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On Dislocated Metric Topology

Mohamed A. Ahmed, F. M. Zeyada, G. F. Hassan*
Department of Mathematics, Faculty of Science, Assiut University, Qesm Than Asyut, Egypt
Email: mahmed68@yahoo.com, zeyada1@hotmail.com

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ABSTRACT

In this paper, we give a comment on the dislocated-neighbourhood systems due to Hitzler and Seda [1]. Also, we recover the open sets of the dislocated topology.

Keywords: Generalized Topology; Dislocated Neighbourhood Systems; Dislocated Metric

1. Introduction

In recent years, the role of topology is of fundamental importance in quantum particle physics and in logic programming semantics (see, e.g. [2-6]). Dislocated metrics were studied under the name of metric domains in the context of domain theory (see, [7]). Dislocated topologies were introduced and studied by Hitzler and Seda [1].

Now, we recall some definitions and a proposition due to Hitzler and Seda [1] as follows.

Definition 1.1. Let X be a set. $d: X\times X\to[0,\infty)$ is called a **distance function**. Consider the following conditions, for all $x,y,z\in X$,

(d_1) $d(x,x)=0$;
(d_2) if $d(x,y)=0$, then $x=y$;
(d_3) $d(x,y)=d(y,x)$;
(d_4) $d(x,y)\le d(x,z)+d(z,y)$.

If d satisfies conditions (d_1) - (d_4), then it is called a **metric** on X. If it satisfies conditions (d_2) - (d_4), then it is called a **dislocated metric** (or simply d-metric) on X.

Definition 1.2. Let X be a set. A distance function d is called a **partial metric** on X if it satisfies (d_3) and the conditions:

(d_5) $x=y$ if and only if $d(x,x)=d(x,y)=d(y,y)$;
(d_6) $d(x,x)\le d(x,y)$;
(d_7) $d(x,z)\le d(x,y)+d(y,z)-d(y,y)$,
for each $x,y,z\in X$.

It is obvious that any partial metric is a d-metric.

Definition 1.3. An **(open $\epsilon-$) ball** in a d-metric space (X,d) with centre $x\in X$ is a set of the form $B_\epsilon(x)=\{y\in X: d(x,y)<\epsilon\}$, where $\epsilon>0$.

It is clear that $B_\epsilon(x)$ may be empty in a d-metric space (X,d) because the centre x of the ball $B_\epsilon(x)$ doesn't belong to $B_\epsilon(x)$.

Definition 1.4. Let X be set. A relation $R\subseteq X\times P(X)$ is called a ***d*-membership relation** (on X) if it satisfies the following property for all $x\in X$ and $A,B\subseteq X$: xRA and $A\subseteq B$ implies xRB.

It is noted that the "d-membership"-relation is a generalization of the membership relation from the set theory.

In the sequel, any concept due to Hitzler and Seda will be denoted by "HS".

Definition 1.5. Let X be a nonempty set. Suppose that R is a d-membership relation on X and $u_x\ne\varnothing$ is a collection of subsets of X for each $x\in X$. We call (u_x,R) **a *d*-neighbourhood system** (d-nbhood system) for x if it satisfies the following conditions:

(Ni) if $U\in u_x$, then $x<U$;
(Nii) if $U,V\in u_x$, then $U\bigcap V\in u_x$;
(Niii) if $U\in u_x$, then there is a $V\subseteq U$ with $V\in u_x$ such that for all yRV we have $U\in u_y$;
(Niv) if $U\in u_x$ and $U\subseteq V$, then $V\in u_x$.

Each $U\in u_x$ is called an **HS-*d*-neighborhood** (HS d-nbhood) of x. The ordered triple (X,u,R) is called an **HS-*d*-topological space** where $u=\{u_x: x\in X\}$.

