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$L(2,1)$-Labeling of the Brick Product Graphs

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Abstract

A k-$L(2,1)$-labeling for a graph G is a function $f:V(G)\to\{0,1,\cdots,k\}$ such that $|f(u)-f(v)|\ge 2$ whenever $uv\in E(G)$ and $|f(u)-f(v)|\ge 1$ whenever u and v are at distance two apart. The λ-number for G, denoted by $\lambda(G)$, is the minimum k over all k-$L(2,1)$-labelings of G. In this paper, we show that $Br(2\ell,m,r)\le 6$ for $\ell=9$ or 11, which confirms Conjecture 6.1 stated in [X. Li, V. Mak-Hau, S. Zhou, The $L(2,1)$-labelling problem for cubic Cayley graphs on dihedral groups, J. Comb. Optim. (2013) 25: 716-736] in the case when $\ell=9$ or 11. Moreover, we show that $Br(2\ell,m,r)=5$ if 1) either $\ell\equiv 0 \pmod 6$, m is odd, $r=3$, or 2) $\ell\equiv 0 \pmod 3$, m is even (mod 2), $r=0$.

Keywords

Graph Labeling, Brick Product Graph, $L(2,1)$-Labeling, Frequency Assignment Problem

1. Introduction

Let $G=(V,E)$ be a graph. For two vertices u and v in G, the distance between u and v is the number of the edges of the shortest path between u and v. A k-$L(2,1)$-labeling for a graph G is a function $f:V(G)\to\{0,1,\cdots,k\}$ such that $|f(u)-f(v)|\ge 2$ whenever $uv\in E(G)$ and $|f(u)-f(v)|\ge 1$ whenever u and v are at distance two apart. The λ-number for G, denoted by $\lambda(G)$, is the minimum k over all k-$L(2,1)$-labelings of G. This labeling problem of graphs was proposed by Griggs and Roberts [1] which is a variation of the frequency assignment problem introduced by Hale [2]. The frequency assignment problem asks for assigning frequencies to transmitters in a broadcasting network with the aim of avoiding undesired interference. One of the graph theoretical models of the frequency assignment problem is the notion of distance constrained labeling of graphs [3] [4] [5].

The *L*(2,1)-labeling problem was studied very extensively in the literature and has attracted much attention. Griggs and Yeh [6] proposed a conjecture, which is called the Δ^2-conjecture, that $\lambda(G) \le \Delta^2$ for any graph with $\Delta \ge 2$, where Δ is the maximum degree of *G*, and they also proved that $\lambda(G) \le \Delta^2 + 2\Delta$. Later, it was shown that $\lambda(G) \le \Delta^2 + \Delta$ by Chang and Kuo [7], $\lambda(G) \le \Delta^2 + \Delta - 1$ by Král' and Škrekovski [8], and then $\lambda(G) \le \Delta^2 + \Delta - 2$ by Goncalves [9]. Until now, this conjecture is still open. Nevertheless, it is still interesting to study the Δ^2-conjecture, which has been confirmed for several classes of graphs, such as chordal graphs, outerplanar graphs, generalized Petersen graphs, Hamiltonian graphs with $\Delta \le 3$, two families of Hamming graphs etc (see [10]). Havet *et al.* obtained a result implying that the Δ^2-conjecture is true for graphs with sufficiently large Δ. Thus, we may need to study the *L*(2,1)-labelling problem for graphs with small Δ. Motivated with this, the *L*(2,1)-labelling problem for the brick product graphs was studied [10].

Let $\ell \ge 2$, $m \ge 1$ and $r \ge 0$ be integers such that $m + r$ is even. Let $C_{2\ell}$ be a cycle of length 2ℓ. The (m,r)-*brick-product* of $C_{2\ell}$, denoted by $Br(2\ell,m,r)$, is the graph with adjacency defined in two cases. For $m = 1, r \ge 3$ must be odd and $Br(2\ell,1,r)$ is obtained from the cycle $C_{2\ell} = (v_0, v_1, v_2, \cdots, v_{2\ell-1}, v_0)$ by adding chords joining v_{2i} and v_{2i+r} for $i = 0,1,\cdots,\ell-1$, where subscripts are taken modulo 2ℓ. In the general case where $m \ge 2$, $Br(2\ell,m,r)$ is obtained by first taking the vertex-disjoint union of *m* copies of $C_{2\ell}$, denoted by

$$C_{2\ell}(i) = (v_{i,0}, v_{i,1}, \cdots, v_{i,2\ell-1}, v_{i,0}), i = 0,1,\cdots,m-1. \quad (1)$$

