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A Real p-Homogeneous Seminorm with Square Property Is Submultiplicative

Mohammed El Azhari
Department of Mathematics, Ecole Normale Supérieure, Rabat, Morocco
Email: mohammed.elazhari@yahoo.fr

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ABSTRACT

We give a functional representation theorem for a class of real p-Banach algebras. This theorem is used to show that every p-homogeneous seminorm with square property on a real associative algebra is submultiplicative.

Keywords: Functional Representation; p-Homogeneous Seminorm; Square Property; Submultiplicative

1. Introduction

J. Arhippainen [1] has obtained the following result:

Theorem 1 of [1]. Let q be a p-homogeneous seminorm with square property on a complex associative algebra A. Then

1) Ker(q) is an ideal of *A*;
2) The quotient algebra *A*/Ker(*q*) is commutative;
3) q is submultiplicative;
4) $q^{\frac{1}{p}}$ is a submultiplicative seminorm on *A*.

This result is a positive answer to a problem posed in [2] and considered in [3-5]. The proofs of (3) and (4) depend on (2) which is obtained by using a locally bounded version of the Hirschfeld-Zelazko Theorem [1, Lemma 1]. This method can not be used in a real algebra; if q is the usual norm defined on the real algebra H of quaternions, $\mathrm{Ker}(q)=\{0\}$ and $\mathrm{H}/\mathrm{Ker}(q)\cong \mathrm{H}$ is noncommutative, then the assertion (2) does not hold in the real case.

The purpose of this paper is to provide a real algebra analogue of the above Arhippainen Theorem, and this improves the result in [6]. Our method is based on a functional representation theorem which we will establish; it is an extension of the Abel-Jarosz Theorem [7, Theorem 1] to real p-Banach algebras. We also give a functional representation theorem for a class of complex p-Banach algebras. As a consequence, we obtain the main result in [8].

2. Preliminaries

Let A be an associative algebra over the field K = R or C. Let $p\in\left]0,1\right]$, a map $\|.\|:A\to[0,\infty[$ is a p-homogeneous seminorm if for a,b in A and α in K, $\|a+b\|\le\|a\|+\|b\|$ and $\|\alpha a\|=|\alpha|^p\|a\|$. Moreover, if $\|a\|=0$ imply that $a=0$, $\|.\|$ is called a p-homogeneous norm. A 1-homogeneous seminorm (resp.norm) is called a seminorm (resp.norm). $\|.\|$ is submultiplicative if $\|ab\|\le\|a\|\|b\|$ for all a,b in *A*. $\|.\|$ has the square property if $\|a^2\|=\|a\|^2$ for all $a\in A$. If $\|.\|$ is a submultiplicative p-homogeneous norm on *A*, then $(A,\|.\|)$ is called a p-normed algebra, we denote by M(*A*) the set of all nonzero continuous multiplicative linear functionals on *A*. A complete p-normed algebra is called a p-Banach algebra. A uniform p-normed algebra is a p-normed algebra $(A,\|.\|)$ such that $\|a^2\|=\|a\|^2$ for all $a\in A$. Let A be a complex algebra with unit *e*, the spectrum of an element $a\in A$ is defined by

$$Sp(a)=\{\alpha\in C,\alpha e-a\notin A^{-1}\}$$

where A^{-1} is the set of all invertible elements of *A*. Let *A* be a real algebra with unit *e*, the spectrum of $a\in A$ is defined by

$$Sp(a)=\left\{s+it\in C,(a-se)^2+t^2e\notin A^{-1}\right\}.$$

Let *A* be an algebra, the spectral radius of an element $a\in A$ is defined by $r(a)=\sup\{|\alpha|,\alpha\in Sp(a)\}$. Let $(A,\|.\|)$ be a p-normed algebra, the limit $\lim_{n\to\infty}\|a^n\|^{\frac{1}{pn}}$ exists for each $a\in A$, and if A is complete, we have

$r(a)=\lim_{n\to\infty}\|a^n\|^{\frac{1}{pn}}$ for all $a\in A$. A $*$-algebra is a complex algebra with a mapping $*:A\to A, a\to a^*$, such that, for a,b in A and $\alpha\in C$,

$$\left(a^*\right)^*=a,\left(a+b\right)^*=a^*+b^*,$$

$$\left(\alpha a\right)^*=\bar{\alpha}a^*,\left(ab\right)^*=b^*a^*.$$

The map $*$ is called an involution on A. An element $a\in A$ is said to be hermitian if $a^*=a$. The set of all hermitian elements of A is denoted by H(*A*).

