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A Remark on the Uniform Convergence of Some Sequences of Functions

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Abstract

We stress a basic criterion that shows in a simple way how a sequence of real-valued functions can converge uniformly when it is more or less evident that the sequence converges uniformly away from a finite number of points of the closure of its domain. For functions of a real variable, unlike in most classical textbooks our criterion avoids the search of extrema (by differential calculus) of their general term.

Keywords

Sequence of Functions, Uniform Convergence, Metric, Boundedness

1. Introduction

Let *X* be a nonempty set, $f : X \to \mathbb{R}$ be a function and $\{f_n\}_{n\in\mathbb{N}}$ be a sequence of real-valued functions from *X* into $\mathbb{R}$. Recall [1]-[3] that the sequence $\{f_n\}_{n\in\mathbb{N}}$ is said to converge uniformly to *f* on *X*, if

$$\lim_{n\to+\infty}\left(\sup\left\{\left|f_n(x)-f(x)\right| : x\in X\right\}\right)=0.$$

Obviously, if $\{f_n\}_{n\in\mathbb{N}}$ converges uniformly to *f* on *X*, then for each $x\in X$ fixed, the sequence $\{f_n(x)\}_{n\in\mathbb{N}}$ converges to $f(x)$; that is, $\{f_n\}_{n\in\mathbb{N}}$ converges pointwise to *f*. It is also obvious that when *X* is finite and $\{f_n\}_{n\in\mathbb{N}}$ converges pointwise to *f* on *X*, then $\{f_n\}_{n\in\mathbb{N}}$ converges uniformly to *f* on *X*. However this converse doesn't hold in general for an arbitrary (infinite) set *X*; *i.e.*, the pointwise convergence may not imply the uniform convergence when *X* is an arbitrary (infinite) set.

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One can observe that in the mathematical literature, there are very few known results that give conditions under which a pointwise convergence implies the uniform convergence. Concerning sequences of continuous functions defined on a compact set, we have the following facts:

Proposition A. (Dini's Theorem) [4]

If *K* is a compact metric space, $f: K \to \mathbb{R}$ a continuous function, and $\{f_n\}_{n\in\mathbb{N}}$ a monotone sequence of continuous functions from *K* into $\mathbb{R}$ that converges pointwise to *f* on *K*, then $\{f_n\}_{n\in\mathbb{N}}$ converges uniformly to *f* on *K*.

Proposition B. [5]

If *E* is a Banach space and $\{T_n\}_{n\in\mathbb{N}}$ is a sequence of bounded linear operators of *E* that converges pointwise to a bounded linear operator *T* of *E*, then for every compact set $K \subset E$, $\{T_n\}_{n\in\mathbb{N}}$ converges uniformly to *T* on *K*.

(For the sake of completeness, we give the proof of this proposition in the Appendix Section).

Therefore our aim is to highlight a new basic criterion that shows in some way how a sequence of real-valued functions can converge uniformly when it is more or less obvious that the sequence converges uniformly away from a finite number of points of the closure of its domain. In the case of sequences of functions of a real variable, our criterion avoids, unlike in most classical textbooks [3] [6], the search of extrema (by differential calculus) of their general terms. Several examples that satisfy the criterion are given.

2. The Main Result (Remark)

2.1. Theorem

Let (E,d) be a metric space and $\Omega \neq \varnothing$ be a subset of *E*. Consider a sequence $\{f_n\}_{n\in\mathbb{N}}$ of functions defined from Ω to $\mathbb{R}$.

Suppose that there exists a function *f* from Ω to $\mathbb{R}$, some points $a_1,\cdots,a_k \in \overline{\Omega}$, some positive real numbers $r_1,\cdots,r_k$ and a nonnegative constant *M* such that

$$\left|f_n(x) - f(x)\right| \le M\prod_{i=1}^{k}\left[d(x,a_i)\right]^{r_i};\ \forall x\in\Omega \text{ and for all } n\in\mathbb{N}. \quad \text{(D)}$$

Suppose furthermore that for each $\varepsilon > 0$, $\{f_n\}_{n\in\mathbb{N}}$ converges uniformly to *f* on $\Omega \setminus \bigcup_{i=1}^{k} B(a_i,\varepsilon)$; where $B(a_i,\varepsilon)$ denotes the open ball of *E* centered at a_i and with radius ε.

