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The Normal Meromorphic Functions Family Concerning Higher Order Derivative and Shared Set by One-Way

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Abstract

Let *F* be a meromorphic functions family on the unit disc Δ, If for every $f \in F$ (the zeros of *f* is a multiplicity of at least *k*) and if $f(z)=0$ then $\left|f^{(k)}(z)\right| \le c$ and $\bar{E}_{f^{(k)}}(S) \subseteq \bar{E}_f(S)$ ($S=\{a,b\}$), then *F* is normal on Δ.

Keywords: Meromorphic Function, Normality Criterion, Shared Set by One-Way, Higher Order Derivative

1. Introduction and Results

First, we introduce the following definition:

Definition: For a,b are two distinct complex values, we have set $S=\{a,b\}$ and

$$\bar{E}_f(S) = \bar{E}_f(a,b) = \{z : (f(z)-a))(f(z)-b) = 0, z \in D\}$$

If $\bar{E}_f(S)=\bar{E}_g(S)$, we call that *f* and *g* share *S* in *D*; If $\bar{E}_f(S) \subseteq \bar{E}_g(S)$, we call that *f* and *g* share *S* by One-way in *D*.

For the normal meromorphic functionsfamilyconcerning one-way sharing set, W. H. Zhang proved the following result [1]:

Theorem A. Let *F* be a family of meromorphic functions in the unit disc Δ, *a* and *b* be two distinct nonzero complex values. $S=\{a,b\}$, If for every $f \in F$, all of whose zeros is multiple, $\bar{E}_{f'}(S)=\bar{E}_f(S)$, then *F* is normal on Δ.

W. H. Zhang continued considering the relation between normality and the shared set, and proved the next result [2]:

Theorem B. Let *F* be meromorphic functions family in the unit disk Δ, *a* and *b* be two distinct nonzero complex values. If for every $f \in F$, all of whose zeros is multiplicity $k+1$ at least (*k* is a positive integer), $\bar{E}_{f^{(k)}}(S)=\bar{E}_f(S)$, then *F* is normal in Δ.

In 2008, F. J. Lv got following theorem in [3]:

Theorem C. Let *F* be a family of meromorphic function in the unit disk Δ, a and b is two distinct nonzero complex values, *k* is positive integer, $\Delta S=\{a,b\}$. If for every $f \in F$, all of whose zeros have multiplicity $k+1$ at least, $\bar{E}_{f^{(k)}}(S) \subseteq \bar{E}_f(S)$, then *F* is normal in Δ.

In this paper,we continue to discuss about normality theorem of meromorphic functions families concerning higher order derivative and shared set by one-way, and obtain main results as follow.

Theorem. Let *F* be a meromorphic functions family on the unit disc Δ, a and b is two distinct nonzero complex values, *k* is positive integer, $S=\{a,b\}$, If for every $f \in F$, all of whose zeros have multiplicity *k* at least, If

1) exists $c>0$,

such that $f(z)=0 \Rightarrow \left|f^{(k)}(z)\right| \le c$;

2) $\bar{E}_{f^{(k)}}(S) \subseteq \bar{E}_f(S)$;

then *F* is normal in Δ.

Remark: Through the following example, all of whose zeros have multiplicity *k* at least is necessary.

Example: Let

$$S=\{1,.-1\},\ \omega_1 \ne \omega_2 \text{ and } \omega_i^k = 1,\ i=1,2.$$

a family of meromorphic function in the unit disk Δ be $F=\{f_n(z)\}$ there

$$f_n(z) = n\left(e^{\omega_1 z} - e^{\omega_2 z}\right), n=1,2,3,\cdots.$$

Obvously $f_n(z)=f_n^{(k)}(z)$ and

$$f_n^{(l)}(z)=n\left(\omega_1^l e^{\omega_1 z}-\omega_1^l e^{\omega_1 z}\right),\ (l=1,2,3,\cdots,k-1),$$

Hence $z=0$ satisfy 1) and 2) and all of whose zeros of $f_n(z)$ have multiplicity $k-1$ at most, Since

$$f^{\#}(0)=\frac{|f'(0)|}{1+|f(0)|^2}=n|\omega_1-\omega_2|\to\infty(n\to\infty),$$

F is not normal at $z=0$ by Marty Theorem.

2. Lemmas

Lemma 1 [4]. Let F be meromorphic functions families in the unit disk Δ, all of whose zeros have multiplicity k at least. If for every $f\in F$, there exists $A>0$, such that $\left|f^{(k)}(z)\right|\le A$ when eve $f(z)=0$. If F is not normal in Δ, then for every $0\le\alpha\le k$, there exists $0<r<1$, $|z_n|<r$, $f_n\in F$, $\rho_n\to 0$, such that $g_n(\zeta)=\frac{f_n(z_n+\rho_n\zeta)}{\rho_n^{\alpha}}$ converges locally and uniformly to a noncontant meromorphic function $g(\zeta)$, and $g^{\#}(\zeta)\le g^{\#}(0)=kA+1$. where $g^{\#}(\zeta)=\frac{|g'(\zeta)|}{1+|g(\zeta)|^2}$ is said to be Spherical derivative of *g*.

Lemma 2 [5]. Let $f(z)$ be nonconstant meromorphic function in *C*

$$\Theta(a,f)=1-\overline{\lim_{r\to\infty}}\frac{\overline{N}\left(r,\frac{1}{f-a}\right)}{T(r,f)},\ (a\ne\infty),$$

$$\Theta(\infty,f)=1-\overline{\lim_{r\to\infty}}\frac{\overline{N}(r,f)}{T(r,f)}$$

then $\sum_{a\in C}\left[\Theta(a,f)\right]+\Theta(\infty,f)\le 2$.