3. A Functional Representation Theorem for a Class of Real p-Banach Algebras

We will need the following result due to B. Aupetit and J. Zemanek ([9,10]), their algebraic approach works for real p-Banach algebras.

Theorem 3.1. Let A be a real p-Banach algebra with unit. If there is a positive constant α such that $r(ab)\le\alpha r(a)r(b)$ for all a,b in A, then for every irreducible representation π of A on a real linear space E, the algebra $\pi(A)$ is isomorphic (algebraically) to its commutant in the algebra $L(E)$ of all linear transformations on E.

Let A be a real p-Banach algebra with unit such that $\|a\|^{\frac{1}{p}}\le mr(a)$ for some positive constant m and all $a\in A$. Let $X(A)$ be the set of all nonzero multiplicative linear functionals from A into the noncommutative algebra H of quaternions. For $a\in A$, we consider the map $J(a):X(A)\to H, J(a)x=x(a)$ for all $x\in X(A)$. We endow $X(A)$ with the weakest topology such that all the functions $J(a), a\in A$, are continuous. The map $J:A\to C\left(X(A),H\right), a\to J(a)$, is a homomorphism from A into the real algebra of all continuous functions from $X(A)$ into H.

Theorem 3.2. If π is an irreducible representation of A, then $\pi(A)$ is isomorphic to R, C or H.

Proof. Let $a,b\in A$ and $n\ge1$, we have

$$\left\|(ab)^n\right\|\le\|a\|^n\|b\|^n,$$

then

$$\left\|(ab)^n\right\|^{\frac{1}{pn}}\le\|a\|^{\frac{1}{p}}\|b\|^{\frac{1}{p}}.$$

Letting $n\to\infty$, we obtain $r(ab)\le m^2r(a)r(b)$. Let π be an irreducible representation of A on a real linear space E. By Theorem 3.1, $\pi(A)$ is isomorphic to its commutant Q in the algebra $L(E)$ of all linear transformations on E. Let y_0 be a fixed nonzero element in E. For $y\in E$, we consider

$$\|y\|_E=\inf\left\{\|a\|, a\in A \text{ and } \pi(a)y_0=y\right\}.$$

By the same proof as in [11, Lemma 6.5], $\|.\|_E$ is a p-norm on E and Q is a real division p-normed algebra of continuous linear operators on E. By [12], Q is isomorphic to R, C or H.

Proposition 3.3. A is semisimple and $X(A)$ is a nonempty set which separates the elements of *A*.

Proof. By the condition $\|a\|^{\frac{1}{p}}\le mr(a)$ for all $a\in A$, we deduce that A is semisimple. Let a be a nonzero element in A, since A is semisimple, there is an irreducible representation π of A such that $\pi(a)\ne0$. By Theorem 3.2, there is $\varphi:\pi(A)\to H$ an isomorphism (into). We consider the map $T=\varphi o\pi, T:A\to H$ is a multiplicative linear functional. Moreover,

$$T(a)=\varphi\left(\pi(a)\right)\ne0$$

since $\pi(a)\ne0$ and φ is injective.

Proposition 3.4.

1) $|x(a)|\le\|a\|^{\frac{1}{p}}$ for all $a\in A$ and $x\in X(A)$;

2) An element a is invertible in A if and only if $J(a)$ is invertible in $C\left(X(A),H\right)$;

3) $Sp(a)=Sp\left(J(a)\right)$ for all $a\in A.$

Proof. (1): Since H is a real uniform Banach algebra under the usual norm

$$|.|,|x(a)|=r_H\left(x(a)\right)\le r_A(a)\le\|a\|^{\frac{1}{p}}$$

for all $a\in A$ and $x\in X(A)$.

