in \Gamma(E, \pi, N)$ , for any  $S, T \in \Gamma(E, \pi, N)$ .

**Proposition 3.2** If  $(E, \pi, N)$  is an  $IDS$  of the Lie algebroid  $((F, \nu, N), [\cdot]_F, (\rho, Id_N))$  and  $\{S_1, \dots, S_r\}$  is a base of the  $F(N)$ -submodule  $(\Gamma(E, \pi, N), +, \cdot)$ , then  $(E, \pi, N)$  is involutive if and only if

$$[S_a, S_b]_F \in \Gamma(E, \pi, N), \text{ for any } a, b \in \overline{1, r}.$$

### 4. Exterior Differential Calculus

Let  $((F, \nu, N), [\cdot]_F, (\rho, Id_N))$  be a Lie algebroid. We denote  $\Lambda^q(F, \nu, N)$  the set of differential forms of degree  $q$ . If  $\Lambda(F, \nu, N) = \bigoplus \Lambda^q(F, \nu, N)$ , then we obtain the exterior differential algebra  $(\Lambda(F, \nu, N), +, \cdot, \wedge)$ .

**Definition 4.1** For any  $z \in \Gamma(F, \nu, N)$  the application

$$\Lambda(F, \nu, N) \xrightarrow{L_z} \Lambda(F, \nu, N),$$

defined by

$$L_z(f) = [\Gamma(\rho, Id_N)z](f),$$

for any  $f \in F(N)$  and

$$L_z\omega(z_1, \dots, z_q) = [\Gamma(\rho, Id_N)z](\omega(z_1, \dots, z_q)) - \sum_{i=1}^q \omega(z_1, \dots, [z, z_i]_F, \dots, z_q),$$

for any  $\omega \in \Lambda^q(F, \nu, N)$  and  $z_1, \dots, z_q \in \Gamma(F, \nu, N)$ , is called the *covariant Lie derivative with respect to the section  $z$* .

**Theorem 4.1** If  $z \in \Gamma(F, \nu, N)$ ,  $\omega \in \Lambda^q(F, \nu, N)$  and  $\theta \in \Lambda^r(F, \nu, N)$ , then

$$L_z(\omega \wedge \theta) = L_z\omega \wedge \theta + \omega \wedge L_z\theta. \tag{4.1}$$

**Definition 4.2** If  $z \in \Gamma(F, \nu, N)$ , then the application

$$\Lambda(F, \nu, N) \xrightarrow{i_z} \Lambda(F, \nu, N) \\ \omega \in \Lambda^q(F, \nu, N) \mapsto i_z\omega \in \Lambda^{q-1}(F, \nu, N)$$

defined by  $i_z(f) = 0$ , for any  $f \in F(N)$  and

$$i_z\omega(z_2, \dots, z_q) = \omega(z, z_2, \dots, z_q),$$

for any  $z_2, \dots, z_q \in \Gamma(F, \nu, N)$ , is called the *interior product associated to the section  $z$* .

**Theorem 4.2** If  $z \in \Gamma(F, \nu, N)$ , then for any  $\omega \in \Lambda^q(F, \nu, N)$  and  $\theta \in \Lambda^r(F, \nu, N)$  we obtain the equality

$$i_z(\omega \wedge \theta) = i_z\omega \wedge \theta + (-1)^q \omega \wedge i_z\theta. \tag{4.2}$$

**Theorem 4.3** For any  $z, \nu \in \Gamma(F, \nu, N)$  we obtain

$$L_\nu \circ i_z - i_z \circ L_z = i_{[\nu, z]_F}. \tag{4.3}$$

**Theorem 4.4** The application

$$\Lambda^q(F, \nu, N) \xrightarrow{d^F} \Lambda^{q+1}(F, \nu, N) \\ \omega \mapsto d^F\omega$$

defined by

$$d^F f(z) = L_z(f),$$

for any  $z \in \Gamma(F, \nu, N)$  and

$$d^F\omega(z_0, z_1, \dots, z_q) = \sum_{i=1}^q (-1)^i [\Gamma(\rho, Id_N)z_i](\omega(z_0, z_1, \dots, \hat{z}_i, \dots, z_q)) + \sum_{i < j} (-1)^{i+j} \omega([z_i, z_j]_F, z_0, z_1, \dots, \hat{z}_i, \dots, \hat{z}_j, \dots, z_q)$$

for any  $z_0, z_1, \dots, z_q \in \Gamma(F, \nu, N)$ , is unique having the following property:

$$L_z = d^F \circ i_z - i_z \circ d^F, \forall z \in \Gamma(F, \nu, N). \tag{4.4}$$

This application is called the *exterior differentiation operator of the exterior differential algebra of the Lie algebroid  $((F, \nu, N), [\cdot]_F, (\rho, Id_N))$* .