3. Proof of Theorem

Suppose that F be not normal in Δ, then by Lemma 1 we get that there exists $f_n\in F$, $z_n\in\Delta$ and $\rho_n\to 0^+$ such that $g_n(\zeta)=\frac{f_n(z_n+\rho_n\zeta)}{\rho_n^k}$ converges locally and uniformly to a noncontant meromorphic function $g(\zeta)$ [6], and $g\#(\zeta)\le g\#(0)=kc+1$. We claim that the following conclusions hold.

1) zeros of $g(\zeta)$ have multiplicity k at least, and $g(\zeta)=0\Rightarrow\left|g^{(k)}(\zeta)\right|\le c$;

2) $g^{(k)}(\zeta)\ne a, g^{(k)}(\zeta)\ne b$;

In fact, suppose that there exists $\xi_0\in\Delta$, such that $g(\zeta_0)=0$, there exists $\zeta_n\to\zeta_0$,

$g_n(\zeta_n)=\frac{f_n(z_n+\rho_n\zeta_n)}{\rho_n^k}=0$ for sufficiently large *n*,

Hence $f_n(z_n+\rho_n\zeta_n)=0$.

Therefore, the following conclusions is obviously

$$f_n^{(j)}(z_n+\rho_n\zeta_n)=0,\ (j=1,2,3,\cdots,k-1)$$

and $\left|f_n^{(k)}(z_n+\rho_n\zeta_n)\right|\le c$,

Hence

$$g_n^{(j)}(\zeta_n)=\rho_n^{j-k}f_n^{(j)}(z_n+\rho_n\zeta_n)=0,\ (j=1,2,3,\cdots,k-1)$$

and $\left|g_n^{(k)}(\zeta_n)\right|\le c$,

So $g^{(j)}(\zeta_0)=\lim_{n\to\infty}g_n^{(j)}(\zeta_n)=0,\ (j=1,2,3,\cdots,k-1)$

and $\left|g^{(k)}(\zeta_0)\right|\le c$,

Hence, zeros of $g(\zeta)$ have multiplicity k at least, and $g(\zeta)=0\Rightarrow\left|g^{(k)}(\zeta)\right|\le c$,

Therefore, conclusion (1) is hold

In what follow, we complete the proof of the claim 2): Suppose that there exists ζ_0

Such that $g^{(k)}(\zeta_0)=a$, by Hurwitz theorem, there exists $\zeta_n\to\zeta_0$ such that

$g_n^{(k)}(\zeta_n)=f_n^{(k)}(z_n+\rho_n\zeta_n)=a$ for sufficiently large *n*.

By conditions:

$$\overline{E}_{f^{(k)}}(S)\subseteq\overline{E}_f(S),$$

$$f_n(z_n+\rho_n\zeta_n)=a(\text{or } b),$$

Hence

$$g(\zeta_0)=\lim_{n\to\infty}g_n(\zeta_n)=\lim_{n\to\infty}\frac{f_n(z_n+\rho_n\zeta_n)}{\rho_n^k}=\infty,$$

this is a contradictions for $g^{(k)}(\zeta_0)=a$.

If $g^{(k)}(\zeta)\equiv a$, then $g(\zeta)$ is polynomial of degree *k*.

Because, zeros of $g(\zeta)$ have multiplicity *k* at least, then $g(\zeta)=\frac{a}{k!}(\zeta-\zeta_0)^k$

and $g'(\zeta)=\frac{a}{(k-1)!}(\zeta-\zeta_0)^{k-1}$

Obviously

$$g^{\#}(0)=\frac{\left|\frac{a}{(k-1)!}(-\zeta_0)^{k-1}\right|}{1+\left|\frac{a}{k!}(-\zeta_0)^k\right|}\le\begin{cases}\frac{k}{2},|\zeta_0|>1\\ \frac{|a|}{(k-1)!},|\zeta_0|\le 1\end{cases}$$

Because $\left|g^{(k)}(\zeta_0)\right|=|a|\le c$, then

$\frac{|a|}{(k-1)!} \le \frac{c}{(k-1)!} < kc+1$, Suppose $c \ge 1$ is not general, therefore $\frac{k}{2} < kc+1$, hence $|g\#(0)| < kc+1$, this is a con-contradictions for conditions of $g(\zeta)$. Hence $g^{(k)}(\zeta) \ne a$. Similar to prove $g^{(k)}(\zeta) \ne b$.

Therefore

$$\Theta\left(a, g^{(k)}\right) = 1 - \overline{\lim_{r\to\infty}} \frac{\overline{N}\left(r, \frac{1}{g^{(k)}-a}\right)}{T\left(r, g^{(k)}\right)} = 1$$ and

$$\Theta\left(b, g^{(k)}\right) = 1 - \overline{\lim_{r\to\infty}} \frac{\overline{N}\left(r, \frac{1}{g^{(k)}-b}\right)}{T\left(r, g^{(k)}\right)} = 1$$

Since

$$T\left(r, g^{(k)}\right) \ge N\left(r, g^{(k)}\right)$$

$$= k\overline{N}(r,g) + N(r,g) \ge (k+1)\overline{N}(r,g)$$

So

$$\Theta\left(\infty, g^{(k)}\right) = 1 - \overline{\lim_{r\to\infty}} \frac{\overline{N}\left(r, g^{(k)}\right)}{T\left(r, g(k)\right)} \ge 1 - \frac{1}{k+1} = \frac{k}{k+1}$$

Therefore

$$\Theta\left(a, g^{(k)}\right) + \Theta\left(b, g^{(k)}\right) + \Theta\left(\infty, g^{(k)}\right) \ge 2 + \frac{k}{k+1} > 2$$

this is a contradictions for Lemma 2. The proof of Theorem is completed.

4. References

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