(2): The direct implication is obvious. Conversely, let π be an irreducible representation of A. By Theorem 3.2, there is $\varphi:\pi(A)\to H$ an isomorphism (into). Since $\varphi o\pi\in X(A)$ and $J(a)$ is invertible,

$$0\ne J(a)(\varphi o\pi)=\varphi\left(\pi(a)\right),$$

then $\pi(a)\ne0.$ Consequently, a is invertible.

(3): $s+it\in Sp(a)$ iff $(a-se)^2+t^2e\notin A^{-1}$

Iff $J\left((a-se)^2+t^2e\right)\notin C\left(X(A),H\right)^{-1}$ by (2)

Iff $\left(J(a)-sJ(e)\right)^2+t^2J(e)\notin C\left(X(A),H\right)^{-1}$

Iff $s+it\in Sp\left(J(a)\right).$

Proposition 3.5. $X(A)$ is a Hausdorff compact space.

Proof. Let x_1, x_2 in $X(A), x_1\ne x_2$, there is an element $a\in A$ such that $x_1(a)\ne x_2(a)$, *i.e.* $J(a)x_1\ne J(a)x_2$, so $X(A)$ is Hausdorff. Let $a\in A$ and

$$K_a=\left\{q\in H,|q|\le\|a\|^{\frac{1}{p}}\right\},$$

K_a is compact in H. Let K be the topological product of K_a for all $a\in A, K$ is compact by the Tychonoff Theorem. By Proposition 3.4(1), $X(A)$ is a subset of K. It is easy to see that the topology of

$X(A)$ is the relative topology from K and that $X(A)$ is closed in K. Then $X(A)$ is compact.

Theorem 3.6. The map

$$J: A \to C(X(A), H), a \to J(a),$$

is an isomorphism (into) such that

$$m^{-1}\|a\|^{\frac{1}{p}} \le \|J(a)\|_s \le \|a\|^{\frac{1}{p}}$$

for all $a \in A$, where $\|.\|_s$ is the supnorm on $C(X(A), H)$. If $m = 1$, we have $\|a\|^{\frac{1}{p}} = \|J(a)\|_s$ for all $a \in A$.

Proof. By Proposition 3.3, J is an injective homomorphism. Let $a \in A$, by Proposition 3.4(3),

$$r(a) = r(J(a)) = \|J(a)\|_s$$

since $C(X(A), H)$ is a real uniform Banach algebra under the supnorm $\|.\|_s$. Moreover, $\|J(a)\|_s \le \|a\|^{\frac{1}{p}}$ by Proposition 3.4(1). Then

$$m^{-1}\|a\|^{\frac{1}{p}} \le r(a) = \|J(a)\|_s \le \|a\|^{\frac{1}{p}}.$$

As an application, we obtain an extension of the Kulkarni Theorem [13, Theorem 1] to real p-Banach algebras.

Theorem 3.7. Let a be an element in A such that $Sp(a) \subset R$, then a belongs to the center of A.

Proof. By Theorem 3.6, $J: A \to C(X(A), H)$ is an isomorphism (into). Let $a \in A$ with $Sp(a) \subset R$. Let $x \in X(A)$ and $x(a) = s + t$ where $s \in R$ and

$$t = t_1 i + t_2 j + t_3 k.$$

Suppose that $t \ne 0$. We have

$$(x(a) - s)^2 = t^2 = -(t_1^2 + t_2^2 + t_3^2) = -|t|^2,$$

Then

$$(x(a) - s)^2 + |t|^2 = 0.$$

Consequently

$$s + i|t| \in Sp(x(a)) \subset Sp(a)$$

with $|t| \ne 0$, a contradiction. Then

$$J(a) \in C(X(A), R)$$

and

$$J(a)J(b) = J(b)J(a)$$

for all b in A, *i.e.* $J(ab - ba) = 0$ for all b in A. Since J is injective, $ab - ba = 0$ for all b in A.