Proposition 1.1. Let (X,d) be a d-metric space. Define the d-membership relation R as the relation $\{(x,A):\text{there }\epsilon>0\text{ for which }B_\epsilon(x)\subseteq A\}$. For each $x\in X$, let u_x be the collection of all subsets A of X such that xRA. Then (u_x,R) is an HS d-nbhood system for x for each $x\in X$, *i.e.*, (X,u,R) is an HS

*Current address: Department of Mathematics, Faculty of Science, Northern Boarders University, Arar, KSA.

d-topological neighbourhood space.

The present paper is organized as follows. In Section 2, we redefine the dislocated neighbourhood systems given due to Hitzler and Seda [1]. Section 3 is devoted to define the concept of dislocated topological space by open sets. In Section 4, we study topological properties of dislocated closure and dislocated interior operation of a set using the concept of open sets. Finally, in Section 5, we study some further properties of the well-known notions of dislocated continuous functions and dislocated convergence sequence via *d*-topologies.

2. Redefinition of Definition 1.5.

In Proposition 1.1, it is proved that (X,u,R) is an HS *d*-topological neighbourhood space. We remark that Property (Niii) can be replaced by the following condition:

(Niii) $*$ If $U \in u_x$, then for each $yRU, U \in u_y$.

One can easily verifies that (X,u,R) satisfies (Niii) *.

According to the above comment, we introduce a redefinition of the concept of the dislocated-neighbourhood systems due to Hitzler and Seda [1] as follows.

Definition 2.1. Let X be a nonempty set. Suppose that R is a *d*-membership relation on X and $u_x \neq \varnothing$ be a collection of subsets of X for each $x \in X$. We call (u_x, R) **a *d**-neighbourhood system** (*d**-nbhood system) for x if it satisfies the following conditions:

(Ni) if $U \in u_x$, then xRU;

(Nii) if $U, V \in u_x$, then $U \bigcap V \in u_x$;

(Niii)* if $U \in u_x$ and yRU, then $U \in u_y$;

(Niv) if $U \in u_x$ and $U \subseteq V$, then $V \in u_x$.

Each $U \in u_x$ is called a ***d**-neighborhood** of x. If $u = \{u_x : x \in X\}$, then (X,u,R) is called a ***d**-topological neighborhood space**.

Now, we state the following theorem without proof.

Theorem 2.1. Let (X,d) be a *d*-metric space. Define the *d*-membership relation R as the relation xRA iff there exists $\epsilon > 0$ for which $B_\epsilon(x) \subseteq A$. Assume that $u_x = \{A : A \subseteq X \text{ and } xRA\}$ and $u = \{u_x : x \in X\}$. Then (X,u,R) is a *d**-topological neighborhood space.

3. Dislocated-Topological Space

In what follows we define the concept of dislocated-topological space (for short, *d*-topological space) by the open sets and prove that this concept and the concept of d*-topological neighborhood space are the same.

Definition 3.1. Let X be a nonempty set. Suppose that R is a *d*-membership relation and $\tau_x = \{A \subseteq X : xRA\}$ for each $x \in X$. We call τ_x an xd **-topology** on X iff it satisfies the following conditions:

($d\tau_x$1) $X \in \tau_x$;

($d\tau_x$2) $A, B \in \tau_x \Rightarrow A \bigcap B \in \tau_x$;

($d\tau_x$3) $A \subseteq B$ and $A \in \tau_x \Rightarrow B \in \tau_x$.

Each $A \in \tau_x$ is called a $d\tau_x$**-open set**. If τ_x is an xd -topology on X for each $x \in X$, then $\tau = \bigcup_{x \in X} \tau_x$ is called a ***d*-topology** on X. The triple (X, τ_x, R) is called an xd **-topological space** and the triple (X,τ,R) is called a **d-topological space**.

Definition 3.2. Let (X,τ,R) be an xd -topological space. A is called a $d\tau_x$**-closed** iff A^c is a $d\tau_x$-open..

Theorem 3.1. The concepts of *d**-topological neighborhood space and *d*-topological space are the same.

