

On the Order Form of the Fundamental Theorems of Asset Pricing

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Abstract

In this article, we provide an order-form of the First and the Second Fundamental Theorem of Asset Pricing both in the one-period market model for a finite and infinite state-space and in the case of multi-period model for a finite state-space and a finite time-horizon. The space of the financial positions is supposed to be a Banach lattice. We also prove relevant results in the case where the space of the financial positions is not ordered by a lattice cone.

Keywords

Strictly Positive Extension, Positive Projection, Sublattice, Complete Market, Incomplete Market

1. Some Remarks on Previous Work about the Fundamental Asset Pricing Theorems

The First Fundamental Theorem of Asset Pricing states that the absence of arbitrage for a stochastic process X is equivalent to the existence of an equivalent martingale measure for X . It was shown in [1] that for a locally bounded \mathbb{R}^d -valued semi-martingale X the condition of **No Free Lunch with Vanishing Risk** is equivalent to the existence of an equivalent local martingale measure for the process X . It was proved in [2] that the local boundedness assumption on X may be dropped under the notion of **equivalent σ -martingale measure**. The work [3], also discussed in [4], is still essential in this topic and actually this work's results rely on what Kreps established as the viable market model consisted by an incomplete market and a linear price system on it. In the present work we are going to resolve the so-called Strictly Positive Extension Property from the financial aspect. The presence of heavy-tails in continuous time models and the possible change of frame from L^p spaces to Orlicz spaces in order to fit the modelling requirements, oblige us to search for more general versions of the two FTAPs, mostly relied on the geometry of these spaces. Recently, in [5], a Fundamental Theorem of Asset Pricing

ing and a Super-Replication Theorem in a model-independent framework are both proposed. But these theorems are proved in the setting of finite, discrete time and a market consisting of a risky asset S , as well as options written on this risky asset, too. Notions like the one of the strictly positive projection or that of the filtration are alike the ones met in [6]. A difference between our notion of strictly positive projection and the equivalent notion in [6] is that ours is weaker. That's because if $x > 0$ implies $Px > 0$, this implies $Px = 0 \Rightarrow x = 0$, because if $x > 0$, it would be $Px > 0$. An important difference between the article of Troitsky and ours is that we extend the framework of Definitions so as to include cases of non-discrete time spaces. Another one is that we apply these notions in order to provide a new version of the two FTAP, while in [6] an important ordered-space theory of martingales in Banach lattices is developed. Finally, markets subspaces are taken to be sublattices because of the fact that we may include layers of call and put options written on an initial market space, as we remarked in [7]. The present paper is organized as follows: First, we provide some useful notions and definitions and examples for them, as well. Next, we prove the Order Form of the FTAP in the Banach-lattice case and in the next sections we provide the analog of these results in the finite-models case. We also explain the application of our results on the Black-Scholes-Merton model. We also compare them to the Example developed in [4]. The case of non-lattice cones is examined in the last section of the paper, in relation with the classes of reflexive and strongly reflexive cones, mentioned in [8]. The role of the existence of an unconditional basic sequence in a Banach space is also quoted in this section independently from the results provided in [8], as an important condition for the extraction of results concerning FTAP. This condition is not irrelevant to ([9], Th. 1.1), about Lindelöf Properties of weak topology, but here it mainly concerns the construction of a Strictly Positive Projection Operator. On the other side, in the paper [10] ideals of $L^0(\mu)$ are used in order to deduce an FTAP-like result ([10], Lem. 1), while our results refer to sublattices.

2. Useful Notions and Preliminaries

We consider two periods of time (0 and 1) and a non-empty set of states of the world Ω which is supposed to be an infinite set. The true state $\omega \in \Omega$ that the investors face is contained in some $A \in \mathcal{F}$, where \mathcal{F} is some σ -algebra of subsets of Ω which gives the information about the states that may occur at time-period 1. A financial position is a \mathcal{F} -measurable random variable $x: \Omega \rightarrow \mathbb{R}$. This random variable is the profile of this position at time-period 1. We suppose that the probability of any state of the world to occur is given by a probability measure $\mu: \mathcal{F} \rightarrow [0,1]$. The financial positions are supposed to lie in some subspace E of $L^0(\Omega, \mathcal{F}, \mu)$, being a Banach lattice.

Definition 1 An *incomplete market* in E is some sublattice M of E . A *complete market* in E is some sublattice M of E , such that $M = E$.

It is well-known that we define the positive cone F_+ of a subspace F of an ordered vector space to be the set $F_+ = F \cap E_+$, where E_+ denotes the positive cone of E .

Definition 2 A *positive projection* $P: E \rightarrow F$ is a projection, which maps each element of E to some element of its subspace F , such that $P(E_+) \subseteq F_+$. A positive projection is called *strictly positive*, if $P(x) \in F_+$, $P(x) = 0 \Leftrightarrow x \in E_+, x = 0$.

We also recall the notion of *random field*.

Definition 3 A *random field* is a map $X: \mathcal{T} \times \Omega \rightarrow E$ where E is a Banach lattice, \mathcal{T} is a topological space and $X_t(\omega) \in E$, for any $t \in \mathcal{T}$. Such a random field X is called *associated to the pair* (\mathcal{T}, E) .

We also may provide the notion of the *filtration* in the frame of random fields:

Definition 4 A *filtration* associated to the pair (\mathcal{T}, E) is a net of projections $(P_a)_{a \in A}$, where $P_a: E \rightarrow E_a$, where E_a is a sublattice of E and if $b \succeq a, P_a P_b = P_a$. A is a directed set, by some binary relation \succeq , called *direction*.

Definition 5 A binary relation \succeq on A is called *direction* on A , if it is reflexive and transitive on A , while for any $a, b \in A$ there is a $c \in A$, such that $c \succeq a, b$.

Definition 6 If $b \succeq a$ and $b \neq a$, this is denoted by $b \succ a, a, b \in A$.

Definition 7 A filtration $(P_a)_{a \in A}$ is called *strictly positive* if $P_a(x) = 0, x \in E_+ \Leftrightarrow x = 0, a \in A$.

We also give the definition of the *adapted* random field under this frame.

Definition 8 A random field $X: A \times \Omega \rightarrow E$, where $A \subseteq \mathcal{T}$ is called *adapted to the filtration* $(P_a)_{a \in A}$, being associated to the pair (\mathcal{T}, E) if $X_a \in E_a$ for any $a \in A$, where A is a directed subset of \mathcal{T} by some

binary relation \succeq , which is reflexive, transitive and every pair has an upper bound.

