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Geodesic Lightlike Submanifolds of Indefinite Sasakian Manifolds*

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Abstract

In this paper, we study geodesic contact CR-lightlike submanifolds and geodesic screen CR-lightlike (SCR) submanifolds of indefinite Sasakian manifolds. Some necessary and sufficient conditions for totally geodesic, mixed geodesic, $\bar{D}$-geodesic and D'-geodesic contact CR-lightlike submanifolds and SCR submanifolds are obtained.

Keywords: CR-Lightlike Submanifolds, Sasakian Manifolds, Totally Geodesic Submanifolds

1. Introduction

A submanifold M of a semi-Riemannian manifold $\bar{M}$ is called lightlike submanifold if the induced metric on M is degenerate. The general theory of a lightlike submanifold has been developed by Kupeli [1] and Bejancu-Duggal [2].

The geometry of CR-lightlike submanifolds of indefinite Kaehler manifolds was studied by Guggal and Bejancu [2]. The geodesic CR-lightlike submanifolds in indefinite Kaehler manifolds were studied by Sahin and Günes [3,4].

Lightlike submanifold of indefinite Sasakian manifolds can be defined according to the behavior of the almost contact structure, and contact CR-lightlike submanifolds and screen CR-lightlike (SCR) submanifolds of indefinite Sasakian manifolds were studied by Duggal and Sahin in [5]. The study of the geometry of submanifolds of indefinite Sasakian manifolds has been developed by [6] and others.

In this paper, geodesic contact CR-lightlike submanifolds and geodesic screen CR-lightlike (SCR) submanifolds of indefinite Sasakian manifolds are considered. Some necessary and sufficient conditions for totally geodesic, mixed geodesic, $\bar{D}$-geodesic and D'-geodesic contact CR-lightlike submanifolds and SCR submanifolds are obtained.

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2. Preliminaries

A submanifold M^m immersed in a semi-Riemannian manifold $(\bar{M}^{m+n}, \bar{g})$ is called a lightlike submanifold if it admits a degenerate metric g induced from $\bar{g}$ whose radical distribution $RadTM$ is of rank r where $1 \le r \le m$, $RadTM = TM \cap TM^{\perp}$, where

$$TM^{\perp} = \bigcup_{x \in M} \left\{ u \in T_x\bar{M} \mid \bar{g}(u,v) = 0, \forall\, v \in T_x\bar{M} \right\}.$$

Let $S(TM)$ be a screen distribution which is semi-Riemannian complementary distribution of $RadTM$ in TM, *i.e.* $TM = RadTM \perp S(TM)$. As $S(TM)$ is a nondegenerate vector subbundle of $T\bar{M}|_M$, we put $T\bar{M}|_M = S(TM) \perp S(TM)^{\perp}$.

We consider a nondegenerate vector subbundle of $S(TM)$, which is a complementary vector bundle of $RadTM$ in $TM^{\perp}$. Since, for any local basis $\{\xi_i\}$ of $RadTM$, there exists a local frame $\{N_i\}$ of sections with value in the orthogonal complement of $S(TM^{\perp})$ such that $\bar{g}(\xi_i, N_i) = \delta_{ij}$ and $\bar{g}(N_i, N_j) = 0$, there exists a lightlike, transversal vector bundle $ltr(TM)$ locally spanned by $\{N_i\}$. Let $tr(TM)$ be the complementary (but not orthogonal) vector bundle to TM in $T\bar{M}|_M$.

Then

$$tr(TM) = ltr(TM) \perp S(TM^{\perp}),$$

$$T\bar{M} = S(TM) \perp \left[RadTM \oplus ltr(TM) \right] \perp S(TM^{\perp}).$$

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Now, let $\bar{\nabla}$ be the levi-Civita connection on $\bar{M}$, we have

$$X\left(\bar{g}(Y,Z)\right)=\bar{g}\left(\bar{\nabla}_X Y,Z\right)+\bar{g}\left(Y,\bar{\nabla}_X Z\right),\quad \forall X,Y,Z\in\Gamma(TM), \tag{2.1}$$

$$\bar{\nabla}_X Y=\nabla_X Y+h(X,Y),\ \forall X,Y\in\Gamma(TM), \tag{2.2}$$

$$\bar{\nabla}_X V=-A_V X+\nabla^t_X V,\forall X\in\Gamma(TM),\quad V\in\Gamma\left(tr(TM)\right), \tag{2.3}$$