Proof. Let $d^*TNS(X)$ be the family of all *d**-topological neighbourhood systems on X and let $dT(X)$ be the family of all *d*-topologies on X. The proof is complete if we point out a bijection between $d^*TNS(X)$ and $dT(X)$. Let $H : d^*TNS(X) \to dT(X)$ and $K : dT(X) \to d^*TNS(X)$ be functions defined as follows: $H((X,u,R)) = (X,\tau,R)$, where $\tau_x = u_x$ for each $x \in X$ and $K((X,\tau,R)) = (X,u,R)$, where $u_x = \tau_x$ for each $x \in X$. One can easily verifies that these functions are well defined, $HoK = id_{dT(X)}$ and $KoH = id_{d^*TNS(X)}$.

The following counterexample illustrates that the statement: xRA iff xRA^c may not be true.

Counterexample 3.1. Let $X = \{x,y,z\}$ and $R = \{(x,\{x\}),(x,\{x,y\}),(x,\{x,z\}),(x,\{y,z\}),(x,X)\}$. Then R is a *d*-membership relation. Since $(x,\{y,z\}) \in R$, then $xR\{x\}^c$, *i.e.* $\exists\, A = \{x\} \subseteq X$ such that xRA and xRA^c.

We get the following theorem without proof.

Theorem 3.2. Let X be a nonempty set. Suppose that R is a *d*-membership relation and $F_x = \{A \subseteq X : xRA^c\}$ for each $x \in X$. Assume that F_x satisfies the following conditions:

(dF_x1) $\varnothing \in F_x$;

(dF_x2) $A, B \in F_x \Rightarrow A \bigcup B \in F_x$;

(dF_x3) $A \subseteq B$ and $B \in F_x \Rightarrow A \in F_x$.

Then (X,τ,R) is a *d*-topology on X, where $\tau_x = \{A^c : A \subseteq X \text{ and } A \in F_x\}$. If (X,τ,R) is a *d*-topological space, then for each $x \in X$ the family F_x of all $d\tau_x$-closed sets satisfies the conditions (dF_x1)-(dF_x3).

4. Dislocated Closure and Dislocated Interior Operations

In the sequel we define the dislocated closure and dislocated interior operations of a set and study some topological properties of dislocated closure and dislocated interior operation.

Definition 4.1. Let (X,τ_x,R) be an xd -topological

space. The $d\tau_x$ **-interior** of a subset A of X is denoted and defined by:
$d\tau_x - int(A) = \bigcup\{B : B \subseteq A \text{ and } B \in \tau_x\}$.

Remark 4.1. From Definition 4.1, if $\varnothing \notin \tau_x$, then $d\tau_x - int(\varnothing)$ is undefined. If $\varnothing \in \tau_x$, then $d\tau_x - int(\varnothing)$ is defined.

Theorem 4.1. Let (X, τ_x, R) be an xd -topological space.

(A) If $\varnothing \in \tau_x$, then $d\tau_x - int(A) = A$ for each $A \subseteq X$.

(B) If $\varnothing \notin \tau_x$, then

(i) $d\tau_x - int(X) = X$;

(ii) $d\tau_x - int(A) \subseteq A$ for each $A \subseteq X$;

(iii) $d\tau_x - int(A \cap B) = (d\tau_x - int(A)) \cap (d\tau_x - int(B))$ for each $A, B \in P(X)$;

(iv) $d\tau_x - int(A) = A$ or $\varnothing$ for each $A \subseteq X$.

(v) $d\tau_x - int(d\tau_x - int(A)) = d\tau_x - int(A)$ if $\varnothing \in \tau_x$ or $d\tau_x - int(A) = A$.

Corollary 4.1. (1) If $d\tau_x - int(A) = A$, then $d\tau_x - int(A)$ is a $d\tau_x$-open.

(2) If $A \in \tau_x$, then $d\tau_x - int(A) = A$.

Theorem 4.2. If $\theta : P(X) - \{\varnothing\} \to P(X)$ such that the conditions B(i), B(iii) and B(iv) are satisfied then $\tau_x = \{A : \theta(A) = A \text{ and } A \neq \varnothing\}$ is an xd -topology on X. The d -membership relation is defined as xRA iff $A \in \tau_x$.