Definition 9 A random field $X : A \times \Omega \rightarrow E$, where $A \subseteq \mathcal{T}$, has the **Martingale Property** if it is adapted to a filtration $(P_a)_{a \in A}$, being associated to the pair (\mathcal{T}, E) , while $P_a X_b = X_a, b \succeq a$.

Definition 10 A random field $X : A \times \Omega \rightarrow E$, where $A \subseteq \mathcal{T}$, has the **Strictly Positive Martingale Property** if it is adapted to a filtration $(P_a)_{a \in A}$, being associated to the pair (\mathcal{T}, E) , it has the Martingale Property, while the filtration $(P_a)_{a \in A}$ is consisted by strictly positive projections.

We give some examples for the previously mentioned notions.

Example 11 If \mathcal{A} is a sub-algebra of the σ -algebra \mathcal{F} of Ω , then since $L^p(\Omega, \mathcal{A}, \mu)$ is a sublattice of $L^p(\Omega, \mathcal{F}, \mu), 1 \leq p < \infty$, then $L^p(\Omega, \mathcal{A}, \mu)$ is an incomplete market of financial positions in $L^p(\Omega, \mathcal{F}, \mu), 1 \leq p < \infty$.

Example 12 The subspace of partially linear functions M in the space $C[0,1]$ is a complete market in $C[0,1]$, due to the Stone-Weierstrass Theorem. We notice that the partially linear functions defined on $[0,1]$ is actually the sublattice generated by the bi-set of functions $\{\phi_1, \phi_2\}$, where $\phi_1(t) = 1, t \in [0,1], \phi_2(t) = t, t \in [0,1]$. We notice that in this case, the span of this bi-set is a lattice-subspace of $C[0,1]$ (see also [11]).

Example 13 A finite-dimensional sublattice M of $C[0,1]$ is an incomplete market in $C[0,1]$. As a lattice-subspace, it actually has a positive basis with nodes ([11], Pr. 2.2), hence the equivalent positive projection $P_M : C[0,1] \rightarrow M$ is defined as follows:

$$P_M(x)(t) = \sum_{i=1}^n \frac{x(t_i)}{b(t_i)} b_i(t), t \in [0,1],$$

where $n = \dim M$ and $t_1, t_2, \dots, t_n \in [0,1]$ are the nodes of the positive basis of M .

Example 14 A sequence of sublattices $(M_n)_{n \in \mathbb{N}}$ of $C[0,1]$ characterized by increasing non-terminal parts of the sequence

$$\{t_1, t_2, \dots, t_n, t_{n+1}, \dots\} \subseteq [0,1],$$

which has different terms in the sense $t_n \neq t_m, n \neq m$ is a filtration of $C[0,1]$, since t_1 is the node for the one-dimensional subspace $M_1, \{t_1, t_2\}$ is the set of nodes of the positive basis of M_2 and so on.

Example 15 An increasing net of sub- σ -algebras $(\mathcal{F}_a)_{a \in A}$ of Ω , being a non-empty set, where A is a non-empty directed set, induces as it is well-known the existence of a filtration in $L^1(\Omega, \mathcal{F}, \mu)$, where $(\Omega, \mathcal{F}, \mu)$ is supposed to be a probability space associated to the measurable space (Ω, \mathcal{F}) . The relevant net of sublattices is:

$$(L^1(\Omega, \mathcal{F}, \mu))_{a \in A}.$$

A may denote a set of cardinals, where if we start from a certain cardinal number a , then the cardinality of σ -algebra \mathcal{F}_a as a class of objects is at most equal to 2^a and it is surely greater than $a + 1$.

Example 16 The filtration of the Example 14 is not strictly positive. This holds because if we pick a sublattice M_{n_1} whose positive basis' nodes is the set:

$$\{t_1, t_2, \dots, t_{n_1}\} \subseteq [0,1].$$

If $x(t_i) = 0, i = 1, 2, \dots, n_1$, this does not imply $x = 0$ if $x \geq 0$. For example, $x(t) = \prod_{i=1}^{n_1} (t - t_i)^2 \geq 0$, but $x(t_i) = 0, i = 1, 2, \dots, n_1$.

Example 17 If E is a Banach lattice with order continuous norm and B is a projection band, namely $E = B \oplus B^d$, then B is norm-closed. The projection $P_B : E \rightarrow B$ is strictly positive since it is positive and $P_B(x) = 0, x \in E_+$, implies that since $x = x_1 + x_2$, where $x_1 \in B$ and $x_2 \in B^d, x_1 = 0, x_2 \wedge 0 = 0$ and $x_2 \geq 0$, hence $x_2 = 0$ and finally $x = 0$. The same situation is valid for **Kantorovich-Banach spaces** (or else **KB-spaces**), in which $E^{**} = E \oplus E^d$. Such examples of spaces are reflexive Banach lattices like $L^p(\Omega, \mathcal{F}, \mu), 1 < p < \infty$ and AL -spaces.

Example 18 Let us consider a Banach lattice E which has a Schauder basis:

$$\{e_1, e_2, \dots, e_n, \dots\}, n \in \mathbb{N},$$

which is moreover a **positive basis**. Also, suppose that:

$$M_n = P_n E = [e_1, e_2, \dots, e_n], n \in \mathbb{N},$$

are finite-dimensional sublattices of E . Then,

$$(P_n)_{n \in \mathbb{N}}$$

is a filtration, because $P_n P_m = P_n, m \leq n$ and $P_n x = 0 \Leftrightarrow x = 0, x \in E_+$, since $\{e_1, e_2, \dots, e_n\}$ is a positive basis of M_n itself. This is the case for $E = c_0, \ell^1, \ell^p, 1 < p < \infty$.

3. Order Versions for the Fundamental Theorems of Asset Pricing

In the proof of the two next Theorems we use the following:

Lemma 19 *A positive projection $P : E \rightarrow M$, where E is a Banach lattice and M is a positive sublattice of it, is a continuous operator.*

Proof: Obvious, because every positive operator from a Banach lattice into to a locally solid Riesz space, is continuous.