Next, for each pair $(i,j) \in \{0,1,\cdots,m-2\} \times \{0,1,\cdots,2\ell-1\}$ such that *i* and *j* have the same parity, an edge is added to join $v_{i,j}$ and $v_{i+1,j}$. Finally, for odd $j = 1,3,\cdots,2\ell-1$, an edge is added to join $v_{0,j}$ and $v_{m-1,j+r}$, where the second subscript is modulo 2ℓ.

Li *et al.* [10] proposed the following conjecture:

Conjecture 1. [10] $\lambda(Br(2\ell,m,r)) = 5$ or 6 for all brick products $Br(2\ell,m,r)$ with $m \ge 2$ and $m + r \equiv 0 (\bmod 2\ell)$

Shao *et al.* [11] confirmed the above conjecture, *i.e.* it was proved that

Theorem 1. [11] $\lambda(Br(2\ell,m,r)) \le 6$ if 1) ℓ is even, or 2) $\ell \ge 5$ is odd and $0 \le r \le 8$.

Therefore, Conjecture 1 is still open for odd ℓ and $r > 8$.

In this paper, we show that $Br(2\ell,m,r) \le 6$ for $\ell = 9$ or 11, which confirms Conjecture 6.1 stated in [X. Li, V. Mak-Hau, S. Zhou, The *L*(2,1)-labelling problem for cubic Cayley graphs on dihedral groups, J. Comb. Optim. (2013) 25: 716-736] in the case when $\ell = 9$ or 11. Moreover, we show that $Br(2\ell,m,r) = 5$ if 1) either $\ell \equiv 0$ (mod 6), *m* is odd, $r = 3$, 2) or $\ell \equiv 0$ (mod 3), *m* is even (mod 2), $r = 0$.

2. Main Results

From the definition of the brick product graph, it is clear that

Fact 1. $Br(2\ell, m, r)$ is isomorphic to $Br(2\ell, m, 2\ell - r)$.

2.1. Some Results on the Upper Bound 6 of λ-Number

In [6], it was shown that

Lemma 1. [6] The λ-number of any connected cubic graph is at least 5.

Proposition 1. Let $\ell = 9$. Then $\lambda(Br(2\ell, m, r)) \le 6$ for all $m \ge 3$.

By Theorem 1, we have $\lambda(Br(2\ell, m, r)) \le 6$ for all $m \ge 3$ and $r \le 8$. Together with Fact 1, we only need to consider $r = 9$. Let