Proof. The desired result is obtained from the following:

(I) ($d\tau_x$1) $X \in \tau_x$ since $\theta(X) = X$;

($d\tau_x$2) $A, B \in \tau_x \Rightarrow \theta(A) = A$ and

$$\begin{aligned}\theta(B) = B &\Rightarrow \theta(A \cap B) = \theta(A) \cap \theta(B) \\ &= A \cap B \Rightarrow A \cap B \in \tau_x\end{aligned};$$

($d\tau_x$3) $A \subseteq B$ and $A \in \tau_x \Rightarrow A \neq \varnothing$, $\theta(A) = A \Rightarrow \theta(B) = B$ (from B(iii)-(iv)).

(II) xRA and $A \subseteq B \Rightarrow A \in \tau_x$ and $A \subseteq B \Rightarrow B \in \tau_x$ (from I ($d\tau_x$3)).

Definition 4.2. Let (X, τ_x, R) be an xd -topological space. The $d\tau_x$ **-closure** of a subset A of X is denoted and defined by:
$d\tau_x - cl(A) = \bigcap\{B : A \subseteq B \text{ and } \in F_x\}$.

If $\varnothing \notin \tau_x$, then $d\tau_x - cl(X)$ is undefined but if $\varnothing \in \tau_x$, then $d\tau_x - cl(X)$ is defined.

Theorem 4.3. Let (X, τ_x, R) be an xd -topological space. Then for each $A \subseteq X$,

$$d\tau_x - cl(A^c) = (d\tau_x - int(A))^c.$$

Proof.

$$\begin{aligned}(d\tau_x - cl(A^c))^c &= \left(\bigcap\{B : A^c \subseteq B \text{ and } \in F_x\}\right)^c = \bigcup\{B^c : B^c \subseteq A \text{ and } B^c \in \tau_x\} \\ &= \bigcup\{H : H \subseteq A \text{ and } H \in \tau_x\} = d\tau_x - int(A)\end{aligned}$$

From Theorems 4.1 and 4.3, we obtain the following theorem without proof.

Theorem 4.4. Let (X, τ_x, R) be an xd -topological space.

(A) If $\varnothing \notin \tau_x$, then $d\tau_x - cl(A) = A$ for each $A \subseteq X$.

(B) If $\varnothing \notin \tau_x$, then

(i) $d\tau_x - cl(\varnothing) = \varnothing$;

(ii) $d\tau_x - cl(A) \supseteq A$ for each $A \subseteq X$;

(iii) $d\tau_x - cl(A \cup B) = (d\tau_x - cl(A)) \cup (d\tau_x - cl(B))$;

(iv) $d\tau_x - cl(A) = A$ or X for each $A \subseteq X$;

(v) $d\tau_x - cl(d\tau_x - cl(A)) = d\tau_x - cl(A)$ if $\varnothing \in \tau_x$ or $d\tau_x - cl(A) = A$.

Corollary 4.2. (1) If $d\tau_x - cl(A) = A$, then $d\tau_x - cl(A)$ is a $d\tau_x$-closed.

(2) If $A \in F_x$, then $d\tau_x - cl(A) = A$.

5. Dislocated Continuous Functions and Dislocated Convergence Sequences via *d*-Topologies

Now, we define the dislocated continuous functions and dislocated convergence sequences. We also obtain a decomposition of dislocated continuous function and dislocated convergence sequences.

Definition 5.1. Let (X, d_1) and (X, d_2) be dislocated-metric spaces. A function $f : X \to Y$ is called ***d*-continuous** at $x_0 \in X$ iff $\forall \epsilon > 0, \exists \delta(\epsilon) > 0$ such that $d_1(x, x_0) < \delta(\epsilon) \Rightarrow d_2(f(x), f(x_0)) < \epsilon$. We say f is ***d*-continuous** iff f is *d*-continuous at each $x_0 \in X$

Theorem 5.1. Let (X, d_1) and (Y, d_2) be dislocated-metric spaces and $f : X \to Y$ be any function. Assume that (X, τ, R) (resp. (Y, σ, R)) be the *d*-topological space obtained from (X, d_1) (resp. (Y, d_2)). Then the following statements are equivalent:

(1) f is *d*-continuous at $x_0 \in X$.