4. A Functional Representation Theorem for a Class of Complex p-Banach Algebras

Let $\|.\|$ be a submultiplicative p-homogeneous seminorm on a complex algebra A. For $a \in A, |a|$ is defined as follows:

$$|a| = \inf \sum_{i=1}^{n} \|a_i\|^{\frac{1}{p}},$$

where the infimum is taken over all decompositions of a satisfying the condition $a = \sum_{i=1}^{n} a_i$, $a_1, \cdots, a_n \in A$. By [14, Theorem 1], $|.|$ is a submultiplicative seminorm on A, it is called the support seminorm of $\|.\|$. Also, it is shown [14] the following result:

Theorem 2 of [14]. Let A be a complex algebra, $\|.\|$ a submultiplicative p-homogeneous seminorm on A, and $|.|$ the support seminorm of $\|.\|$. Then

$$\lim_{n\to\infty} \|a^n\|^{\frac{1}{pn}} = \lim_{n\to\infty} |a^n|^{\frac{1}{n}}$$

for all $a \in A$.

In the proof of this theorem, Xia Dao-Xing uses the following inequality: If $a = a_1 + \cdots + a_m$ and $n \ge 1$, then

$$\|a^n\| \le \sum_{\alpha_1+\cdots+\alpha_m=n} \left(\frac{n!}{\alpha_1!\cdots\alpha_m!}\right)^p \|a_1\|^{\alpha_1} \cdots \|a_m\|^{\alpha_m}.$$

If the algebra is commutative,

$$a^n = (a_1 + \cdots + a_m)^n = \sum_{\alpha_1+\cdots+\alpha_m=n} \frac{n!}{\alpha_1!\cdots\alpha_m!} a_1^{\alpha_1} \cdots a_m^{\alpha_m},$$

then

$$\|a^n\| \le \sum_{\alpha_1+\cdots+\alpha_m=n} \left(\frac{n!}{\alpha_1!\cdots\alpha_m!}\right)^p \|a_1\|^{\alpha_1} \cdots \|a_m\|^{\alpha_m}.$$

This inequality is not justified in the noncommutative case; if the algebra is noncommutative, we only have

$$\|a^n\| \le \sum_{\alpha_1+\cdots+\alpha_m=n} \frac{n!}{\alpha_1!\cdots\alpha_m!} \|a_1\|^{\alpha_1} \cdots \|a_m\|^{\alpha_m}.$$

For the sequel, we will use Theorem 2 of [14] in the commutative case.

Theorem 4.1. Let $(A, \|.\|)$ be a complex p-normed algebra such that $\|a\|^2 \le m\|a^2\|$ for some positive constant m and all $a \in A$. Then $|a| \le \|a\|^{\frac{1}{p}} \le m^{\frac{1}{p}}|a|$ and $|a|^2 \le m^{\frac{2}{p}}|a^2|$ for all $a \in A$, where $|.|$ is the support seminorm of $\|.\|$.

Proof. The completion B of $(A, \|.\|)$ is a p-Banach algebra such that $\|b\|^2 \le m\|b^2\|$ for all $b \in B$, it is commutative by [1, Lemma 1], so A is commutative. By induction, $\|a\| \le m^{1-2^{-n}} \|a^{2^n}\|^{2^{-n}}$ for all $a \in A$ and $n \ge 1$, then $\|a\| \le m \lim_{n\to\infty} \|a^n\|^{\frac{1}{n}}$ for all $a \in A$. By the commutative version of [14, Theorem 2], we have

$$|a| \le \|a\|^{\frac{1}{p}} \le m^{\frac{1}{p}} \lim_{n\to\infty} \|a^n\|^{\frac{1}{pn}}$$
$$= m^{\frac{1}{p}} \lim_{n\to\infty} |a^n|^{\frac{1}{n}} \le m^{\frac{1}{p}} |a|$$

for all $a \in A.$ From the above inequalities,

$$|a|^2 \le \|a\|^{\frac{2}{p}} \le \left(m\|a^2\|\right)^{\frac{1}{p}} \le m^{\frac{2}{p}} |a^2|.$$

Corollary 4.2. Let $(A, \|.\|)$ be a complex uniform p-normed algebra. Then $|a| = \|a\|^{\frac{1}{p}}$ for all $a \in A.$