Then the sequence of functions $\{f_n\}_{n\in\mathbb{N}}$ converges uniformly to *f* on Ω.

Proof

Let $\varepsilon > 0$ be arbitrarily fixed (it may be sufficiently small in order to be meaningful). Then for every natural number *n*, we have

$$\sup_{x\in\Omega}\left|f_n(x)-f(x)\right| \le \max\left\{M\varepsilon^{r_1+\cdots+r_k},\ \sup_{\substack{x\in\Omega,\\ d(x,a_i)\ge\varepsilon;i=1,\cdots,k}}\left|f_n(x)-f(x)\right|\right\}.$$

Thus

$$\limsup_{n\to+\infty}\left(\sup_{x\in\Omega}\left|f_n(x)-f(x)\right|\right) \le M\varepsilon^{r_1+\cdots+r_k} \qquad \forall\varepsilon>0$$

by the uniform convergence of $\{f_n\}_n$ on $\Omega \setminus \bigcup_{i=1}^{k} B(a_i,\varepsilon)$.

And so

$$\limsup_{n\to+\infty}\left(\sup_{x\in\Omega}\left|f_n(x)-f(x)\right|\right) = 0;$$

i.e.,

$$\lim_{n\to+\infty}\left(\sup_{x\in\Omega}\left|f_n(x)-f(x)\right|\right) = 0.$$

2.2. Observation

The boundedness condition (D) of the above theorem can not be removed as shown by the sequence of functions defined from $[0,1]$ into $\mathbb{R}$ as follows:

$$f_n(x) = nx(1-x)^n; \quad x \in [0,1] \subset \mathbb{R};$$

where $\mathbb{R}$ is equipped with its standard metric. Indeed, $\{f_n\}_{n\in\mathbb{N}}$ converges uniformly to 0 on $[\varepsilon,1]$ for each $\varepsilon \in (0,1)$, but with $k=1$ and $a_1 = 0$ there is no positive number r for which the condition (D) is satisfied since

$$\forall r > 0, \quad \sup_{n\in\mathbb{N}} \left(\sup_{0<x\le 1} \frac{|f_n(x)|}{|x|^r} \right) = \infty.$$

And we can see that $\{f_n\}_{n\in\mathbb{N}}$ does not converge uniformly to 0 on $[0,1]$ since

$$\lim_{n\to\infty} \left(\sup_{0\le x\le 1} |f_n(x)| \right) = \lim_{n\to\infty} \left(\frac{n}{n+1} \right)^{n+1} = \frac{1}{\mathrm{e}} \ne 0.$$

3. Examples

We give some examples that illustrate the theorem.

(1) Let (E,d) be an infinite metric space and let $a \in E$ be fixed. Denote by φ the function defined from E into $\mathbb{R}$ by

$$\varphi(x) = d(x,a), \quad \forall x \in E.$$

Then the sequence of functions $\{\varphi_n\}_{n\in\mathbb{N}}$ defined by

$$\varphi_n(x) = \frac{n[d(x,a)]^2 + d(x,a) - \min\{1, d(x,a)\}}{1 + nd(x,a)}, \quad \forall x \in E,$$

converges uniformly to φ on E.

(2) Given an infinite metric space (E,d), $a \in E$ and $\alpha \in (0,+\infty)$, we have that

i) the sequence of functions $\{u_n\}_{n\in\mathbb{N}}$ defined by

$$u_n(x) = \frac{[d(x,a)]^\alpha}{[1+d(x,a)]^n} \quad \forall x \in E,$$

converges uniformly to 0 on E,

ii) the sequence of functions $\{v_n\}_{n\in\mathbb{N}}$ defined by

$$v_n(x) = [d(x,a)]^\alpha \exp(-nd(x,a)) \quad \forall x \in E,$$

converges uniformly to 0 on E.