where $\{\nabla_X Y, A_V X\}$ and $\{h(X,Y),\nabla^t_X V\}$ belong to $\Gamma(TM)$ and $\Gamma\left(tr(TM)\right)$,

respectively. Using the projectors $l:tr(TM)\to S(TM)$ and $s:tr(TM)\to ltr\left(TM^{\perp}\right)$, from [1], we have

$$\bar{\nabla}_X Y=\nabla_X Y+h^l(X,Y)+h^s(X,Y),\forall X,Y\in\Gamma(TM), \tag{2.4}$$

$$\bar{\nabla}_X N=-A_N X+\nabla^l_X N+D^s(X,N),\forall N\in\Gamma\left(ltr(TM)\right), \tag{2.5}$$

$$\bar{\nabla}_X W=-A_W X+\nabla^s_X W+D^l(X,W),\forall W\in\Gamma\left(S\left(TM^{\perp}\right)\right). \tag{2.6}$$

Denote the projection of TM to $S(TM)$ by P, we have the decomposition

$$\nabla_X PY=\nabla^*_X PY+h^*(X,PY), \tag{2.7}$$

$$\nabla_X \xi=-A^*_\xi X+\nabla^{*t}_X\xi, \tag{2.8}$$

for any $X,Y\in\Gamma(TM),\xi\in\Gamma(RadTM),N\in\Gamma\left(ltr(TM)\right)$. From the above equations we have

$$\bar{g}\left(h^l(X,Y),\xi\right)=g\left(A^*_\xi X,Y\right), \tag{2.9}$$

$$\bar{g}\left(h^*(X,PY),N\right)=g\left(A_N X,PY\right), \tag{2.10}$$

$$\bar{g}\left(h^l(X,\xi),\xi\right)=0, A^*_\xi\xi=0. \tag{2.11}$$

Definition 2.1 *A* $(2n+1)$*-dimensional Semi-Riemannian manifold* $\left(\bar{M},\bar{g}\right)$ *is called a contact metric manifold if there is a* $(1,1)$ *tensor field* ϕ*, a vector field* V*, called the characteristic vector field, and its dual* 1*-form* η *such that*

$$\bar{g}(\phi X,\phi Y)=\bar{g}(X,Y)-\varepsilon\eta(X)\eta(Y),\bar{g}(V,V)=\varepsilon, \tag{2.12}$$

$$\phi^2(X)=-X+\eta(X)V,\bar{g}(X,V)=\varepsilon\eta(X), \tag{2.13}$$

$$d\eta(X,Y)=\bar{g}(X,\phi Y),\forall X,Y\in\Gamma(TM), \tag{2.14}$$

where $\varepsilon=\pm1$.

From the above definiton, it follows that

$$\phi V=0,\eta\circ\phi=0,\eta(V)=1. \tag{2.15}$$

The $(\phi,V,\eta,\bar{g})$ is called a contact metric structure of $\bar{M}$. If $N_\phi+d\eta\otimes V=0$, we say that $\bar{M}$ has a normal contact structure, where N_ϕ is the Nijenhuis tensor field of ϕ. A normal contact metric manifold is called a Sasakian manifold for which we have

$$\bar{\nabla}_X V=-\phi X. \tag{2.16}$$

$$\left(\bar{\nabla}_X\phi\right)Y=\bar{g}(X,Y)V-\varepsilon\eta(Y)X. \tag{2.17}$$

Let $\left(M,g,S(TM),S\left(TM^{\perp}\right)\right)$ be a lightlike submanifold of $\left(\bar{M},\bar{g}\right)$. For any vector field X tangent to M, we put

$$\phi X=PX+QX, \tag{2.18}$$

where PX and QX are the tangential and the transversal parts of ϕX, respectively.

Let's suppose V is a spacelike vector field so that $\varepsilon=1$, it's similar when V is a timelike vector field.