Theorem 4.3. Let $(A, \|.\|)$ be a complex p-Banach algebra with unit such that $\|a\|^2 \le m\|a^2\|$ for some positive constant m and all $a \in A.$ Then the Gelfand map $G : A \to C(M(A))$ is an isomorphism (into) such that

$$m^{-\frac{2}{p}} \|a\|^{\frac{1}{p}} \le m^{-\frac{1}{p}} |a| \le \|G(a)\|_s \le |a| \le \|a\|^{\frac{1}{p}}$$

for all $a \in A,$ where $\|.\|_s$ is the supnorm on $C(M(A)).$

Proof. A is commutative by [1, Lemma 1]. By Theorem 4.1, $|a| \le \|a\|^{\frac{1}{p}} \le m^{\frac{1}{p}} |a|$ for all $a \in A$, then $(A, |.|)$ is a complex commutative Banach algebra with unit. Clearly $M(A) = M(A, \|.\|) = M(A, |.|)$ is a nonempty compact space. As in the proof of Theorem 4.1, we have

$$|a| \le m^{\frac{1}{p}} \lim_{n\to\infty} |a^n|^{\frac{1}{n}}$$
$$= m^{\frac{1}{p}} \sup\{|f(a)|, f \in M(A)\}$$
$$= m^{\frac{1}{p}} \|G(a)\|_s \le m^{\frac{1}{p}} |a|.$$

Let $a \in A,$ from the above inequalities,

$$m^{-\frac{2}{p}} \|a\|^{\frac{1}{p}} \le m^{-\frac{1}{p}} |a| \le \|G(a)\|_s \le |a| \le \|a\|^{\frac{1}{p}}.$$

Corollary 4.4. Let $(A, \|.\|)$ be a complex uniform p-Banach algebra with unit. Then the Gelfand map $G : A \to C(M(A))$ is an isomorphism (into) such that

$$|a| = \|a\|^{\frac{1}{p}} = \|G(a)\|_s$$

for all $a \in A.$

Theorem 4.5. Let $(A, \|.\|)$ be a complex p-normed $*$-algebra with unit such that

1) $\|a\|^2 \le m\|a^2\|$ for some positive constant m and all $a \in A;$

2) Every element in $H(A)$ has a real spectrum in the completion B of $A.$

Then the involution $*$ is continuous on A and the Gelfand map $G : B \to C(M(B))$ is a $*$-isomorphism such that $m^{-\frac{2}{p}} \|b\|^{\frac{1}{p}} \le \|G(b)\|_s \le \|b\|^{\frac{1}{p}}$ for all b in $B.$

Proof. By Theorem 4.3, it remains to show that the involution $*$ is continuous on A, $G(b^*) = G(b)^*$ for all $b \in B,$ and G is surjective. Let $h \in H(A),$

$$Sp_B(h) = \{f(h), f \in M(B)\} \subset R$$

by (2). Let $a \in A,$ we have $a = h_1 + ih_2$ with $h_1, h_2 \in H(A)$. Let $f \in M(B),$

$$f(a^*) = f(h_1 - ih_2) = f(h_1) - if(h_2)$$
$$= \left(f(h_1) + if(h_2)\right)^* = f(h_1 + ih_2)^* = f(a)^*$$

since $f(h_1)$ and $f(h_2)$ are real. Then $G(a^*) = G(a)^*$ for all $a \in A.$ By Theorem 4.3,

$$m^{-\frac{2}{p}} \|a^*\|^{\frac{1}{p}} \le \|G(a^*)\|_s$$
$$= \|G(a)^*\|_s = \|G(a)\|_s \le \|a\|^{\frac{1}{p}}$$

for all $a \in A,$ then $\|a^*\| \le m^2 \|a\|$ for all $a \in A.$ Consequently, the involution $*$ is continuous on A and can be extended to a continuous involution on B which we will also denote by $*$. Let $b \in B,$ there exists a sequence $(a_n)_n$ in A such that $a_n \to b$. Since the involution on B and the Gelfand map $G : B \to C(M(B))$ are continuous, we have

$$G(a_n^*) \to G(b^*)$$

and

$$G(a_n)^* \to G(b)^*,$$

then

$$G(b^*) = G(b)^*.$$

By the Stone-Weierstrass Theorem, we deduce that G is surjective.

As a consequence, we obtain the main result in [8].

Corollary 4.6. Let A be a complex uniform p-normed $*$-algebra with unit such that every element in $H(A)$ has a real spectrum in the completion B of $A.$ then B is a commutative C^*-algebra.