**Theorem 4.5** The exterior differentiation operator  $d^F$  given by the previous theorem has the following properties:

1) For any  $\omega \in \Lambda^q(F, \nu, N)$  and  $\theta \in \Lambda^r(F, \nu, N)$  we obtain

$$d^F(\omega \wedge \theta) = d^F\omega \wedge \theta + (-1)^q \omega \wedge d^F\theta. \tag{4.5}$$

2) For any  $z \in \Gamma(F, \nu, N)$  we obtain

$$L_z \circ d^F = d^F \circ L_z. \tag{4.6}$$

3)  $d^F \circ d^F = 0$ .

**Theorem 4.6** (of Maurer-Cartan type) If  $((F, \nu, N), [\cdot]_F, (\rho, Id_N))$  is a Lie algebroid and  $d^F$  is the exterior differentiation operator of the exterior differential  $F(N)$ -algebra  $(\Lambda(F, \nu, N), +, \cdot, \wedge)$ , then we obtain the structure equations of Maurer-Cartan type

$$d^F t^\alpha = -\frac{1}{2} L_{\beta\gamma}^\alpha t^\beta \wedge t^\gamma, \alpha \in \overline{1, p} \tag{MC_1}$$

and

$$d^F x^i = \rho_\alpha^i t^\alpha, i \in \overline{1, n}. \tag{MC_2}$$

where  $\{t^\alpha, \alpha \in \overline{1, p}\}$  is the coframe of the vector bundle  $(F, \nu, N)$ .

These equations will be called the *structure equations of Maurer-Cartan type associated to the Lie algebroid  $((F, \nu, N), [\cdot]_F, (\rho, Id_N))$* .

*Proof.* Let  $\alpha \in \overline{1, p}$  be arbitrary. Since

$$d^F t^\alpha(t_\beta, t_\gamma) = -L_{\beta\gamma}^\alpha, \forall \beta, \gamma \in \overline{1, p}$$

it results that

$$d^F t^\alpha = -\sum_{\beta < \gamma} L_{\beta\gamma}^\alpha t^\beta \wedge t^\gamma. \tag{1}$$

Since  $L_{\alpha\beta}^\gamma = -L_{\beta\alpha}^\gamma$  and  $t^\beta \wedge t^\gamma = -t^\gamma \wedge t^\beta$ , for any  $\beta, \gamma \in \overline{1, p}$ , it results that

$$\sum_{\beta < \gamma} L_{\beta\gamma}^\alpha t^\beta \wedge t^\gamma = \frac{1}{2} L_{\beta\gamma}^\alpha t^\beta \wedge t^\gamma, \alpha \in \overline{1, p}. \tag{2}$$

Using the equalities (1) and (2) it results the structure equation (MC<sub>1</sub>).

Let  $i \in \overline{1, n}$  be arbitrary. Since

$$d^F x^i(t_\alpha) = \rho_\alpha^i, \forall \alpha \in \overline{1, p}$$

it results the structure equation (MC<sub>2</sub>). *q.e.d.*

*Remark 4.1* In the particular case of the standard Lie algebroid  $((TN, \tau_N, N), [ \cdot, \cdot ]_{TN}, (Id_{TN}, Id_N))$  we obtain

$$d^{TN} x^i = dx^i, i \in \overline{1, n}, \quad (MC_2)'$$

where  $\{dx^i, i \in \overline{1, n}\}$  is the coframe of the vector bundle  $(TN, \tau_N, N)$ .