3. Geodesic Invariant Lightlike Submanifolds

Definition 3.1 *Let* $\left(M,g,S(TM),S\left(TM^{\perp}\right)\right)$ *be a lightlike submanifold, tangent to the structure vector field* V, $V\in S(TM)$*, immersed in an indefinite Sasakian manifold* $\left(\bar{M},g\right)$*, we say that* M *is an invariant submanifolds of* $\bar{M}$ *if the following conditions are satisfied*

$$\phi(RadTM)=RadTM,\ \phi\left(S(TM)\right)=S(TM). \tag{3.1}$$

From (2.16), (2.17), (2.18) and (2.4) we have

$$h^l(X,V)=h^s(X,V)=0,\bar{\nabla}_X V=\nabla_X V=-PX, \tag{3.2}$$

$$h^l(X,\phi Y)=\phi h(X,Y)=h(\phi X,Y),\ \forall X,Y\in\Gamma(TM). \tag{3.3}$$

From (3.1) and (2.12) we have

$$\phi ltr(TM)=ltr(TM),\ \phi\left(S\left(TM^{\perp}\right)\right)=S\left(TM^{\perp}\right). \tag{3.4}$$

Theorem 3.1 *Let* $\left(M,g,S(TM),S\left(TM^{\perp}\right)\right)$ *be an invariant lightlike submanifold of an indefinite Sasakian manifold* $\bar{M}$*, then* M *is totally geodesic if and only if* h^l *and* h^s *of* M *are parallel.*

Proof. Suppose h^l is parallel, for any $X,Y,Z\in\Gamma(TM)$, we have

$$\left(\bar{\nabla}_X h^l\right)(Y,V)=\bar{\nabla}_X h^l(Y,V)-h^l\left(\bar{\nabla}_X Y,V\right)-h^l\left(Y,\bar{\nabla}_X V\right)=0.$$

By (3.2), we have

$$h^l(Y,V)=h^l\left(\bar{\nabla}_X Y,V\right)=0,$$

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so $h^l\left(Y,\overline{\nabla}_X Y\right)=0$. That is to say $h^l\left(Y,PX\right)=0$.

In a similar way, we can get $h^s\left(Y,PX\right)=0$. Thus, M is totally geodesic.

Conversely, if $h^l\left(X,Y\right)=h^s\left(X,Y\right)=0$, since

$$\left(\overline{\nabla}_X h^l\right)(Y,Z)=\overline{\nabla}_X h^l\left(Y,Z\right)-h^l\left(\overline{\nabla}_X Y,Z\right)-h^l\left(Y,\overline{\nabla}_X Z\right)=0,$$

$$\left(\overline{\nabla}_X h^s\right)(Y,Z)=\overline{\nabla}_X h^s\left(Y,Z\right)-h^s\left(\overline{\nabla}_X Y,Z\right)-h^s\left(Y,\overline{\nabla}_X Z\right)=0,$$

so h^l and h^s are parallel, which completes the proof.

4. Geodesic Contact CR-Lightlike Submanifolds

Definition 4.1 *Let* $\left(M,g,S(TM),S\left(TM^{\perp}\right)\right)$ *be a lightlike submanifold, tangent to the structure vector field* V *, immersed in an indefinite Sasakian manifold* $\left(\overline{M},\overline{g}\right)$. *We say that* M *is a contact CR-lightlike submanifold of* $\overline{M}$ *if the following conditions are satisfied* [(*A*)] *RadTM is a distribution on* M *such that* $RadTM\cap\phi(RadTM)=\{0\}$. [(*B*)] *There exist vector bundles* D_0 *and* D' *over* M *such that*

$$S\left(TM\right)=\left\{\phi\left(RadTM\right)\oplus D'\right\}\perp D_0\perp V,$$

$$\phi D_0=D_0,\phi D'=L_1\perp L_2,$$

where D_0 is non-degenerate and $L_1=ltr\left(TM\right)$, L_2 is a vector subbundle of $S\left(TM^{\perp}\right)$. So we have the decomposition

$$TM=\left\{D\perp\oplus D'\right\}\perp V,D=RadTM\perp\phi\left(RadTM\right)\perp D_0.$$

If we denote $\hat{D}=D\perp V$, then we have

$$TM=\hat{D}\oplus D',\phi\hat{D}=\hat{D}.$$

Definition 4.2 *A contact CR-lightlike submanifold of an indefinite Sasakian manifold is called* $\hat{D}$ *-geodesic contact CR-lightlike submanifold if its second fundamental form* h *satisfied* $h\left(X,Y\right)=0$, *for any* $X,Y\in\Gamma\left(\hat{D}\right)$.