(3) Let (E,d) be an infinite metric space and Ω be a bounded and infinite subset of E, let a and b be two different points of $\overline{\Omega}$ and let α and β be two fixed positive numbers.

i) Consider the sequence of functions $\{f_n\}_{n\in\mathbb{N}}$ defined by

$$f_n(x) = \frac{[d(x,a)]^\alpha [d(x,b)]^\beta}{1 + nd(x,a)d(x,b)}, \quad x \in \Omega.$$

Then $\{f_n\}_{n\in\mathbb{N}}$ converges uniformly to 0 on Ω.

ii) Consider the sequence of functions $\{g_n\}_{n\in\mathbb{N}}$ defined by

$$g_n(x)=\frac{[d(x,a)]^{\alpha}[d(x,b)]^{\beta}}{[1+d(x,a)d(x,b)]^{n}},\quad x\in\Omega.$$

Then $\{g_n\}_{n\in\mathbb{N}}$ converges uniformly to 0 on Ω.

iii) Consider the sequence of functions $\{h_n\}_{n\in\mathbb{N}}$ defined by

$$h_n(x)=[d(x,a)]^{\alpha}[d(x,b)]^{\beta}\exp(-nd(x,a)d(x,b)),\quad x\in\Omega.$$

Then $\{h_n\}_{n\in\mathbb{N}}$ converges uniformly to 0 on Ω.

(4) In real analysis, we can recover the facts that each of the following sequences converges uniformly to 0 on their respective domains:

$$(1-x)x^n;\quad 0\le x\le 1,\quad n=1,2,3,\cdots.\quad (1-x)^n\sqrt{x};\quad 0\le x\le 1,\quad n=1,2,3,\cdots.$$

$$\sin^n x\cos x;\quad 0\le x\le\frac{\pi}{2},\quad n=1,2,3,\cdots.\quad \cos^n x\sin x;\quad 0\le x\le\frac{\pi}{2},\quad n=1,2,3,\cdots.$$

$$xe^{-nx};\quad x\ge 0,\quad n=1,2,3,\cdots.$$

Justifications (Proofs) of the examples

(1) For every $n\in\mathbb{N}$, we have

$$|\varphi_n(x)-\varphi(x)|=\frac{\min\{1,d(x,a)\}}{1+nd(x,a)},\quad \forall x\in E.$$

Therefore, on the one hand, for each $\varepsilon>0$, we have

$$|\varphi_n(x)-\varphi(x)|\le\frac{1}{1+n\varepsilon},\quad \forall x\in E\setminus B(a,\varepsilon),\quad \forall n\in\mathbb{N},$$

showing that $\{\varphi_n\}_{n\in\mathbb{N}}$ converges uniformly to φ on $E\setminus B(a,\varepsilon)$.

On the other hand, we have

$$|\varphi_n(x)-\varphi(x)|\le d(x,a),\quad \forall x\in E \text{ and } \forall n\in\mathbb{N}$$

fulfilling condition (D) of the above theorem.

Thus $\{\varphi_n\}_{n\in\mathbb{N}}$ converges uniformly to φ on *E*.

(2) i) On the one hand, for each $\varepsilon>0$, we have for all $x\in E\setminus B(a,\varepsilon)$ and for all $n\in\mathbb{N}$ with $n>\alpha$:

$$|u_n(x)|=u_n(x)=\frac{[d(x,a)]^{\alpha}}{[1+d(x,a)]^{\alpha}}\times\frac{1}{[1+d(x,a)]^{n-\alpha}}\le\frac{1}{(1+\varepsilon)^{n-\alpha}}$$

and so $\{u_n\}_{n\in\mathbb{N}}$ converges uniformly to 0 on $E\setminus B(a,\varepsilon)$.

On the other hand, we have

$$|u_n(x)|\le[d(x,a)]^{\alpha},\quad \forall x\in E \text{ and } \forall n\in\mathbb{N}$$

fulfilling condition (D) of the above theorem.

Thus $\{u_n\}_{n\in\mathbb{N}}$ converges uniformly to 0 on *E*.

ii) The uniform convergence of $\{v_n\}_{n\in\mathbb{N}}$, follows that of $\{u_n\}_{n\in\mathbb{N}}$ since

$$0\le v_n(x)\le u_n(x),\quad \forall x\in E \text{ and } \forall n\in\mathbb{N}.$$

Observe that the uniform convergence of $\{v_n\}_{n\in\mathbb{N}}$ could also be proved using directly the above theorem.