$$P_3 = \begin{bmatrix} 1 & 3 & 2 \\ 4 & 0 & 4 \\ 2 & 6 & 6 \\ 0 & 4 & 0 \\ 6 & 1 & 2 \\ 4 & 3 & 5 \\ 2 & 0 & 0 \\ 6 & 6 & 4 \\ 1 & 3 & 2 \\ 5 & 5 & 0 \\ 0 & 2 & 6 \\ 3 & 4 & 1 \\ 1 & 6 & 3 \\ 4 & 2 & 5 \\ 6 & 0 & 0 \\ 2 & 5 & 2 \\ 4 & 1 & 4 \\ 6 & 6 & 0 \end{bmatrix}, P_5 = \begin{bmatrix} 4 & 6 & 4 & 6 & 2 \\ 2 & 2 & 0 & 3 & 5 \\ 0 & 4 & 5 & 1 & 0 \\ 6 & 6 & 2 & 6 & 2 \\ 4 & 1 & 0 & 4 & 5 \\ 2 & 5 & 3 & 2 & 0 \\ 6 & 0 & 1 & 6 & 6 \\ 3 & 3 & 5 & 4 & 2 \\ 5 & 1 & 2 & 0 & 5 \\ 0 & 6 & 4 & 3 & 1 \\ 4 & 2 & 1 & 5 & 6 \\ 6 & 0 & 6 & 0 & 4 \\ 1 & 5 & 4 & 2 & 1 \\ 3 & 3 & 1 & 5 & 3 \\ 0 & 6 & 6 & 0 & 0 \\ 4 & 1 & 3 & 4 & 6 \\ 2 & 5 & 5 & 2 & 1 \\ 0 & 3 & 1 & 0 & 4 \end{bmatrix}, P_7 = \begin{bmatrix} 2 & 4 & 2 & 4 & 3 & 1 & 0 \\ 0 & 6 & 0 & 6 & 0 & 6 & 2 \\ 5 & 2 & 5 & 2 & 5 & 3 & 4 \\ 3 & 4 & 1 & 4 & 1 & 0 & 6 \\ 1 & 6 & 3 & 6 & 6 & 2 & 3 \\ 5 & 2 & 5 & 2 & 0 & 4 & 0 \\ 0 & 4 & 0 & 4 & 5 & 1 & 5 \\ 6 & 6 & 3 & 6 & 3 & 6 & 3 \\ 2 & 0 & 5 & 2 & 1 & 4 & 1 \\ 5 & 4 & 1 & 0 & 6 & 2 & 5 \\ 3 & 6 & 6 & 3 & 3 & 0 & 3 \\ 0 & 1 & 4 & 1 & 5 & 4 & 1 \\ 6 & 3 & 0 & 6 & 0 & 2 & 6 \\ 1 & 5 & 2 & 2 & 4 & 5 & 0 \\ 4 & 0 & 6 & 0 & 6 & 3 & 2 \\ 2 & 2 & 4 & 3 & 1 & 0 & 4 \\ 0 & 5 & 0 & 5 & 4 & 2 & 1 \\ 6 & 1 & 6 & 1 & 6 & 5 & 3 \end{bmatrix}.$$

We use the pattern P_m to label $Br(18, m, 9)$ for $m \in \{3, 5, 7\}$, and P_m induces a 6-$L(2,1)$-labeling of $Br(18, m, 9)$. Therefore, the case $m < 9$ is settled.