Definition 4.3 *A contact CR-lightlike submanifold of an indefinite Sasakian manifold is called mixed geodesic contact CR-lightlike submanifold if its second fundamental form* h *satisfied* $h\left(X,Z\right)=0$, *for any* $X\in\Gamma\left(\hat{D}\right)$ *and* $Z\in\Gamma\left(D'\right)$.

Definition 4.4 *A contact CR-lightlike submanifold of an indefinite Sasakian manifold is called* D' *-geodesic contact CR-lightlike submanifold if its second fundamental form* h *satisfied* $h\left(Z,U\right)=0$, *for any* $Z,U\in\Gamma\left(D'\right)$.

Theorem 4.1 *Let* M *be a contact CR-lightlike submanifold of an indefinite Sasakian manifold* $\overline{M}$. *Then* M *is totally geodesic if and only if* $\overline{g}\left(Y,A_w X\right)=\overline{g}\left(Y,D^l\left(X,W\right)\right)$, $\nabla_X\phi Y$ *has no components in* ϕL_1, $Y\in\Gamma\left(TM-span\{V\}\right)$ *or* X *has no components in* ϕL_1.

Proof. We know that M is totally geodesic if and only if $h\left(X,Y\right)=0$, for any $X,Y\in\Gamma\left(TM\right)$. By the definition of the second fundamental form, $h\left(X,Y\right)=0$ is equivalent to $\overline{g}\left(h\left(X,Y\right),\xi\right)=0,\overline{g}\left(h\left(X,Y\right),W\right)=0$, for any $\xi\in\Gamma\left(RadTM\right),W\in\Gamma\left(S\left(TM^{\perp}\right)\right)$.

From (2.4) and (2.7) we have

$$\begin{aligned}
\overline{g}\left(h\left(X,Y\right),\xi\right)&=\overline{g}\left(\overline{\nabla}_X Y,\xi\right)\\
&=\overline{g}\left(\phi\overline{\nabla}_X Y,\phi\xi\right)+\eta\left(\overline{\nabla}_X Y\right)\eta\left(\xi\right)\\
&=\overline{g}\left(\phi\overline{\nabla}_X Y,\phi\xi\right) \qquad (4.1)\\
&=\overline{g}\left(\overline{\nabla}_X\phi Y,\phi\xi\right)+\overline{g}\left(\overline{g}\left(X,Y\right)V+\eta\left(Y\right)X,\phi\xi\right)\\
&=\overline{g}\left(\nabla_X\phi Y,\phi\xi\right)+\eta\left(Y\right)\overline{g}\left(X,\phi\xi\right)
\end{aligned}$$

and

$$\begin{aligned}
\overline{g}\left(h^s\left(X,Y\right),W\right)&=\overline{g}\left(\overline{\nabla}_X Y,W\right)\\
&=X\left(\overline{g}\left(Y,W\right)\right)-\overline{g}\left(Y,\overline{\nabla}_X W\right)\\
&=-\overline{g}\left(Y,\overline{\nabla}_X W\right) \qquad (4.2)\\
&=-\overline{g}\left(Y,-A_W X+\nabla^s_X W+D^l\left(X,W\right)\right)\\
&=\overline{g}\left(Y,A_W X\right)-\overline{g}\left(Y,D^l\left(X,W\right)\right).
\end{aligned}$$

Thus, from (4.1) and (4.2), the proof is completed.

Theorem 4.2 *Let* M *be a contact CR-lightlike submanifold of an indefinite Sasakian manifold* $\overline{M}$. *Then* M *is mixed geodesic if and only if* $A_{\varphi Y}X$ *has no components in* $\phi RadTM\perp L_2$.

Proof. By the definition, M is mixed geodesic if and only if

$$\overline{g}\left(h\left(X,Y\right),\xi\right)=0,\ \overline{g}\left(h\left(X,Y\right),W\right)=0.$$