5. The Main Result

Theorem 5.1. Let A be a real associative algebra. Every p-homogeneous seminorm q with square property on A is submultiplicative and $q^{\frac{1}{p}}$ is a submultiplicative seminorm on $A.$

Proof. By [1], there exists a positive constant m such that $q(ab) \le mq(a)q(b)$ for all $a, b \in A$. $\mathrm{Ker}(q)$ is an ideal of $A,$ the norm $|.|$ on the quotient algebra $A/\mathrm{Ker}(q)$ defined by $|a + \mathrm{Ker}(q)| = q(a)$ is a p-norm with square property. Define

$$\|a + \mathrm{Ker}(q)\| = m|a + \mathrm{Ker}(q)|$$

for all $a \in A.$ Let $a, b \in A,$

$$\|ab+\operatorname{Ker}(q)\| = m|ab+\operatorname{Ker}(q)|$$
$$\le m^2 |a+\operatorname{Ker}(q)||b+\operatorname{Ker}(q)|$$
$$= \|a+\operatorname{Ker}(q)\|\|b+\operatorname{Ker}(q)\|,$$

then $(A/\operatorname{Ker}(q), \|.\|)$ is a real p-normed algebra. Let $a \in A$,

$$\|a^2+\operatorname{Ker}(q)\| = m|a^2+\operatorname{Ker}(q)|$$
$$= m|a+\operatorname{Ker}(q)|^2$$
$$= m^{-1}\left(m|a+\operatorname{Ker}(q)|\right)^2$$
$$= m^{-1}\|a+\operatorname{Ker}(q)\|^2$$

i.e.

$$\|a+\operatorname{Ker}(q)\|^2 = m\|a^2+\operatorname{Ker}(q)\|.$$

The completion B of $(A/\operatorname{Ker}(q), \|.\|)$ satisfies also the property $\|b\|^2 = m\|b^2\|$ for all $b \in B$, and by induction $\|b\| = m^{1-2^{-n}}\left\|b^{2^n}\right\|^{2^{-n}}$ for all $b \in B$ and $n \ge 1$, then $\|b\| = mr(b)^p$ for all $b \in B$. We consider two cases:

B is unital: By section 3, $X(B)$ is a nonempty compact space and the map $J : B \to C(X(B), H)$ is an isomorphism (into). By Proposition 3.4(3), $r(b) = r(J(b))$ for all $b \in B$. Let $b \in B$,

$$\|b\| = mr(b)^p = mr(J(b))^p = m\|J(b)\|_s^p$$

since $C(X(B), H)$ is a real uniform Banach algebra under the supnorm $\|.\|_s$. Then $|b| = m^{-1}\|b\| = \|J(b)\|_s^p$ for all $b \in A/\operatorname{Ker}(q)$, so $|.|$ is submultiplicative and $|.|^{\frac{1}{p}}$ is a submultiplicative norm. Consequently, q is submultiplicative and $q^{\frac{1}{p}}$ is a submultiplicative seminorm.

B is not unital: Let B_1 be the algebra obtained from B by adjoining the unit. By the same proof of [15, Lemma 2] which works for real p-Banach algebras, there exists a p-norm N on B_1 such that

1) (B_1, N) is a real p-Banach algebra with unit;

2) $N(b)^{\frac{1}{p}} \le m^3 r_{B_1}(b)$ for all $b \in B_1$;

3) N and $\|.\|$ are equivalent on B.

By section 3, $X(B_1)$ is a nonempty compact space and the map $J : B_1 \to C(X(B_1), H)$ is an isomorphism (into). Let $b \in B$,

$$\|b\| = mr_B(b)^p = mr_{B_1}(b)^p$$

by (3)

$$= mr(J(b))^p \text{ by Proposition 3.4(3)}$$
$$= m\|J(b)\|_s^p \text{ by the square property of the supnorm.}$$

Then $|b| = m^{-1}\|b\| = \|J(b)\|_s^p$ for all $b \in A/\operatorname{Ker}(q)$, so $|.|$ is submultiplicative and $|.|^{\frac{1}{p}}$ is a submultiplicative norm. Consequently, q is submultiplicative and $q^{\frac{1}{p}}$ is a submultiplicative seminorm.

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