As  $d^{TN} \circ d^{TN} = 0$ . and  $L_{jk}^i = 0$ , for all  $i, j, k \in \overline{1, n}$  we obtain

$$d^F(dx^i) = 0 = -\frac{1}{2} L_{jk}^i dx^j \wedge dx^k, i \in \overline{1, n}. \quad (MC_1)'$$

These equations are the *structure equations of Maurer-Cartan type associated to the standard Lie algebroid*  $((TN, \tau_N, N), [ \cdot, \cdot ]_{TN}, (Id_{TN}, Id_N))$ .

**Theorem 4.7** (of Frobenius type) *Let  $(E, \pi, N)$  be an IDS of the Lie algebroid  $((F, \nu, N), [ \cdot, \cdot ]_F, (\rho, Id_N))$ . If  $\{\Theta^{r+1}, \dots, \Theta^p\}$  is a base of  $F(N)$ -submodule  $(\Gamma(E^0, \pi^0, N), +, \cdot)$ , then the IDS  $(E, \pi, N)$  is involutive if and only if it exists  $\Omega_\beta^\alpha \in \Lambda^1(F, \nu, N)$ ,  $\alpha, \beta \in \overline{r+1, p}$  so that*

$$d^F \Theta^\alpha = \sum_{\beta \in \overline{r+1, p}} \Omega_\beta^\alpha \wedge \Theta^\beta, \alpha \in \overline{r+1, p}.$$

*Proof.* Let  $\{S_1, \dots, S_r\}$  is a base of the  $F(N)$ -submodule  $(\Gamma(E, \pi, N), +, \cdot)$ .

Let  $\{S_{r+1}, \dots, S_p\} \subseteq \Gamma(F, \nu, N)$  so that  $\{S_1, \dots, S_r, S_{r+1}, \dots, S_p\}$  is a base of the  $F(N)$ -module  $(\Gamma(F, \nu, N), +, \cdot)$ .

Let  $\{\Theta^1, \dots, \Theta^r\} \subseteq \Gamma\left(\overset{*}{F}, \overset{*}{\nu}, N\right)$  so that

$\{\Theta^1, \dots, \Theta^r, \Theta^{r+1}, \dots, \Theta^p\}$  is a base of the  $F(N)$ -module  $\left(\Gamma\left(\overset{*}{F}, \overset{*}{\nu}, N\right), +, \cdot\right)$ .

For any  $a, b \in \overline{1, r}$  and  $\alpha, \beta \in \overline{r+1, p}$  we have the equalities:

$$\begin{aligned} \Theta^a(S_b) &= \delta_b^a, & \Theta^\alpha(S_b) &= 0, \\ \Theta^a(S_\beta) &= 0, & \Theta^\alpha(S_\beta) &= \delta_\beta^\alpha. \end{aligned}$$

We remark that the set of the 2-forms

$$\left\{ \Theta^a \wedge \Theta^b, \Theta^a \wedge \Theta^\beta, \Theta^\alpha \wedge \Theta^b, \Theta^\alpha \wedge \Theta^\beta; \right. \\ \left. a, b \in \overline{1, r} \wedge \alpha, \beta \in \overline{r+1, p} \right\}$$

is a base of the  $F(N)$ -module  $(\Lambda^2(F, \nu, N), +, \cdot)$ .

Therefore, we have

$$\begin{aligned} d^F \Theta^\alpha &= \sum_{b < c} A_{bc}^\alpha \Theta^b \wedge \Theta^c + \sum_{b, \gamma} B_{b\gamma}^\alpha \Theta^b \wedge \Theta^\gamma \\ &+ \sum_{\beta < \gamma} C_{\beta\gamma}^\alpha \Theta^\beta \wedge \Theta^\gamma, \end{aligned} \quad (1)$$

where,  $A_{bc}^\alpha, B_{b\gamma}^\alpha$  and  $C_{\beta\gamma}^\alpha$ ,  $a, b, c \in \overline{1, r}$ ,  $\alpha, \beta, \gamma \in \overline{r+1, p}$  are real local functions so that  $A_{bc}^\alpha = -A_{cb}^\alpha$  and  $C_{\beta\gamma}^\alpha = -C_{\gamma\beta}^\alpha$ .