$$\forall\ x\in\Gamma\left(\hat{D}\right),\ Y\in\Gamma\left(D'\right).$$

Then we have

$$\begin{aligned}
\overline{g}\left(h\left(X,Y\right),\xi\right)&=\overline{g}\left(\overline{\nabla}_X Y,\xi\right)\\
&=\overline{g}\left(\phi\overline{\nabla}_X Y,\phi\xi\right)+\eta\left(\overline{\nabla}_X Y\right)\eta\left(\xi\right)\\
&=\overline{g}\left(\phi\overline{\nabla}_X Y,\phi\xi\right)\\
&=\overline{g}\left(\overline{\nabla}_X\phi Y,\phi\xi\right)+\overline{g}\left(\overline{g}\left(X,Y\right)V+\eta\left(Y\right)X,\phi\xi\right)\\
&=\overline{g}\left(\overline{\nabla}_X\varphi Y,\phi\xi\right)+\eta(Y)\overline{g}\left(X,\phi\xi\right)\\
&=-\overline{g}\left(A_{\varphi Y}X,\phi\xi\right)+\eta(Y)\overline{g}\left(X,\phi\xi\right)\\
&=-\overline{g}\left(A_{\varphi Y}X,\phi\xi\right)
\end{aligned}$$

and

$$\bar{g}(h(X,Y),W) = \bar{g}(\bar{\nabla}_X Y, W)$$
$$= \bar{g}(\phi\bar{\nabla}_X Y, \phi W) + \eta(\bar{\nabla}_X Y)\eta(W)$$
$$= \bar{g}(\phi\bar{\nabla}_X Y, \phi W)$$
$$= \bar{g}(\bar{\nabla}_X \phi Y, \phi W) + \bar{g}(\bar{g}(X,Y)V + \eta(Y)X, \phi W)$$
$$= \bar{g}(\bar{\nabla}_X \phi Y, \phi W)$$
$$= -\bar{g}(A_{\phi Y} X, \phi W).$$

Thus, the proof of the theorem is complete.

Theorem 4.3 *Let* M *be a contact CR-lightlike submanifold of an indefinite Sasakian manifold* $\bar{M}$. *Then* M *is* $\hat{D}$*-geodesic if and only if* $\nabla^*_X \phi\xi \in \Gamma(\phi RadTM \perp \phi L_2)$, $\nabla_X Y$ *has no components in* $\phi L_2, \forall X, Y \in \Gamma(\hat{D})$.

Proof. M is $\hat{D}$-geodesic if and only if $\bar{g}(h^l(X,Y),\xi) = 0$, $\bar{g}(h^s(X,Y),W) = 0$, for any $X, Y \in \Gamma(\hat{D}), \xi \in \Gamma(RadTM)$ and $W \in \Gamma(S(TM^\perp))$.

Then we have

$$\bar{g}(h(X,Y),\xi) = \bar{g}(\bar{\nabla}_X Y, \xi)$$
$$= \bar{g}(\phi\bar{\nabla}_X Y, \phi\xi) + \eta(\bar{\nabla}_X Y)\eta(\xi)$$
$$= \bar{g}(\phi\bar{\nabla}_X Y, \phi\xi)$$
$$= \bar{g}(\bar{\nabla}_X \phi Y, \phi\xi) + \bar{g}(\bar{g}(X,Y)V + \eta(Y)X, \phi\xi)$$
$$= \bar{g}(\bar{\nabla}_X \phi Y, \phi\xi)$$
$$= -\bar{g}(\phi Y, \bar{\nabla}_X \phi\xi)$$
$$= -\bar{g}(\phi Y, \nabla^*_X \phi\xi)$$

and

$$\bar{g}(h^s(X,Y),W) = \bar{g}(\bar{\nabla}_X Y, W)$$
$$= \bar{g}(\phi\bar{\nabla}_X Y, \phi W) + \eta(\bar{\nabla}_X Y)\eta(W)$$
$$= \bar{g}(\phi\bar{\nabla}_X Y, \phi W)$$
$$= \bar{g}(\bar{\nabla}_X \phi Y, \phi W) + \bar{g}(\bar{g}(X,Y)V + \eta(Y)X, \phi W)$$
$$= \bar{g}(\bar{\nabla}_X \phi Y, \phi W)$$
$$= \bar{g}(\nabla_X Y, \phi W).$$

Thus the assertions of the theorem follows.

Theorem 4.4 *Let* M *be a contact CR-lightlike submanifold of an indefinite Sasakian manifold* $\bar{M}$. *Then* M *is* D'*-geodesic if and only if* $A_W X$, $A^*_\xi X$ *have no components in* $\phi L_2 \perp \phi(RadTM)$ $\forall X, Y \in \Gamma(D')$.