Theorem 20 *(Order 1st Fundamental Theorem of Asset Pricing) Let E be a Banach lattice and M be a sublattice of E . If M admits a strictly positive projection, then every strictly positive and continuous functional $f : M \rightarrow \mathbb{R}$, admits a strictly positive, continuous extension on E . Also, if E is a Banach lattice and M is a sublattice of E such that every strictly positive and continuous functional $f : M \rightarrow \mathbb{R}$, admits a strictly positive, continuous extension on E , then M admits a strictly positive projection.*

Proof: The adjoint operator of the strictly positive projection $P : E \rightarrow M$ is an injection. Hence $P^*(f) : E \rightarrow \mathbb{R}$ is a continuous, strictly positive functional of E . This is due to the duality:

$$\langle x, P^*(f) \rangle = \langle P(x), f \rangle, x \in E.$$

For the proof of the opposite, we have the following: We define the projection $P_M : E \rightarrow M$ as follows. $P_M(x) = x, x \in M, P_M(x) = 0, x \notin M$. P_M is a positive operator from a Banach lattice into a locally solid Riesz space. Hence it is continuous. By duality for some f strictly positive, continuous functional f of M ,

$$\langle x, P_M^*(f) \rangle = \langle P_M(x), f \rangle, x \in E.$$

Hence if we suppose that there is some $x_0 \in E_+ \setminus \{0\}$ such that $g(x_0) = P_M^*(f)(x_0) = 1$, while $P_M(x_0) = 0$. But this leads to a contradiction.

Corollary 21 *If E is a Banach lattice which has the **Strictly Positive Martingale Property** with respect to some filtration $(P_a)_{a \in A}$, where A is a directed set. If $a \in A$ such that $P_a E = M_a$, then every strictly positive and continuous functional $f : M_a \rightarrow \mathbb{R}$, admits a strictly positive, continuous extension on E .*

Corollary 22 *Let E be a Banach lattice of financial positions and M be an incomplete market, such that (M, f) is a market model. If M admits a strictly positive projection, then for every price system $f : M \rightarrow \mathbb{R}$, the market model (M, f) is viable.*

The existence of a strictly positive projection may be replaced by the **Strictly Positive Martingale Property** with respect to some filtration in the statement of the above Theorem. The term viable is the one established in the seminal work of D.M. Kreps (see [3], p. 18-19).

Theorem 23 *(Order 2nd Fundamental Theorem of Asset Pricing) Let E be a Banach lattice and M be a dense sublattice of E . If M admits a strictly positive projection, then every strictly positive and continuous functional $f : M \rightarrow \mathbb{R}$, admits a unique strictly positive, continuous extension on E . Also, let E be a Banach lattice and M be a sublattice of E such that M admits a strictly positive projection. Moreover, every strictly positive and continuous functional $f : M \rightarrow \mathbb{R}$, admits a unique strictly positive, continuous extension on E . Then M is dense in E .*

Proof: Since M is a dense sublattice of E , the adjoint (linear by the duality $\langle x, P^*(f) \rangle = \langle P(x), f \rangle, x \in E$) operator $P^* : M^* \rightarrow E^*$ of the strictly positive projection $P : E \rightarrow M$ is a surjection. Hence for any $g^* \in E^*$, there is some $h^* \in M^*$, such that $g^* = P^*(h^*)$, or else by duality relations:

$$\langle x, P^*(h_g^*) \rangle = \langle x, g^* \rangle = \langle P(x), h_g^* \rangle, x \in E.$$

For the converse, we have that for any $g^* \in E^*$, there is some $h^* \in M^*$, such that $g^* = P^*(h_g^*)$, or else by duality relations:

$$\langle x, P^*(h_g^*) \rangle = \langle x, g^* \rangle = \langle P(x), h_g^* \rangle, x \in E.$$

where $P: E \rightarrow M$ is a strictly positive projection. This implies that $P^*: M^* \rightarrow E^*$ is a surjection, which is equivalent to the fact that M is dense in E .

Corollary 24 *If E is a Banach lattice which has the **Strictly Positive Martingale Property** with respect to some filtration $(P_a)_{a \in A}$, where A is a directed set. If a_0 is an element of A such that $P_{a_0} E = M_{a_0}$ is a dense sublattice of E , then every strictly positive and continuous functional $f: M_{a_0} \rightarrow \mathbb{R}$, admits a unique strictly positive, continuous extension on E .*

Corollary 25 *Let E be a Banach lattice of financial positions and M be a complete market, such that (M, f) is a market model. If M admits a strictly positive projection, then for every price system $f: M \rightarrow \mathbb{R}$, the market model (M, f) is viable.*

The term **viable** is the one established in the seminal work of D.M. Kreps (see [3], pp. 18-19).

We may notice that our Theorem does not make any reference to the No-Free Lunch Condition, but it simply extends the **No-Arbitrage** Property all over the space E . Theorem 23 is the analog of the usual 2nd FTAP, which implies that the (local) Equivalent Martingale Measures' set of a complete market is a singleton, while under this class of market spaces the uniqueness of the (strictly positive) extension of a price system all over the space of financial positions is achieved under no presence of the No-Free Lunch Condition, too.

Let us see some Examples which confirm the connection of the above Theorems to well-known models of Mathematical Finance.

Example 26 *Let $(\Omega, \mathcal{F}, \mu)$ be a probability space endowed with an m -dimensional Brownian motion $B = (B(t))_{t \in [0, T]}$ ($m \in \mathbb{N}$), where $T > 0$. Denote by $\mathbb{F} = (\mathcal{F}_t)_{t \in [0, T]}$ the filtration that this Brownian motion generates, i.e., $\mathcal{F}_t = \sigma(B(u), u \in [0, t])$. We assume a financial market consisting of $n+1$ assets whose prices are modelled by an \mathbb{F} -adapted, $(n+1)$ -dimensional Itô process $X = (X(t))_{t \in [0, T]}$ ($n \in \mathbb{N}$) of the form $X(\cdot) = (X_0(\cdot), \dots, X_n(\cdot))$ where:*

$$dX_0(t) = r(t)X_0(t)dt,$$

$$dX_i(t) = m_i(t, X_t)dt + \sigma_i(t, X_t)dB(t),$$

$$X_0(0) = 1, \mu - a.e.,$$

$$X_i(0) = \theta_i, \mu - a.e.,$$

where $\sigma_i(\cdot)$ is the i -th row ($i = 1, 2, \dots, n$) of the $n \times m$ -matrix process $\sigma(\cdot)$. The process $X_0(\cdot)$ represents the price of a riskless asset (where $r(\cdot)$ is the interest rate process which is supposed to have bounded values), while the i -th component $X_i(\cdot)$, of the process $X(\cdot)$, represents the evolution of the price of the i -th asset (stock). The price of the riskless asset may be used as numeraire. Suppose that $W \in L^2(\Omega, \mathcal{F}_T, \mu)$. If $(Z_t)_{t \in [0, T]}$ is a stochastic exponential, then as it is well-known, the following relation holds:

$$\mathbb{E}_Q(W | \mathcal{F}_t) = \frac{1}{Z_t} \mathbb{E}_\mu(Z_T W | \mathcal{F}_t), \mu, Q, a.e., t < T$$

where Q is the probability measure defined on \mathcal{F}_T as follows: $Q(A) = \int_A Z_T d\mu, A \in \mathcal{F}_T$, according to the Girsanov-Cameron-Martin Theorem. Taking mean values over μ we have:

$$\mathbb{E}_\mu(W Z_T) = \mathbb{E}_\mu \left(\frac{1}{Z_t} \mathbb{E}_\mu(Z_T W | \mathcal{F}_t) \right),$$

which in terms of evaluation maps' values is interpreted as follows:

$$\langle XZ_T, 1 \rangle = \left\langle \left(\mathbb{E}_\mu (Z_T X | \mathcal{F}_t) \right), \frac{1}{Z_t} \right\rangle.$$

The equivalent Riesz pairs are:

$$\langle L^2(\Omega, \mathcal{F}_t, \mu), L^2(\Omega, \mathcal{F}_t, \mu) \rangle, \langle L^2(\Omega, \mathcal{F}_T, \mu), L^2(\Omega, \mathcal{F}_T, \mu) \rangle,$$

where the strictly positive projection $P : L^2(\Omega, \mathcal{F}_T, \mu) \rightarrow L^2(\Omega, \mathcal{F}_t, \mu)$ is $P(W) = \mathbb{E}_\mu(W | \mathcal{F}_t)$, the strictly positive linear functional $f : L^2(\Omega, \mathcal{F}_t, \mu) \rightarrow \mathbb{R}$ is $f(W) = \left\langle W, \frac{1}{Z_t} \right\rangle$ and its strictly positive extension $P^*(f) : L^2(\Omega, \mathcal{F}_T, \mu) \rightarrow \mathbb{R}$ is $P^*(f)(W) = \langle W, 1 \rangle$. This Example gives also a Hilbert space taste, due to the presence of L^2 -spaces, see also [12].

Example 27

$$\mathbb{E}_Q(X | \mathcal{F}_t) = \frac{1}{Z_t} \mathbb{E}_\mu(Z_T X | \mathcal{F}_t), \mu, Q, a.e., t < T,$$

holds for the unique possible change of measure Q , if the market is complete for example in the Black-Scholes model and this arises independently from the unique solution of the **market-price-of-risk equation**.

Finally, we may revisit the Example constructed in [4], in order to quote it.

Example 28 *The actual form of the elements of the subspace M of ℓ^∞ is described by the following strictly positive projection:*

$$P : \ell^\infty \rightarrow M, (x_1, x_2, x_3, x_4, \dots) \mapsto (x_1, x_2, x_1, x_2, x_3, x_4, x_3, x_4, \dots).$$

M is a sublattice of ℓ^∞ under the usual component-wise ordering. Also, $\pi = \left(\frac{1}{2^n} \right)_{n \in \mathbb{N}}$, while according to

Theorem 3, a strictly positive extension of π all over ℓ^∞ exists, through duality relation $f(x) = \pi(P(x))$, $x \in \ell^\infty$.

4. The Finite-State, One Period-Model Case

We will show how the above Theorems 3, 23 are applied in finite -state space models.

Let us consider the two-date market model in which the number of states of the world is denoted by S , while the time-periods are denoted by 0 and 1, respectively. We also consider an incomplete market of primitive assets whose time-period -1 payoffs are the positive, linearly independent vectors $y_1 = x_1, y_2 = x_2, \dots, y_J = x_J$ of \mathbb{R}^S , whose span is denoted by X . We suppose that X contains the riskless asset 1, while $J < S$, which implies standard incompleteness. We also assume a time-period 0, no-arbitrage price $q = (q^1, q^2, \dots, q^J)$ for the primitive assets. As it is well-known from ([7], p. 4), $F_1(X)$ is identified to the **sublattice $S(X)$ of \mathbb{R}^S generated by X** . We also remind of the following **Projection Basis Theorem** for sublattices of \mathbb{R}^S , which arises from both ([13], Th. 3.7), ([14], Th. 9).

Theorem 29 *Let X be a J -dimensional subspace of \mathbb{R}^S with $J < S$ generated by the positive elements y_1, y_2, \dots, y_J in which the riskless bond 1 is a marketed asset ($1 \in X$). Suppose that the range $R(\beta)$ of the basic function β of the elements y_1, y_2, \dots, y_J is the finite set $\{P_1, P_2, \dots, P_\mu\}$ of the simplex Δ_{J-1} of \mathbb{R}^J (note that $\mu \geq J$). Suppose that the first J vectors of this set are linearly independent. If we suppose that the vectors $y_{J+i}, i = 1, 2, \dots, \mu - J$ are such that $y_{J+i}(s) = y(s), s \in I_{J+i}$ and $y_{J+i}(s) = 0, s \notin I_{J+i}$, where $y(s) = y_1(s) + y_2(s) + \dots + y_J(s)$ where $s \in \Omega$ and $I_{J+i} = \{s \in \Omega | \beta(s) = P_{J+i}\}, i = 1, 2, \dots, \mu - J$ (which are the vectors indicated by ([13], Th. 3.7), then,*

- 1) $S(X) = X \oplus [y_{J+1}, y_{J+2}, \dots, y_\mu]$.
- 2) $b_i = 2y_i, i = J + 1, J + 2, \dots, \mu$.
- 3) If $b_i = \tilde{b}_i + b'_i, i = 1, 2, \dots, J$ with $\tilde{b}_i \in X$ and $b'_i \in [y_{J+1}, y_{J+2}, \dots, y_\mu]$ then the vectors \tilde{b}_i defined by:

$$(\tilde{b}_1, \tilde{b}_2, \dots, \tilde{b}_J)^T = A^{-1} \cdot (y_1, y_2, \dots, y_J)^T$$

where A is the $J \times J$ matrix whose columns are the vectors $P_i, i = 1, 2, \dots, J$ are a basis of X called projection basis. This basis has the property: The J first coordinates of an element $x \in X$ in the positive basis of $S(X)$ coincide with the coordinates of the expansion of x in the basis $\{\tilde{b}_i, i = 1, 2, \dots, J\}$.