$$Q_9 = \left[\begin{array}{cccc|ccccc} 1 & 6 & 4 & 6 & 0 & 2 & 6 & 2 & 1 \\ 5 & 3 & 1 & 3 & 5 & 4 & 1 & 0 & 3 \\ 2 & 0 & 5 & 0 & 1 & 6 & 3 & 6 & 6 \\ 4 & 6 & 2 & 6 & 3 & 2 & 0 & 1 & 4 \\ 1 & 3 & 4 & 1 & 0 & 4 & 5 & 3 & 2 \\ 6 & 0 & 6 & 3 & 6 & 6 & 1 & 0 & 5 \\ 4 & 2 & 1 & 5 & 4 & 2 & 4 & 2 & 3 \\ 0 & 5 & 3 & 2 & 0 & 5 & 0 & 5 & 0 \\ 3 & 1 & 6 & 4 & 6 & 1 & 6 & 3 & 2 \\ 5 & 4 & 2 & 1 & 3 & 4 & 2 & 1 & 6 \\ 0 & 6 & 0 & 6 & 0 & 6 & 0 & 4 & 3 \\ 2 & 1 & 5 & 4 & 2 & 3 & 5 & 2 & 0 \\ 5 & 3 & 3 & 1 & 5 & 1 & 1 & 6 & 6 \\ 0 & 0 & 6 & 6 & 0 & 6 & 4 & 4 & 2 \\ 4 & 2 & 4 & 2 & 4 & 2 & 2 & 0 & 0 \\ 1 & 5 & 1 & 5 & 1 & 0 & 6 & 6 & 4 \\ 6 & 3 & 6 & 0 & 6 & 3 & 3 & 1 & 2 \\ 4 & 0 & 2 & 2 & 4 & 5 & 0 & 4 & 6 \end{array}\right], Q_{11} = \left[\begin{array}{cccc|ccccccc} 6 & 1 & 4 & 6 & 0 & 6 & 6 & 4 & 5 & 1 & 2 \\ 4 & 5 & 2 & 1 & 4 & 3 & 1 & 1 & 3 & 6 & 0 \\ 0 & 3 & 0 & 3 & 6 & 0 & 4 & 6 & 0 & 2 & 5 \\ 2 & 1 & 4 & 5 & 2 & 5 & 2 & 2 & 5 & 4 & 1 \\ 4 & 6 & 6 & 1 & 4 & 1 & 6 & 4 & 1 & 6 & 3 \\ 0 & 3 & 0 & 3 & 6 & 3 & 0 & 0 & 3 & 2 & 5 \\ 5 & 1 & 2 & 5 & 0 & 5 & 5 & 2 & 5 & 0 & 0 \\ 3 & 6 & 4 & 1 & 3 & 1 & 3 & 4 & 1 & 4 & 6 \\ 0 & 2 & 0 & 6 & 6 & 4 & 6 & 0 & 6 & 2 & 3 \\ 4 & 5 & 3 & 2 & 0 & 0 & 2 & 5 & 3 & 0 & 5 \\ 6 & 0 & 1 & 5 & 3 & 5 & 4 & 1 & 1 & 6 & 2 \\ 2 & 3 & 6 & 0 & 6 & 1 & 6 & 6 & 4 & 3 & 0 \\ 4 & 1 & 2 & 4 & 2 & 4 & 3 & 0 & 0 & 5 & 6 \\ 0 & 6 & 0 & 6 & 0 & 0 & 5 & 4 & 2 & 1 & 4 \\ 5 & 3 & 5 & 3 & 4 & 2 & 1 & 6 & 5 & 3 & 2 \\ 2 & 0 & 2 & 0 & 6 & 6 & 3 & 3 & 1 & 0 & 5 \\ 4 & 6 & 6 & 4 & 3 & 1 & 0 & 5 & 6 & 2 & 1 \\ 0 & 3 & 0 & 2 & 5 & 4 & 2 & 2 & 0 & 4 & 6 \end{array}\right].$$

Now, we consider the case $m \ge 9$. If $m = 4k + 5$ for $k \ge 1$, we obtain a 6-$L(2,1)$-labeling of $Br(18, m, 9)$ by repeating the leftmost four columns of Q_9; If $m = 4k + 7$ for $k \ge 1$, we obtain a 6-$L(2,1)$-labeling of $Br(18, m, 9)$ by repeating the leftmost four columns of Q_{11} (see **Figure 1**). Therefore, $\lambda(Br(2\ell, m, r)) \le 6$ for $\ell = 9$ and $m \ge 3$.

Proposition 2. Let $\ell = 11$. Then $\lambda(Br(2\ell, m, r)) \le 6$ for all $m \ge 3$.

Similar to Proposition 1, we only need to consider the case $r = 9$ and 11.

Case 1: $r = 9$.

We use the following pattern P_m to label $Br(22, m, 9)$ for $m \in \{3, 5\}$, and P_m induces a 6-$L(2,1)$-labeling of $Br(22, m, 9)$. Therefore, the case $m \le 5$ is settled. Now, we consider the case $m \ge 7$. If $m = 4k + 3$ for $k \ge 1$, we obtain a 6-$L(2,1)$-labeling of $Br(22, m, 9)$ by repeating the leftmost four columns of Q_7; If $m = 4k + 5$ for $k \ge 1$, we obtain a 6-$L(2,1)$-labeling of $Br(22, m, 9)$ by repeating the leftmost four columns of Q_9. Therefore, $\lambda(Br(2\ell, m, r)) \le 6$ for $\ell = 11$ and $m \ge 3$.