Proof. M is D'-geodesic if and, only if $\bar{g}(h^l(X,Y),\xi) = 0, \bar{g}(h^s(X,Y),W) = 0$, for any $X, Y \in \Gamma(D'), \xi \in \Gamma(RadTM)$ and $W \in \Gamma(S(TM^\perp))$.

So we have

$$\bar{g}(h(X,Y),\xi) = \bar{g}(\bar{\nabla}_X Y, \xi) = -\bar{g}(Y, \bar{\nabla}_X \xi) = \bar{g}(A^*_\xi X, Y)$$

and

$$\bar{g}(h(X,Y),W) = \bar{g}(\bar{\nabla}_X Y, W) = -\bar{g}(Y, \bar{\nabla}_X W) = \bar{g}(A_W X, Y).$$

Thus the assertions of the theorem follows.

5. Geodesic Contact SCR-Lightlike Submanifolds

Definition 5.1 *Let* $(M, g, S(TM), S(TM^\perp))$ *be a lightlike submanifold, tangent to the structure vector field* V, *immersed in an indefinite Sasakian manifold* $(\bar{M}, \bar{g})$. *We say that* M *is a contact SCR-lightlike submanifold of* $\bar{M}$ *if the following conditions are satisfied*

[(*A*)] *There exist real non-null distributions* D *and* $D^\perp$, *such that*

$$S(TM) = D \perp D^\perp \perp V, \phi(D^\perp) \subset S(TM^\perp),$$
$$D \cap D^\perp = \{0\},$$

where $D^\perp$ is the orthogonal complementary to $D \perp V$ in $S(TM)$. [(B)] $\phi D = D$, $\phi RadTM = RadTM$, $\phi ltr(TM) = ltr(TM)$.

Hence we have the decomposition $TM = \bar{D} \perp D^\perp \perp V$, $\bar{D} = D \perp RadTM$.

Let us denote $\hat{\bar{D}} = \bar{D} \perp V$.

Definition 5.2 *A contact SCR-lightlike submanifold of an indefinite Sasakian manifold is called mixed geodesic contact SCR-lightlike submanifold if its second fundamental form* h *satisfied* $h(X,Y) = 0$, *for any* $X \in \Gamma(\bar{D})$ *and* $Y \in \Gamma(D^\perp)$.

Theorem 5.1 *Let* M *be a contact SCR-lightlike submanifold of an indefinite Sasakian manifold* $\bar{M}$. *Then* M *is totally geodesic if and only if*

$$(L_\xi \bar{g})(X,Y) = (L_W \bar{g})(X,Y) = 0, \ \forall \ X, Y \in \Gamma(TM),$$
$$\xi \in \Gamma(RadTM), \ W \in \Gamma(S(TM^\perp)).$$

Proof. We know M is totally geodesic if and only if

$$\bar{g}(h(X,Y),\xi) = 0, \ \bar{g}(h(X,Y),W) = 0.$$
$$\forall \ X \in \Gamma(\hat{D}), \ Y \in \Gamma(D').$$

From (2.1) and Lie derivative we obtain

$$\begin{aligned}
&\bar{g}(h(X,Y),\xi)=\bar{g}(\bar{\nabla}_X Y,\xi)\\
&=X(\bar{g}(Y,\xi))-\bar{g}(Y,\bar{\nabla})X\xi\\
&=\bar{g}(Y,[\xi,X])-\bar{g}(Y,\bar{\nabla}_\xi X)\\
&=\bar{g}(Y,[\xi,X])-\xi(\bar{g}(X,Y))+\bar{g}(X,\bar{\nabla}_\xi Y)\\
&=\bar{g}(Y,[\xi,X])-\xi(\bar{g}(X,Y))+\bar{g}(X,[\xi,Y])+\bar{g}(\bar{\nabla}_Y\xi,X)\\
&=-(L_\xi\bar{g})(X,Y)-\bar{g}(\xi,\bar{\nabla}_Y X)\\
&=-(L_\xi\bar{g})(X,Y)-\bar{g}(h(X,Y),\xi).
\end{aligned}$$

Hence we have $2\bar{g}(h(X,Y),\xi)=-(L_\xi\bar{g})(X,Y)$.
In a similar way, we can get

$$2\bar{g}(h(X,Y),W)=-(L_W\bar{g})(X,Y),$$

thus the proof is completed.