(2) $\forall u \in \sigma_{f(x_0)}, f^{-1}(u) \in \tau_{x_0}$.

(3) $\forall v \in V_{f(x_0)}, \exists u \in u_{x_0}$ such that $f(u) \subset v$, where $V_{f(x_0)}$ and u_{x_0} are the d*-topological neighborhood systems obtained from (X, d_1) and (Y, d_2) respectively.

(4) $\forall \epsilon > 0 \, \exists \delta > 0$ such that $f(B_\delta(x_0)) \subseteq B_\epsilon(f(x_0))$.

Proof. ((1)⇒(2)): Let $u \in \sigma_{f(x_0)}$. Then $\exists \epsilon > 0$ such that $B_\epsilon(f(x_0)) \subseteq u$. Thus $\exists \delta(\epsilon) > 0$ such that $d_1(x, x_0) < \delta(\epsilon) \Rightarrow d_2(f(x), f(x_0)) < \epsilon$, *i.e.*, $\forall x \in B_{\delta(\epsilon)}(x_0)$, $f(x) \in B_\epsilon(f(x_0)) \subseteq u$, then

$B_{\delta(\epsilon)}(x_0) \subseteq f^{-1}(u)$. Hence $f^{-1}(u) \in \tau_{x_0}$.

((2)⇒(1)): Let $\epsilon > 0$. Suppose that for each $\delta > 0$, $\exists x \in X$ such that $d_1(x, x_0) < \delta(\epsilon) \Rightarrow d_2(f(x), f(x_0)) \geq \varepsilon$. Now, $B_\epsilon(f(x_0)) \in \sigma_{f(x_0)}$. From the assumption $f^{-1}(B_\epsilon(f(x_0))) \in \tau_{x_0}$, *i.e.*, $\exists \delta(\epsilon) > 0$ such that $x \in B_\delta(x_0) \subseteq f^{-1}(B_\epsilon(f(x_0)))$. Then $f(x) \in B_\epsilon(f(x_0))$. The contradiction demands that f is *d*-continuous at $x_0 \in X$.

(1) ⇔ (4) and (2) ⇔ (3) are immediate.

Definition 5.2. Let (X, d) be a *d*-metric space. A sequence (x_n) ***d*-converges** to $x \in X$ if $\forall \epsilon > 0\ \exists n_0 \in N$ such that $\forall n \geq n_0$, $d(x_n, x) < \epsilon$.

Theorem 5.2. Let (X, d) be a *d*-metric space and (X, τ, R) be the *d*-topological space obtained from it. Then the sequence (x_n) *d*-converges to $x \in X$ iff $\forall u \in \tau_x, \exists n_0 \in N$ such that for each $n \geq n_0, x_n \in u$.

Proof. (⇒:) Let $u \in \tau_x$. Then there exists $\epsilon > 0$ such that $B_\epsilon(x) \subseteq u$. From the assumption $\exists n_0 \in N$ such that $\forall n \geq n_0, (x_n, x) < \epsilon$. Thus $x_n \in B_\epsilon(x)$ for each $n \geq n_0$. So $x_n \in u$ for each $n \geq n_0$.

(⇐:) Let $\epsilon > 0$. Since $B_\epsilon(x) \subseteq B_\epsilon(x)$, then $B_\epsilon(x) \in \tau_x$. Thus $\exists n_0 \in N$ such that for each $n \geq n_0, x_n \in B_\varepsilon(x)$,*i.e.*, $d(x_n, x) < \epsilon$ for each $n \geq n_0$. Hence $\lim_{n\to\infty} d(x_n, x) = 0$.

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