(3) Note that for all natural number *n*, we have

$$0\le h_n\le g_n\le f_n$$

because

$$e^{-nt} \le \frac{1}{(1+t)^n} \le \frac{1}{1+nt}, \quad \forall t \ge 0 \text{ and } \forall n \in \mathbb{N},$$

following from

$$1+nt \le (1+t)^n \le e^{nt}, \quad \forall t \ge 0 \text{ and } \forall n \in \mathbb{N}.$$

Therefore it suffices to prove that $\{f_n\}_{n\in\mathbb{N}}$ converges uniformly to 0 on Ω, although each of these three sequences can be handled directly with the above theorem.

Let δ be the diameter of Ω.

Then on the one hand, for each $\varepsilon > 0$, we have for all $x \in \Omega \setminus (B(a,\varepsilon) \cup B(b,\varepsilon))$ and for all $n \in \mathbb{N}$:

$$|f_n(x)| = f_n(x) = \frac{[d(x,a)]^\alpha [d(x,b)]^\beta}{1+nd(x,a)d(x,b)} \le \frac{\delta^{\alpha+\beta}}{1+n\varepsilon^2},$$

and so $\{f_n\}_{n\in\mathbb{N}}$ converges uniformly to 0 on $\Omega \setminus (B(a,\varepsilon) \cup B(b,\varepsilon))$.

On the other hand, we have

$$|f_n(x)| \le [d(x,a)]^\alpha [d(x,b)]^\beta, \quad \forall x \in \Omega \text{ and } \forall n \in \mathbb{N}$$

showing condition (D) of the above theorem.

Thus $\{f_n\}_{n\in\mathbb{N}}$ converges uniformly to 0 on Ω and we are done.

(4) i) Let us set $\psi_n(x) = (1-x)x^n; 0 \le x \le 1$, with $n = 1,2,3,\cdots$.

On the one hand, we have for every $n \in \mathbb{N}$:

$$|\psi_n(x)| \le |x||x-1| \quad \forall x \in [0,1].$$

On the other hand, we have for every $\varepsilon \in \left(0, \frac{1}{2}\right)$:

$$|\psi_n(x)| \le (1-\varepsilon)^n \quad \forall x \in (\varepsilon, 1-\varepsilon), \quad \text{for all } n \in \mathbb{N},$$

showing that $\{\psi_n\}_n$ converges uniformly to 0 on $(\varepsilon, 1-\varepsilon)$.

Therefore, by taking $E = \mathbb{R}$, $\Omega = [0,1]$, $a_1 = 0$, $a_2 = 1$, $r_1 = r_2 = 1$ and $M = 1$, the above theorem implies that $\{\psi_n\}_n$ converges uniformly to 0 on $[0,1]$.

ii) For $\psi_n(x) = (1-x)^n \sqrt{x}; 0 \le x \le 1$, with $n = 1,2,3,\cdots$.

On the one hand, we have for every $n \in \mathbb{N}$:

$$|\psi_n(x)| \le |x-1|\sqrt{x} \quad \forall x \in [0,1].$$

On the other hand, we have for every $\varepsilon \in \left(0, \frac{1}{2}\right)$:

$$|\psi_n(x)| \le (1-\varepsilon)^n \quad \forall x \in (\varepsilon, 1-\varepsilon), \text{ for all } n \in \mathbb{N},$$

showing that $\{\psi_n\}_n$ converges uniformly to 0 on $(\varepsilon, 1-\varepsilon)$.

Therefore, by taking $E = \mathbb{R}$, $\Omega = [0,1]$, $a_1 = 0$, $a_2 = 1$, $2r_1 = r_2 = 1$ and $M = 1$, the above theorem implies that $\{\psi_n\}_n$ converges uniformly to 0 on $[0,1]$.

iii) For $\psi_n(x) = \sin^n x \cos x; 0 \le x \le \frac{\pi}{2}$ with $n = 1,2,3,\cdots$.