Theorem 5.2 *Let M be a contact SCR-lightlike submanifold of an indefinite Sasakian manifold $\bar{M}$. Then M is mixed geodesic if and only if*

$\nabla^s_X\phi Y\in\Gamma(D^\perp)$, $A_{\phi Y}X\in\Gamma(\hat{\bar{D}})$, *for any* $X\in\Gamma(\hat{\bar{D}})$, $Y\in\Gamma(D^\perp)$.

Proof. For any

$$X\in\Gamma(\hat{\bar{D}}),\ Y\in\Gamma(D^\perp),$$
$$\xi\in\Gamma(RadTM),\ W\in\Gamma(S(TM^\perp))$$

denote by

$$\phi X=P'X+Q'X,\ \phi W=B'W+C'W,$$

where $P'X\in\Gamma(\bar{D})$, $Q'X\in\Gamma(\phi D^\perp)$, $B'W\in\Gamma(D^\perp)$ and $C'W\in\Gamma(S(TM^\perp)-\phi D^\perp)$.

If M is mixed geodesic, then $h(X,Y)=\bar{\nabla}_X Y-\nabla_X Y=0$. From the definition, there exists $W\in\Gamma(S(TM^\perp))$ such that $\phi W=Y$. Thus we have

$$\begin{aligned}
0&=\bar{\nabla}_X\phi W-\nabla_X Y=\phi\bar{\nabla}_X W-\nabla_X Y\\
&=\phi(-A_W X+\nabla^t_X W)-\nabla_X Y\\
&=-P'A_W X-Q'A_W X+B'\nabla^t_X W+C'\nabla^t_X W-\nabla_X Y.
\end{aligned}$$

From the definition of the Q' and C', we know that $Q'A_W X=C'\nabla^t_X W=0$. So we have $\nabla^t_X W\in\Gamma(\phi D^\perp)$, $A_W X\in\Gamma(\hat{\bar{D}})$. From $\phi W=Y$ and (2.13), we have $W=-\phi Y$, thus the proof is completed.

Theorem 5.3 *Let M be a contact SCR-lightlike submanifold of an indefinite Sasakian manifold $\bar{M}$. Then $D^\perp$ defines a totally geodesic foliation if and only if $h^s(X,\phi Z)$ and $h^s(X,\phi N)$ has no components in $\Gamma(\phi(D^\perp))$, $\forall\ X\in\Gamma(D^\perp)$, $Z\in\Gamma(\bar{D})$.*

Proof. From the definition, we have that $D^\perp$ is a totally geodesic foliation if and only if $\nabla_X Y\in\Gamma(D^\perp)$, for any $X,Y\in\Gamma(D^\perp)$, which is equivalent to

$$g(\nabla_X Y,Z)=g(\nabla_X Y,N)=0,$$
$$\forall Z\in\Gamma(\bar{D}),N\in\Gamma(ltr(TM)).$$

Then we have

$$\begin{aligned}
&g(\nabla_X Y,Z)=\bar{g}(\bar{\nabla}_X Y,Z)=-\bar{g}(Y,\bar{\nabla}_X Z)\\
&=-\bar{g}(\phi Y,\phi\bar{\nabla}_X Z)-\eta(Y)\eta(\bar{\nabla}_X Z)\\
&=-\bar{g}(\phi Y,\phi\bar{\nabla}_X Z)\\
&=-\bar{g}(\phi Y,\bar{\nabla}_X\phi Z+g(X,Z)V+\eta(Z)X)\\
&=-\bar{g}(\phi Y,\bar{\nabla}_X\phi Z)\\
&=-\bar{g}(\phi Y,h^s(X,\phi Z))
\end{aligned}$$

and

$$\begin{aligned}
&g(\nabla_X Y,N)=\bar{g}(\bar{\nabla}_X Y,N)\\
&=\bar{g}(\phi\bar{\nabla}_X Y,\phi N)+\eta(\bar{\nabla}_X Y)\eta(N)\\
&=\bar{g}(\phi\bar{\nabla}_X Y,\phi N)\\
&=\bar{g}(\bar{\nabla}_X\phi Y+g(X,Y)V+\eta(Y)X,\phi N)\\
&=\bar{g}(\bar{\nabla}_X\phi Y,\phi N)\\
&=-\bar{g}(\phi Y,\bar{\nabla}_X\phi N)\\
&=-\bar{g}(\phi Y,h^s(X,\phi N)).
\end{aligned}$$

Thus the assertion is proved.

6. References

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