Using the formula

$$\begin{aligned} d^F \Theta^\alpha(S_b, S_c) &= \Gamma(\rho, Id_N)(S_b)(\Theta^\alpha(S_c)) \\ &- \Gamma(\rho, Id_N)(S_c)(\Theta^\alpha(S_b)) - \Theta^\alpha([S_b, S_c]_F) \end{aligned} \quad (2)$$

we obtain that

$$A_{bc}^\alpha = -\Theta^\alpha([S_b, S_c]_F), \quad (3)$$

for any  $b, c \in \overline{1, r}$  and  $\alpha \in \overline{r+1, p}$ .

We admit that  $(E, \pi, N)$  is an involutive IDS of the Lie algebroid  $((F, \nu, N), [ \cdot, \cdot ]_F, (\rho, Id_N))$ . As

$[S_b, S_c]_F \in \Gamma(E, \pi, N)$ , for any  $b, c \in \overline{1, r}$ , it results that  $\Theta^\alpha([S_b, S_c]_F) = 0$  for any  $b, c \in \overline{1, r}$  and  $\alpha \in \overline{r+1, p}$ .

Therefore, for any  $b, c \in \overline{1, r}$  and  $\alpha \in \overline{r+1, p}$ , we obtain  $A_{bc}^\alpha = 0$  and

$$\begin{aligned} d^F \Theta^\alpha &= \sum_{b, \gamma} B_{b\gamma}^\alpha \Theta^b \wedge \Theta^\gamma + \frac{1}{2} C_{\beta\gamma}^\alpha \Theta^\beta \wedge \Theta^\gamma \\ &= \left( B_{b\gamma}^\alpha \Theta^b + \frac{1}{2} C_{\beta\gamma}^\alpha \Theta^\beta \right) \wedge \Theta^\gamma \end{aligned}$$

As  $\Omega_\gamma^\alpha = B_{b\gamma}^\alpha \Theta^b + \frac{1}{2} C_{\beta\gamma}^\alpha \Theta^\beta \in \Lambda^1(F, \nu, N)$ , for any  $\beta, \gamma \in \overline{r+1, p}$  it results the first implication.

Conversely, we admit that it exists  $\Omega_\beta^\alpha \in \Lambda^1(F, \nu, N)$ ,  $\alpha, \beta \in \overline{r+1, p}$  so that

$$d^F \Theta^\alpha = \sum_{\beta \in \overline{r+1, p}} \Omega_\beta^\alpha \wedge \Theta^\beta, \quad (4)$$

for any  $\alpha \in \overline{r+1, p}$ .

Using the affirmations (1), (2) and (4) we obtain that  $A_{bc}^\alpha = 0$ , for any  $b, c \in \overline{1, r}$  and  $\alpha \in \overline{r+1, p}$ .

Using the affirmation (3), we obtain

$\Theta^\alpha([S_b, S_c]_F) = 0$  for any  $b, c \in \overline{1, r}$  and  $\alpha \in \overline{r+1, p}$ .

Therefore, we have  $[S_b, S_c]_F \in \Gamma(E, \pi, N)$ , for any  $b, c \in \overline{1, r}$ . Using the Proposition 3.2, we obtain the second implication. *q.e.d.*

### 5. Exterior Differential Systems

Let  $((F, \nu, N), [ \cdot, \cdot ]_F, (\rho, Id_N))$  be a Lie algebroid.

**Definition 5.1** Any ideal  $(I, +, \cdot)$  of the exterior differential algebra of the Lie algebroid

$((F, \nu, N), [\cdot]_F, (\rho, Id_N))$  closed under differentiation operator  $d^F$ , namely  $d^F I \subseteq I$ , is called *differential ideal of the Lie algebroid*  $((F, \nu, N), [\cdot]_F, (\rho, Id_N))$ .

**Definition 5.2** Let  $(I, +, \cdot)$  be a differential ideal of the Lie algebroid  $((F, \nu, N), [\cdot]_F, (\rho, Id_N))$ . If it exists an *IDS*  $(E, \pi, N)$  so that for all  $k \in N^*$  and  $\omega \in I \cap \Lambda^k(F, \nu, N)$  we have  $\omega(u_1, \dots, u_k) = 0$ , for any  $u_1, \dots, u_k \in \Gamma(E, \pi, N)$ , then we will say that  $(I, +, \cdot)$  is an *exterior differential system (EDS) of the Lie algebroid*  $((F, \nu, N), [\cdot]_F, (\rho, Id_N))$ .