Also, according to what is mentioned in [7] about the completion of an incomplete market X by options and by following the notation we introduced, $y = y_1 + y_2 + \dots + y_J + y_{J+1} + \dots + y_\mu$, where $\mu = \dim F_1(X)$ and $\{y_1, y_2, \dots, y_\mu\}$ is a maximal set of linearly independent, positive vectors of $F_1(X)$. Due to ([7], Th. 21), y_{J+1}, \dots, y_μ are portfolios of **call and put options** written on elements of X , especially since $1 \in X$.

The dimension equation which holds in the case of the no-arbitrage price q , is:

$$\langle W \rangle \oplus \langle W \rangle^\perp = S + 1,$$

where $\langle W \rangle$ denotes the subspace of \mathbb{R}^{S+1} generated by the columns of the payoff matrix $W(q, X)$ of the primitive securities, while $\langle W \rangle^\perp$ denotes the orthogonal subspace of it. Due to the characterization of the absence of arbitrage in the primitive asset market (see [15], Th. 9.2), there is at least one $\pi \in \mathbb{R}_{++}^{S+1}$ such that $\pi \cdot W(q, X) = 0$ where $0 \in \mathbb{R}^J$. This implies that $\langle W \rangle^\perp \neq \{0\}$ in this case, while $q = \pi_1 \cdot X$ if by X we also denote the $S \times J$ matrix whose columns are the vectors x_1, x_2, \dots, x_J . The last relation arises from

$\pi \cdot W(q, X) = 0$ if we suppose that $\pi = (\pi_0, \hat{\pi}_1)$. Then $\pi_0 q = \hat{\pi}_1 \cdot X$ and if we denote $\frac{1}{\pi_0} \hat{\pi}_1 = \pi_1$, we obtain

the last relation. As it is implied in [7] $F_1(X)$ is determined by the positive basis $\{b_1, b_2, \dots, b_\mu\}$ of it.

We also have the following:

Theorem 30 Any $\pi \in \mathbb{R}_+^S$ such that $q = \pi_1 \cdot X$ implies a no-arbitrage price $q(\pi_1) \in \mathbb{R}^\mu$ for which the price $q(\pi_1)(\lambda) = q(\pi_1) \cdot \lambda$ of the portfolio $\lambda \in \mathbb{R}^\mu$ or else the price of the asset $\sum_{i=1}^\mu \lambda_i y_i$ lying in the completion $F_1(X)$ to be equal to the price of the same asset under $q \in \mathbb{R}^J$ if $\sum_{i=1}^\mu \lambda_i y_i \in X$, where $y_i, i = 1, 2, \dots, \mu$ are the vectors indicated by the Projection Basis Theorem.

Proof: Consider the vector $q(\pi_1) = \pi_1 \cdot D \cdot (b_1, b_2, \dots, b_\mu)^T$. The above vector satisfies the following equalities:

$$\begin{bmatrix} \pi_1 \cdot y_1 \\ \pi_1 \cdot y_2 \\ \vdots \\ \pi_1 \cdot y_\mu \end{bmatrix}^T = \pi_1 \cdot \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_\mu \end{bmatrix}^T = \begin{bmatrix} q \\ \pi_1 \cdot y_{J+1} \\ \pi_1 \cdot y_{J+2} \\ \vdots \\ \pi_1 \cdot y_\mu \end{bmatrix}^T.$$

The definition of the vector $q(\pi_1)$ allows us to prove that it is a no-arbitrage price in the subspace generated by the vectors y_1, y_2, \dots, y_μ which is the completion by options $F_1(X)$ of X . If for a portfolio $\lambda \in \mathbb{R}^\mu$ the payoff $\sum_{i=1}^\mu \lambda_i y_i$ lies in the positive cone \mathbb{R}_+^S except $\{0\}$, then:

$$q(\pi_1)(\lambda) = q(\pi_1) \cdot \lambda = \sum_{i=1}^\mu \lambda_i (\pi_1 \cdot y_i) = \pi_1 \cdot \left(\sum_{i=1}^\mu \lambda_i y_i \right) > 0,$$

because $\pi_1 \in \mathbb{R}_{++}^S$. Also, from the Projection Basis Theorem 29, if $\sum_{i=1}^\mu \lambda_i y_i \in X$, this means that:

$$\sum_{i=1}^\mu \lambda_i y_i = \sum_{i=1}^J \lambda_i y_i.$$

Hence $q(\pi_1)(\lambda) = \sum_{i=1}^J \lambda_i \pi_1 \cdot y_i$ in this case, which is equal to the valuation of the portfolio $(\lambda_1, \lambda_2, \dots, \lambda_J)$ of the primitive assets under q . We remind that \mathbb{R}^μ is the space of the financial positions, since $F_1(X)$ is actually equal to this space according to ([7], Pr. 6).

Theorem 31 (First Order Finite Fundamental Theorem of Asset Pricing) For any subspace $X = \text{span}[x_1, x_2, \dots, x_J]$ of \mathbb{R}_+^S , where $\mu = \dim F_1(X)$ and $1 \in X$ and $x_i, i = 1, 2, \dots, J$ are linearly independent, every strictly positive linear functional of X has a strictly positive extension on $F_1(X) = \mathbb{R}^\mu$.

Proof: Every strictly positive functional $f : X \rightarrow \mathbb{R}$ defines a no-arbitrage price $q(f)$ on X as follows: $q_i(f) = f(x_i), i = 1, 2, \dots, J$. According to Theorem 30, p_1 for some $\pi_1 \in \mathbb{R}_{++}^S$ such that $q = \pi_1 \cdot X$ is a strictly positive extension of f on \mathbb{R}^μ , where $p_1(i) = \pi_1(i), i \in \text{supp}(b_j), j = 1, 2, \dots, \mu$, where $\text{supp}(b_j)$ is the support of the vector b_j of the positive basis of $F_1(X)$, see ([7], Th. 6).

Proposition 32 *If we suppose that the vectors of the date-1 payoffs of the primitive assets x_1, x_2, \dots, x_J are linearly independent and $1 \in X$, then $F_1(X) = \mathbb{R}^S$, where $X = \text{span}[x_1, x_2, \dots, x_J]$, except a set of vectors x_1, x_2, \dots, x_J of Lebesgue measure zero in $(\mathbb{R}^S)^J$.*

Proof: In the last part of [7], a brief proof was given about the fact that **resolving markets** have the property $F_1(X) = \mathbb{R}^S$. It is also well-known that resolving matrices are in **general position**, namely the complement of the set of them is a null-set in the vector space of the matrices $S \times J$, whose entries are real numbers. Hence the super-set of all the $S \times J$ -matrices (markets), such that $1 \in X = \text{span}[x_1, x_2, \dots, x_J]$ where x_1, x_2, \dots, x_J are linearly independent and they have the property that $F_1(X) = \mathbb{R}^S$ are also in general position.