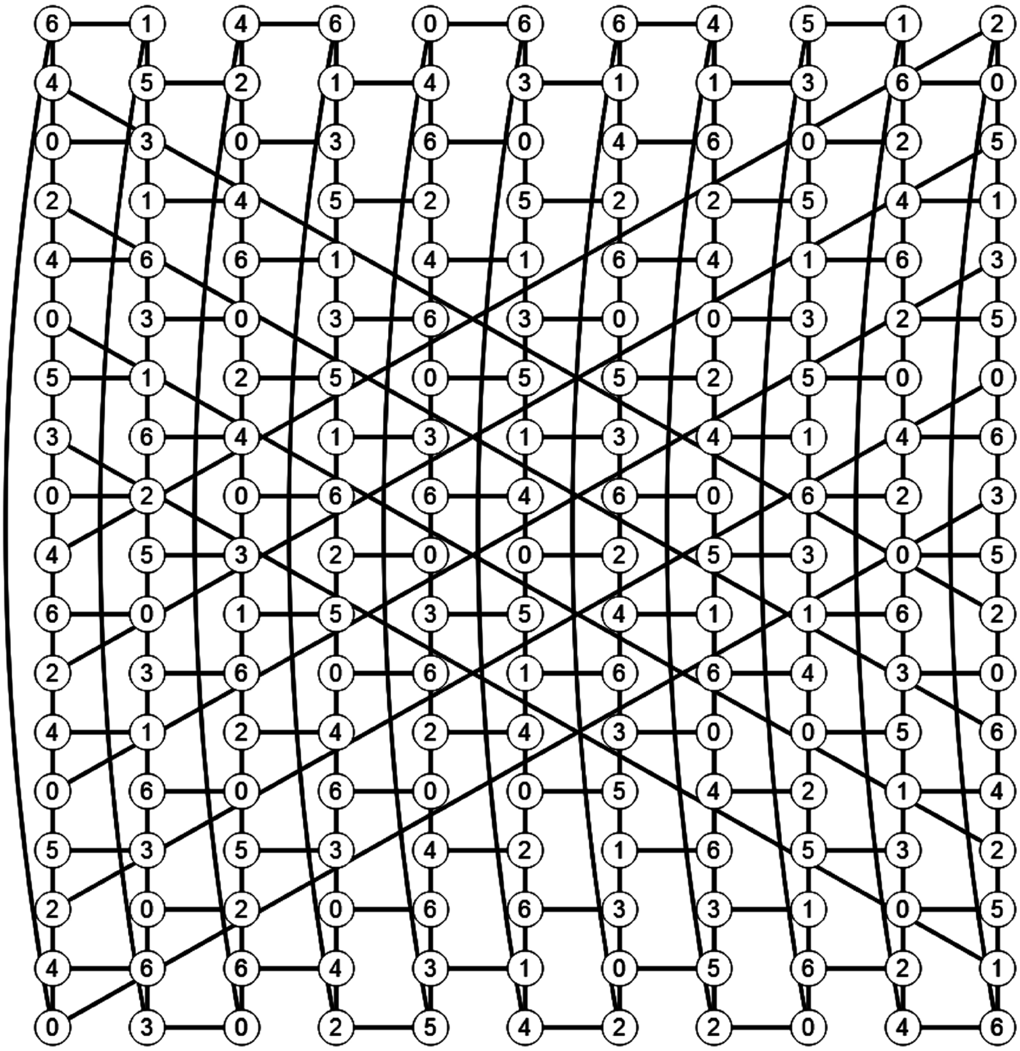

Figure 1. The 6-$L(2,1)$-labeling of $Br(18, 11, 9)$ induced by Q_{11}.