On the one hand, we have for every $n \in \mathbb{N}$:

$$|\psi_n(x)| \le |x|\left|x-\frac{\pi}{2}\right| \quad \forall x \in \left[0, \frac{\pi}{2}\right].$$

On the other hand, we have for every $\varepsilon \in \left(0, \frac{\pi}{4}\right)$:

$$|\psi_n(x)| \le \sin^n\left(\frac{\pi}{2}-\varepsilon\right) = \cos^n \varepsilon \quad \forall x \in \left(\varepsilon, \frac{\pi}{2}-\varepsilon\right), \quad \text{for all } n \in \mathbb{N},$$

showing that $\{\psi_n\}_n$ converges uniformly to 0 on $\left(\varepsilon, \frac{\pi}{2}-\varepsilon\right)$ since $|\cos\varepsilon| < 1$.

Therefore, by taking $E = \mathbb{R}$, $\Omega = \left[0, \frac{\pi}{2}\right]$, $a_1 = 0$, $a_2 = \frac{\pi}{2}$, $r_1 = r_2 = 1$ and $M = 1$, the above theorem implies that $\{\psi_n\}_n$ converges uniformly to 0 on $\left[0, \frac{\pi}{2}\right]$.

iv) For $\psi_n(x) = \cos^n x \sin x;\ 0 \le x \le \frac{\pi}{2}$ with $n = 1, 2, 3, \cdots$.

On the one hand, we have for every $n \in \mathbb{N}$:

$$|\psi_n(x)| \le |x|\left|x-\frac{\pi}{2}\right| \quad \forall x \in \left[0, \frac{\pi}{2}\right].$$

On the other hand, we have for every $\varepsilon \in \left(0, \frac{\pi}{4}\right)$:

$$|\psi_n(x)| \le \cos^n \varepsilon \quad \forall x \in \left(\varepsilon, \frac{\pi}{2}-\varepsilon\right), \quad \text{for all } n \in \mathbb{N},$$

showing that $\{\psi_n\}_n$ converges uniformly to 0 on $\left(\varepsilon, \frac{\pi}{2}-\varepsilon\right)$ since $|\cos\varepsilon| < 1$.

Therefore, by taking $E = \mathbb{R}$, $\Omega = \left[0, \frac{\pi}{2}\right]$, $a_1 = 0$, $a_2 = \frac{\pi}{2}$, $r_1 = r_2 = 1$ and $M = 1$, the above theorem implies that $\{\psi_n\}_n$ converges uniformly to 0 on $\left[0, \frac{\pi}{2}\right]$.

v) The example of $\psi_n(x) = xe^{-nx};\ x \ge 0$, with $n = 1, 2, 3, \cdots$, is a particular case of Example (2)-ii) above with $E = \mathbb{R}_+$, $d(x, y) = |x - y|$ for all $x, y \in \mathbb{R}_+$, $a = 0$ and $\alpha = 1$.

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Appendix

In this section, we prove Proposition B for the sake of completeness.

Proof of Proposition B

Let $\varepsilon > 0$ be given. By the Uniform Boundedness Principle, we have that $\sup_{n\geq 1}\|T_n\| < \infty$. So let $M = \sup_{n\geq 1}\|T_n\|$. Then there exist $a_1, a_2, \cdots, a_m$ such that $K \subset \bigcup_{i=1}^{m} B\left(a_i, \frac{\varepsilon}{2(M+1)}\right)$.

Also, $\forall x \in K, \exists j \in \{1, \cdots, m\} : x \in B\left(a_j, \frac{\varepsilon}{2(M+1)}\right)$. We have that

$$\begin{aligned}\|T_n(x) - T(x)\| &\leq \|T_n(x) - T_n(a_j)\| + \|T_n(a_j) - T(a_j)\| + \|T(a_j) - T(x)\| \\ &\leq \|T_n\|\|x - a_j\| + \|T\|\|a_j - x\| + \|T_n(a_j) - T(a_j)\| \\ &\leq \frac{\varepsilon M}{M+1} + \|T_n(a_j) - T(a_j)\|.\end{aligned}$$

It follows that $\sup_{x\in K}\|T_n(x) - T(x)\| \leq \varepsilon + \max_{1\leq i\leq m}\|T_n(a_i) - T(a_i)\|$ and therefore

$$\sup_{x\in K}\|T_n(x) - T(x)\| \to 0, \text{ as } n \to +\infty.$$