Theorem 33 (Second Order Finite Fundamental Theorem of Asset Pricing) *For almost any subspace $X = \text{span}[x_1, x_2, \dots, x_J]$ of \mathbb{R}^S , where $S = \dim F_1(X)$ and $1 \in X$ and $x_i, i = 1, 2, \dots, J$ are linearly independent, every strictly positive linear functional of X has a unique strictly positive extension on $F_1(X) = \mathbb{R}^S$.*

Proof: Every strictly positive functional $f : X \rightarrow \mathbb{R}$ defines a no-arbitrage price $q(f)$ on X as follows: $q_i(f) = f(x_i), i = 1, 2, \dots, J$. According to Theorem 30, $q(f) = \pi_1 \cdot X$ for a unique $\pi_1 \in \mathbb{R}_{++}^S$.

5. The Finite Multi-Period Model Case

Let us see what happens in the multi-period framework. We consider the event-tree model as it is presented in [15], according to which there is a finite time-horizon $\mathbb{T} = \{0, 1, 2, \dots, T\}$, a family of partitions \mathbb{F} of Ω such that $F_0 = \{\Omega\}, F_T = \{\{\omega\}, \omega = 1, 2, \dots, S\}$ and F_{t+1} is thinner than F_t for any $t = 0, 1, \dots, T-1$ in the sense that for any $\sigma_{t+1} \in F_{t+1}$, there is a $\sigma_t \in F_t$ such that $\sigma_{t+1} \subseteq \sigma_t$. Then the set $\mathbb{D} = \{\xi = (t, \sigma) \mid \sigma \in F_t, t \in \mathbb{T}\}$ is the event-tree corresponding to the family of partitions \mathbb{F} . Every event-tree \mathbb{D} is a model of information revealing along the time-periods of \mathbb{T} . We also consider J assets (financial contracts) whose payoff vectors are $V^1, V^2, \dots, V^J \in \mathbb{R}^{\mathbb{D}}$ and if we denote by n the physical number which is equal to the cardinality of the nodes of the event-tree \mathbb{D} , these are actually vectors of \mathbb{R}^n . We also suppose that the price vectors of the assets are q^1, q^2, \dots, q^J , where $q^j(\xi) = 0$ if $\xi \in \mathbb{D}_T, j = 1, 2, \dots, J$ and the set \mathbb{D}_T denotes the set of nodes of the event-tree corresponding to the time-period T . If we suppose that these price vectors do not provide arbitrage opportunities in the market of the assets $j = 1, 2, \dots, J$, then since the market is incomplete there is at least one node-price vector $\pi \in \mathbb{R}_{++}^n$ such that $\pi \cdot W(q, V) = 0$, where $W(q, V)$ is the payoff matrix of this market as it is indicated in ([15], Ch. 4). In order to simplify things, we may suppose that $\pi(\xi_0) = 1$, where $\xi_0 = (0, \Omega)$. We also suppose that one of the assets of the market is riskless, or else that for any $\xi \in \mathbb{D}$ which corresponds to the same time-period, its payoff is the same. Also, this asset's initial price $q(\xi_0)$ is equal to 1. The submatrix $[V(\xi^+) + q(\xi^+)]$ for any $\xi \in \mathbb{D}^-$ is the $b(\xi) \times J$ -matrix whose rows are the vectors $V(\xi') + q(\xi')$ of \mathbb{R}^J , indicating the payoffs and the ex-payoff price of the J primitive securities at the node $\xi' \in \xi^+$. The cardinality of $\xi^+, \xi \in \mathbb{D}^-$ is denoted by $b(\xi)$.

The market of the securities is *complete* or as it is usually said the securities' markets are **dynamically complete**, if every *contingent claim* $c = (c(\xi), \xi \in \mathbb{D}^+)$ can be replicated by a portfolio $z = (z(\xi), \xi \in \mathbb{D}^-), z(\xi) \in \mathbb{R}^J$. In order to understand the next, we remind of the following,

Definition 34 *The forward-start call option written on a contingent claim $c = (c(\xi), \xi \in \mathbb{D}^+)$ with exercise-price a at the node $\xi \in \mathbb{D}^-$ given that $1_{b(\xi)} \in X(\xi)$, is equal to:*

$$(c(\xi^+) - a 1_{b(\xi)})^+.$$

Definition 35 *The forward-start put option written on a contingent claim $c = (c(\xi), \xi \in \mathbb{D}^+)$ with exercise-price a at the node $\xi \in \mathbb{D}^-$ given that $1_{b(\xi)} \in X(\xi)$, is equal to:*

$$(a 1_{b(\xi)} - c(\xi^+))^+.$$

As a reference for these options we append to ([16], Par. 9.2).

The market is **dynamically complete** if and only if it is **one-period complete** for any non-terminal node

$\xi \in \mathbb{D}^-$, namely if $\text{rank}[V(\xi^+) + q(\xi^+)] = b(\xi)$. Otherwise it is called **incomplete**. For any $\pi \in \mathbb{R}_{++}^n$ such that $\pi \cdot W(q, V) = 0$, and moreover there is a non-terminal node $\xi \in \mathbb{D}^-$ such that for the corresponding submatrix $[V(\xi^+) + q(\xi^+)]$ of $W(q, V)$, $\text{rank}[V(\xi^+) + q(\xi^+)] < b(\xi)$ holds, we may add forward-start options of the form $(x - a1_{b(\xi)})^+, (a1_{b(\xi)} - x)^+$, where $x \in \text{span}[V(\xi^+) + q(\xi^+)]$ to make it complete. In the same way we may talk about the completion by options of the span $X(\xi) = \text{span}[V(\xi^+) + q(\xi^+)]$ with respect to the asset $1_{b(\xi)}$ which may be denoted by $F_{1_{b(\xi)}}(X(\xi))$ for any $\xi \in \mathbb{D}^-$. $1_{b(\xi)}$ is the vector of the Euclidean space $\mathbb{R}^{b(\xi)}$ such that $1_{b(\xi)}(\xi') = 1, \xi' \in \xi^+, \xi \in \mathbb{D}^-$. In a way similar to [7], the dimension of the completion $F_{1_{b(\xi)}}(X(\xi))$ is denoted by $\mu(\xi)$. It is obvious that we may reach a complete market if and only if $\mu(\xi) = b(\xi)$ for any $\xi \in \mathbb{D}^-$. A question which also arises in this case is how the new assets introduced in a submarket $X(\xi_1), \xi_1 \in \mathbb{D}^-$ with $\text{rank}[V(\xi_1^+) + q(\xi_1^+)] < b(\xi_1)$ in order to reach $\mu(\xi_1)$ are priced. The answer is given in the next Theorem, being equivalent to Theorem 29.