$$P_3=\begin{bmatrix}3&0&0\\1&1&3\\0&4&4\\3&3&0\\4&1&1\\2&0&4\\1&3&3\\4&4&1\\3&0&0\\1&1&3\\0&4&4\\3&3&0\\4&1&1\\2&2&4\\3&0&0\\1&1&3\\0&4&2\\3&3&0\\4&1&4\\0&0&2\\1&3&1\\4&2&4\end{bmatrix},P_5=\begin{bmatrix}4&1&1&3&1\\2&0&4&2&4\\1&3&3&0&0\\4&4&1&1&3\\3&0&0&4&2\\1&1&3&3&0\\2&4&4&1&4\\0&0&2&0&2\\1&3&1&3&1\\4&2&4&2&4\\3&0&3&0&0\\1&4&1&1&3\\0&2&2&4&2\\3&3&0&3&0\\4&1&4&1&4\\2&0&2&0&2\\1&3&1&3&3\\4&2&4&4&1\\3&0&0&2&2\\1&1&3&3&0\\0&4&4&1&4\\3&3&0&0&2\end{bmatrix},Q_7=\left[\begin{array}{cccc|ccc}6&0&0&6&6&1&2\\2&4&2&4&0&3&0\\5&1&5&1&2&5&6\\0&3&0&6&4&0&4\\2&5&4&2&1&3&2\\6&0&6&0&6&6&0\\1&4&1&4&4&2&4\\5&2&5&2&0&0&6\\3&0&0&6&3&5&1\\1&6&3&4&1&2&4\\5&2&5&2&5&0&0\\3&4&0&0&3&3&6\\6&1&2&6&6&1&4\\4&3&5&1&4&5&2\\0&6&0&3&0&3&0\\2&2&4&5&2&6&4\\4&0&6&0&4&1&2\\6&5&2&3&6&3&0\\0&3&4&1&2&5&6\\2&6&0&6&0&0&4\\5&1&2&4&5&2&1\\3&3&5&1&3&4&6\end{array}\right],Q_9=\left[\begin{array}{cccc|ccccc}2&6&3&1&2&6&4&6&6\\4&4&0&6&4&0&2&0&4\\0&2&5&2&1&3&5&3&2\\6&6&1&4&6&6&0&6&0\\3&0&3&0&0&2&2&4&3\\5&2&5&2&5&4&6&1&6\\0&6&0&6&3&1&0&3&0\\2&1&4&4&0&6&2&6&2\\6&3&6&1&2&4&5&0&4\\0&5&0&3&5&0&3&2&6\\4&1&2&6&1&6&6&4&1\\2&6&4&0&4&4&2&0&5\\5&3&1&5&2&0&5&3&2\\1&0&6&3&6&6&1&6&4\\4&2&4&0&0&2&4&2&1\\6&5&1&2&5&5&0&0&3\\1&3&6&4&1&3&3&6&5\\5&0&2&0&6&6&1&4&1\\2&6&5&3&2&0&5&0&3\\4&4&0&1&4&4&2&2&6\\6&2&2&6&6&1&6&4&0\\0&0&5&4&0&3&0&1&3\end{array}\right],$$

$$P_3'=\begin{bmatrix}6&2&2\\3&0&4\\1&5&6\\4&3&1\\0&6&4\\3&2&0\\1&5&5\\6&3&1\\2&0&6\\4&5&2\\0&3&4\\6&6&1\\1&4&5\\3&2&0\\0&6&6\\2&4&2\\6&0&5\\3&3&1\\1&5&4\\4&0&2\\2&6&6\\0&4&0\end{bmatrix},P_5'=\begin{bmatrix}4&0&0&2&3\\6&2&5&4&0\\0&4&3&6&5\\5&6&0&1&3\\3&1&2&4&0\\0&4&6&6&2\\6&2&1&3&4\\3&0&4&5&1\\1&5&6&2&3\\6&3&1&4&6\\2&0&5&0&0\\5&6&2&3&5\\0&3&4&6&2\\2&5&0&0&4\\6&1&2&5&1\\4&3&6&3&6\\1&5&4&0&4\\6&0&2&6&1\\2&4&5&3&5\\0&6&0&0&2\\5&1&2&4&4\\2&3&6&6&1\end{bmatrix},Q_7'=\left[\begin{array}{cccc|ccc}5&3&2&6&2&6&4\\2&0&5&0&5&0&2\\4&6&1&3&1&3&6\\0&2&4&6&4&5&0\\3&5&0&2&0&2&3\\1&1&6&4&6&4&1\\5&3&3&0&3&0&6\\2&0&5&6&1&2&4\\4&6&1&3&4&6&0\\1&2&4&5&2&1&5\\3&0&6&0&0&4&3\\6&5&3&4&6&2&0\\0&2&0&2&1&5&6\\4&6&4&6&3&3&1\\1&3&2&0&5&0&5\\6&0&5&4&2&4&2\\2&4&3&1&0&6&6\\0&6&0&6&3&2&4\\5&1&2&4&5&0&0\\2&3&6&0&2&6&3\\4&0&1&5&4&1&5\\1&6&4&3&0&3&0\end{array}\right],Q_9'=\left[\begin{array}{cccc|ccccc}5&3&4&1&4&2&1&6&3\\0&6&0&3&6&0&4&4&1\\4&2&2&5&1&5&2&0&5\\6&0&4&0&3&3&6&6&2\\3&5&6&2&5&1&4&1&4\\0&1&3&4&0&6&0&3&0\\2&4&5&1&2&4&2&5&2\\5&6&0&3&5&0&6&0&6\\1&3&4&6&1&3&1&3&3\\4&5&1&0&4&6&4&6&0\\2&0&6&3&2&0&0&2&4\\5&4&2&1&6&3&6&5&1\\0&6&0&4&4&1&4&0&6\\3&2&5&6&0&6&2&2&4\\1&4&3&1&2&4&0&6&0\\6&6&0&4&6&1&3&4&2\\0&2&5&2&0&5&6&0&6\\4&4&1&6&4&2&4&2&4\\1&6&3&0&1&6&0&6&0\\5&2&5&2&5&4&2&1&3\\0&4&0&4&3&1&6&4&6\\2&1&6&6&0&5&3&2&0\end{array}\right].$$