**Theorem 5.1** (of Cartan type) *The IDS  $(E, \pi, N)$  of the Lie algebroid  $((F, \nu, N), [\cdot]_F, (\rho, Id_N))$  is involutive, if and only if the ideal generated by the  $F(N)$ -submodule  $(\Gamma(E^0, \pi^0, N), +, \cdot)$  is an EDS of the Lie algebroid  $((F, \nu, N), [\cdot]_F, (\rho, Id_N))$ .*

*Proof.* Let  $(E, \pi, N)$  be an involutive *IDS* of the Lie algebroid  $((F, \nu, N), [\cdot]_F, (\rho, Id_N))$ . Let  $\{\Theta^{r+1}, \dots, \Theta^p\}$  be a base of the  $F(N)$ -submodule  $(\Gamma(E^0, \pi^0, N), +, \cdot)$ . We know that

$$I(\Gamma(E^0, \pi^0, N)) = \bigcup_{q \in N} \{ \Omega_\alpha \wedge \Theta^\alpha; \{ \Omega_{r+1}, \dots, \Omega_p \} \subseteq \Lambda^q(F, \nu, N) \}.$$

Let  $q \in N$  and  $\{ \Omega_{r+1}, \dots, \Omega_p \} \subseteq \Lambda^q(F, \nu, N)$  be arbitrary.

Using the Theorems 4.5 and 4.7 we obtain

$$d^F(\Omega_\alpha \wedge \Theta^\alpha) = d^F \Omega_\alpha \wedge \Theta^\alpha + (-1)^q \Omega_\beta \wedge d^F \Theta^\beta = (d^F \Omega_\alpha + (-1)^{q+1} \Omega_\beta \wedge \Omega_\alpha^\beta) \wedge \Theta^\alpha.$$

As  $d^F \Omega_\alpha + (-1)^{q+1} \Omega_\beta \wedge \Omega_\alpha^\beta \in \Lambda^{q+2}(F, \nu, N)$  it results that

$$d^F(\Omega_\alpha \wedge \Theta^\alpha) \in I(\Gamma(E^0, \pi^0, N)).$$

Therefore,  $d^F I(\Gamma(E^0, \pi^0, N)) \subseteq I(\Gamma(E^0, \pi^0, N))$ .

Conversely, let  $(E, \pi, N)$  be an *IDS* of the Lie algebroid  $((F, \nu, N), [\cdot]_F, (\rho, Id_N))$  so that the  $F(N)$ -submodule  $(\Gamma(E^0, \pi^0, N), +, \cdot)$  is an *EDS* of the Lie algebroid  $((F, \nu, N), [\cdot]_F, (\rho, Id_N))$ .

Let  $\{\Theta^{r+1}, \dots, \Theta^p\}$  be a base of the  $F(N)$ -submodule  $(\Gamma(E^0, \pi^0, N), +, \cdot)$ .

As  $d^F I(\Gamma(E^0, \pi^0, N)) \subseteq I(\Gamma(E^0, \pi^0, N))$  it results that it exists  $\Omega_\beta^\alpha \in \Lambda^1(F, \nu, N)$ ,  $\alpha, \beta \in \overline{r+1, p}$  so that

$$d^F \Theta^\alpha = \sum_{\beta \in \overline{r+1, p}} \Omega_\beta^\alpha \wedge \Theta^\beta \in I(\Gamma(E^0, \pi^0, N)).$$

Using the Theorem 4.7 there results that  $(E, \pi, N)$  is an involutive *IDS*. *q.e.d.*

## 6. Acknowledgments

I would like to thank Matsumae International Foundation for the research grant at Tokai University, during April-September 2008. I would also like to thank Professors Hideo SHIMADA and Sorin Vasile SABAU from Tokai University-Japan, for useful discussions and their suggestions. In memory of Prof. Dr. Gheorghe RADU. Dedicated to Acad. Prof. Dr. Doc. Radu MIRON at his 84th anniversary.

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