Theorem 36 For any submarket $X(\xi_1), \xi_1 \in \mathbb{D}^-$ with $\text{rank}[V(\xi_1^+) + q(\xi_1^+)] < b(\xi_1)$ and any $\pi \in \mathbb{R}_{++}^n$ where $\pi \cdot W(q, V) = 0$, where $q = (q^1, q^2, \dots, q^J)$ is a no-arbitrage price vector for the assets $i = 1, 2, \dots, J$. $q(\pi)(\xi_1) \in \mathbb{R}^{\mu(\xi_1)}$ is a price vector which assigns the price $q(\pi)(\xi_1) \cdot \lambda$ to the portfolio $\lambda \in \mathbb{R}^{\mu(\xi_1)}$. Specifically, the price of the asset $\sum_{i=1}^{\mu(\xi_1)} \lambda_i y_i$ lying in the completion $F_{1_{b(\xi_1)}}(X(\xi_1))$ is equal to the price of the same asset under $q \in \mathbb{R}^J$ if $\sum_{i=1}^{\mu(\xi_1)} \lambda_i y_i \in X(\xi_1)$, where $y_i, i = 1, 2, \dots, \mu(\xi_1)$ are the vectors indicated by the Projection Basis Theorem.

Proof: Consider the vector $q(\pi)(\xi_1) = \pi(\xi_1^+) \cdot D \cdot (b_1, b_2, \dots, b_{\mu(\xi_1)})^T$, where $\pi(\xi_1^+) \in \mathbb{R}_{++}^{b(\xi_1)}$. The above vector satisfies the following equalities:

$$\begin{bmatrix} \pi(\xi_1^+) \cdot y_1 \\ \pi(\xi_1^+) \cdot y_2 \\ \vdots \\ \pi(\xi_1^+) \cdot y_{\mu(\xi_1)} \end{bmatrix}^T = \pi(\xi_1^+) \cdot \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_{\mu(\xi_1)} \end{bmatrix}^T = \begin{bmatrix} q(\xi_1) \\ \pi(\xi_1^+) \cdot y_{J+1} \\ \pi(\xi_1^+) \cdot y_{J+2} \\ \vdots \\ \pi(\xi_1^+) \cdot y_{\mu(\xi_1)} \end{bmatrix}^T.$$

The definition of the vector $q(\pi)(\xi_1)$ allows us to prove that it is a no-arbitrage price in the subspace generated by the vectors $y_1, y_2, \dots, y_{\mu(\xi_1)}$ which is the completion by options $F_{1_{b(\xi_1)}}(X(\xi_1))$,

$$X(\xi_1) = \text{span}[V(\xi_1^+) + q(\xi_1^+)].$$

If for a portfolio $\lambda \in \mathbb{R}^{\mu(\xi_1)}$ the payoff $\sum_{i=1}^{\mu(\xi_1)} \lambda_i y_i$ lies in the positive cone $\mathbb{R}_+^{b(\xi_1)}$ except $\{0\}$, then:

$$q(\pi)(\xi_1) \cdot \lambda = \sum_{i=1}^{\mu(\xi_1)} \lambda_i (\pi_1(\xi_1^+) \cdot y_i) = \pi(\xi_1^+) \cdot (\sum_{i=1}^{\mu(\xi_1)} \lambda_i y_i) > 0,$$

because $\pi(\xi_1^+) \in \mathbb{R}_{++}^{b(\xi_1)}$. Also, from the Projection Basis Theorem, if $\sum_{i=1}^{\mu(\xi_1)} \lambda_i y_i \in X(\xi_1)$, this means that $\sum_{i=1}^{\mu(\xi_1)} \lambda_i y_i = \sum_{i=1}^J \lambda_i y_i$. Hence $q(\pi)(\xi_1) \cdot \lambda = \sum_{i=1}^J \lambda_i \pi_1 \cdot y_i$ in this case, which is equal to the valuation of the portfolio $(\lambda_1, \lambda_2, \dots, \lambda_J)$ of the primitive assets under $q(\xi_1)$. This concludes the proof.

Theorem 37 (First Order Event-Tree Fundamental Theorem of Asset Pricing). For any submarket $X(\xi_1), \xi_1 \in \mathbb{D}^-$ with $\text{rank}[V(\xi_1^+) + q(\xi_1^+)] < b(\xi_1)$ and any $\pi \in \mathbb{R}_{++}^n$ with $\pi \cdot W(q, V) = 0$, every strictly positive linear functional of $X(\xi_1), \xi_1 \in \mathbb{D}^-$ has a strictly positive extension on $F_{1_{b(\xi_1)}}(X(\xi_1)) = \mathbb{R}^{\mu(\xi_1)}$.

Proof: If $f(\xi_1): X(\xi_1) \rightarrow \mathbb{R}$ is a strictly positive functional of $X(\xi_1), \xi_1 \in \mathbb{D}^-$, then this implies a no-arbitrage price $q_f(\xi_1) \in \mathbb{R}^J$ and since q is given, $q(\xi_1) = q_f(\xi_1)$. The extension of $f(\xi_1)$ is $p_1(\xi_1)$ for some $\pi_1 \in \mathbb{R}_{++}^{b(\xi_1)}$ such that $\pi_1(\xi_1) q_f(\xi_1) = \pi_1(\xi_1^+) \cdot X(\xi_1)$ is a strictly positive extension of f on $\mathbb{R}^{\mu(\xi_1)}$, where

$p_1(k') = \pi_1(k'), k' \in \text{supp}(b_j), k' = 1, 2, \dots, \mu(\xi_1)$, where $\text{supp}(b_j)$ is the support of the vector b_j of the positive basis of $F_{1_{b(\xi_1)}}(X(\xi_1))$, see ([7], Th. 6).