Case 2: $r=11$.

We use the following pattern P_m' to label $Br(22,m,11)$ for $m\in\{3,5\}$, and P_m' induces a 6-$L(2,1)$-labeling of $Br(22,m,11)$. Therefore, the case $m\le 5$ is settled. Now, we consider the case $m\ge 7$. If $m=4k+3$ for $k\ge 1$, we obtain a

6-$L(2,1)$-labeling of $Br(22,m,11)$ by repeating the leftmost four columns of Q_7'; If $m=4k+5$ for $k\ge 1$, we obtain a 6-$L(2,1)$-labeling of $Br(22,m,11)$ by repeating the leftmost four columns of Q_9'. Therefore, $\lambda(Br(2\ell,m,r))\le 6$ for $\ell=11$ and $m\ge 3$.

From Propositions 1 and 2, we have

Theorem 2. Let $m\ge 3$. Then we have $\lambda(Br(2\ell,m,r))\le 6$ for $\ell=9$ or 11.

2.2. Brick Product Graphs with λ-Number 5

In [10], it was proved that

Theorem 3. Let $\ell,m\ge 2$ and $r\ge 0$ be integers such that $m+r\equiv 0(\bmod 2\ell)$. Then

$$5\le \lambda(Br(2\ell,m,r))\le 7.$$

Moreover, $\lambda(Br(2\ell,m,r))=5$ if and only if one of the following holds:

1) 3 divides ℓ and 6 divides m;

2) 6 divides ℓ and 3 divides m.

Furthermore, if neither 1) nor 2) is satisfied, then $\lambda(Br(2\ell,m,r))=6$ provided that $m=2$ (and ℓ is even or odd), or both ℓ and m are even.

However, Theorem 3 consider the condition that $m+r\equiv 0(\bmod 2\ell)$. There may exist other brick product graphs with λ-number 5 with the condition $m+r\not\equiv 0(\bmod 2\ell)$. We provide some brick product graphs $Br(2\ell,m,r)$ with λ-number 5 in the following:

Theorem 4. Let $\ell\equiv 0(\bmod 3)$, $m\equiv 0(\bmod 2)$ with $m\ge 4$, $r=0$. Then $\lambda(Br(2\ell,m,r))=5$.

Let $m=2k$, $P=\begin{bmatrix}5 & 2\\ 1 & 4\\ 3 & 0\end{bmatrix}$, $P_1=P^k=\underbrace{PP\cdots P}_{k\text{ times}}$ and $Q=\begin{bmatrix}P_1\\ P_1\\ \vdots\\ P_1\end{bmatrix}$, where P_1 is used for $\frac{2\ell}{3}$ times. Then Q induces a 5-$L(2,1)$-labeling of $Br(2\ell,m,r)$, and so $\lambda(Br(2\ell,m,r))\le 5$.

Proposition 3. Let $\ell\equiv 0(\bmod 6)$, $m=3$, $r=3$. Then $\lambda(Br(2\ell,m,r))=5$.

Let $P=\begin{bmatrix}2&0&5\\5&4&2\\3&1&0\\0&5&3\\4&2&1\\1&0&4\\5&3&2\\2&1&5\\0&4&3\\3&2&0\\1&5&4\\4&3&1\end{bmatrix}$, and $Q=\begin{bmatrix}P\\ P\\ \vdots\\ P\end{bmatrix}$, where P is used for $\frac{\ell}{3}$ times. Then Q

induces a 5-$L(2,1)$-labeling of $Br(2\ell,m,r)$, and so $\lambda(Br(2\ell,m,r)) \le 5$.

Proposition 4. Let $\ell \equiv 0 \pmod 6$, $m=5$, $r=3$. Then $\lambda(Br(2\ell,m,r))=5$.

Let $P=\begin{bmatrix} 1&3&4&0&1\\ 5&0&2&3&5\\ 2&4&5&1&2\\ 0&1&3&4&0\\ 3&5&0&2&3\\ 1&2&4&5&1\\ 4&0&1&3&4\\ 2&3&5&0&2\\ 5&1&2&4&5\\ 3&4&0&1&3\\ 0&2&3&5&0\\ 4&5&1&2&4 \end{bmatrix}$, and $Q=\begin{bmatrix} P\\ P\\ \vdots\\ P \end{bmatrix}$, where P is used for $\frac{\ell}{3}$ times.

Then Q induces a 5-L(2,1)-labeling of $Br(2\ell,m,r)$, and so $\lambda(Br(2\ell,m,r)) \le 5$.

Proposition 5. Let $\ell \equiv 0 \pmod 6$, $m=7$, $r=3$. Then $\lambda(Br(2\ell,m,r))=5$.

Let $P=\begin{bmatrix} 1&5&4&2&1&5&4\\ 4&3&1&0&4&3&1\\ 2&0&5&3&2&0&5\\ 5&4&2&1&5&4&2\\ 3&1&0&4&3&1&0\\ 0&5&3&2&0&5&3\\ 4&2&1&5&4&2&1\\ 1&0&4&3&1&0&4\\ 5&3&2&0&5&3&2\\ 2&1&5&4&2&1&5\\ 0&4&3&1&0&4&3\\ 3&2&0&5&3&2&0 \end{bmatrix}$, and $Q=\begin{bmatrix} P\\ P\\ \vdots\\ P \end{bmatrix}$, where P is used for $\frac{\ell}{3}$ times. Then Q induces a 5-L(2,1)-labeling of $Br(2\ell,m,r)$, and so $\lambda(Br(2\ell,m,r)) \le 5$.

Proposition 6. Let $\ell \equiv 0 \pmod 6$, $m=9$, $r=3$. Then $\lambda(Br(2\ell,m,r))=5$.

Let $P=\begin{bmatrix} 1&5&4&2&1&5&4&2&3\\ 4&3&1&0&4&3&1&2&0\\ 2&0&5&3&2&0&5&4&3\\ 5&4&2&1&5&4&2&1&5\\ 3&1&0&4&3&1&0&3&2\\ 0&5&3&2&0&5&3&0&4\\ 4&2&1&5&4&2&1&2&1\\ 1&0&4&3&1&0&4&5&3\\ 5&3&2&0&5&3&2&1&0\\ 2&1&5&4&2&1&5&4&2\\ 0&4&3&1&0&4&3&0&5\\ 3&2&0&5&3&2&0&3&1 \end{bmatrix}$, and $Q=\begin{bmatrix} P\\ P\\ \vdots\\ P \end{bmatrix}$, where P is used for

$\frac{\ell}{3}$ times. Then Q induces a 5-$L(2,1)$-labeling of $Br(2\ell,m,r)$, and so $\lambda(Br(2\ell,m,r)) \le 5$.

By observing the results of Propositions 3 - 6, we propose the following conjecture:

Conjecture 2. Let $\ell \equiv 0 \pmod 6$, $m \equiv 1 \pmod 2$, $r = 3$. Then $\lambda(Br(2\ell,m,r)) = 5$.

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