Theorem 38 (Second Order Finite Fundamental Theorem of Asset Pricing) *If the market is complete, then for any submarket $X(\xi_1), \xi_1 \in \mathbb{D}^-$, every strictly positive linear functional of $X(\xi_1), \xi_1 \in \mathbb{D}^-$ has a unique strictly positive extension on $F_{1_{b(\xi_1)}}(X(\xi_1)) = \mathbb{R}^{b(\xi_1)}$.*

Proof: Since the market is complete, $[V(\xi_1^+) + q(\xi_1^+)] = b(\xi_1), \xi_1 \in \mathbb{D}^-$ and there is a unique $\pi \in \mathbb{R}_{++}^n$ with $\pi \cdot W(q, V) = 0$. If $f(\xi_1): X(\xi_1) \rightarrow \mathbb{R}$ is a strictly positive functional of $X(\xi_1), \xi_1 \in \mathbb{D}^-$, then this implies a no-arbitrage price $q_f(\xi_1) \in \mathbb{R}^J$ and since q is given, $q(\xi_1) = q_f(\xi_1)$. The unique extension of $f(\xi_1)$ is $p_1(\xi_1)$ for some $\pi_1 \in \mathbb{R}_{++}^{b(\xi_1)}$ such that $\pi_1(\xi_1)q_f(\xi_1) = \pi_1(\xi_1^+) \cdot X(\xi_1)$ is a strictly positive extension of f on $\mathbb{R}^{b(\xi_1)}$, where $p_1(k') = \pi_1(k'), k' \in \text{supp}(b_j), k' = 1, 2, \dots, b(\xi_1)$, where $\text{supp}(b_j)$ is the support of the vector b_j of the positive basis of $F_{1_{b(\xi_1)}}(X(\xi_1))$, see ([7], Th. 6), since π_1 is unique.

6. General Cones Revisited

Let us consider a Banach space E of financial positions, partially ordered by a closed cone C , which is not a lattice cone. Such a cone is for example a **Bishop-Phepls** cone, see ([17], pp. 126-127), which is **well-based** and it has also interior points, hence it is not a lattice cone, according to ([17], Th. 4.4.4). Of course, the set of strictly positive functionals of such a cone has not to be empty. This is the reason due to which the Lindelöf Property mentioned in [9] about the weak topology $\sigma(X, Y)$ defined on a dual system $\langle X, Y \rangle$ is important. Of course, there are cones which do not admit continuous strictly positive functionals. Such a cone is the positive cone of an $\ell^p(\Gamma), 1 < p < \infty$ space, where Γ is uncountable.

Also, in this section, the definition of (in)completeness are altered.

Definition 39 *If M is a infinite-dimensional subspace of E ordered by the cone C , a **market** is an infinite-dimensional subspace of E , such that $M = C - C$.*

Definition 40 *A market is **incomplete** if $\bar{M} \neq E$, while it is **complete** if $\bar{M} = E$.*

Then, the following versions of the Second and the First Fundamental Theorem of Asset Pricing are deduced, respectively.

Theorem 41 *Let E be a Banach space with an unconditional basis. Then a non-lattice one exists, which makes E a complete market and every strictly positive functional of this cone admits a unique strictly positive extension.*

Proof: As it is well-known from ([18], Th. 4.2.22), the cone of the unconditional basis $(x_n)_{n \in \mathbb{N}}$ $K = \{x \in E | x = \sum_{n=1}^{\infty} a_n x_n, a_n \geq 0\}$ makes X a Banach lattice under an equivalent norm. According to ([8], Th. 5.7) there is a **strongly reflexive cone** (see [8], Def. 5.1) C in E_+ , such that $E = \overline{C - C}$. Also, since the one-dimensional-subspace projections $P_n, n \in \mathbb{N}$ are continuous, according to ([18], Cor. 4.2.26), the operator $P = \sum_{n=1}^{\infty} \frac{1}{2^n} P_n$ is a continuous projection from E ordered by E_+ (which is also the cone of the positive basis) to $C - C$ being ordered by C . Also, we notice that P is strictly positive in the sense that $Px = 0 \Leftrightarrow x = 0$, whenever $x \in E_+$. Hence, P may be taken as a strictly positive projection, and consequently we may repeat the proof of Theorem 23.

Theorem 42 *Let E be a Banach space with an unconditional basic sequence. Then, for the incomplete market Y arising from the basic sequence, there exists a non-lattice cone C , such that $Y = \overline{C - C}$ and strictly positive functional of this cone admits a strictly positive extension on E .*

Proof: According to ([8], Cor. 5.8) there is a **strongly reflexive cone** (see [8], Def. 5.1) C in E_+ , such that $Y = \overline{C - C}$, while for any unconditional basic sequence it is well-known that (see [18], Th. 4.2.22) its cone Y_+ makes Y a Banach lattice (under an equivalent norm). Also, since the one-dimensional-subspace projections $P_n, n \in \mathbb{N}$ are continuous, according to ([18], Cor. 4.2.26) the operator $P = \sum_{n=1}^{\infty} \frac{1}{2^n} P_n$ is a continuous projection from E ordered by E_+ (which is also the cone of the positive basis) to $C - C$ being ordered by

C . Also, we notice that P is strictly positive in the sense that $Px = 0 \Leftrightarrow x = 0$, whenever $x \in E_+$. Hence, P may be taken as a strictly positive projection, and consequently we may repeat the proof of Theorem 3.

In the proof of ([8], Th. 5.7) the strongly reflexive cone's construction relies exactly on the existence of an unconditional basis for the Banach space E . Then we may understand that the crucial point for the above Theorems is the existence of a basic sequence for the Banach space E . We may remind the seminal work by Bessaga-Pelczynski [19] essentials on this topic.

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Appendix

In this Section, we give some essential notions and results from the theory of partially ordered linear spaces which are used in this paper. For these notions and definitions, see ([17], Ch. 1, Ch. 2, Ch. 3). Let E be a (normed) linear space. A set $C \subseteq E$ satisfying $C + C \subseteq C$ and $\lambda C \subseteq C$ for any $\lambda \in \mathbb{R}_+$ is called *wedge*. A wedge for which $C \cap (-C) = \{0\}$ is called *cone*. A pair (E, \geq) where E is a linear space and \geq is a binary relation on E satisfying the following properties:

- 1) $x \geq x$ for any $x \in E$ (reflexive);
- 2) If $x \geq y$ and $y \geq z$ then $x \geq z$, where $x, y, z \in E$ (transitive);
- 3) If $x \geq y$ then $\lambda x \geq \lambda y$ for any $\lambda \in \mathbb{R}_+$ and $x + z \geq y + z$ for any $z \in E$, where $x, y \in E$ (compatible with the linear structure of E), is called *partially ordered linear space*. The binary relation \geq in this case is a *partial ordering* on E . The set $P = \{x \in E \mid x \geq 0\}$ is called (*positive*) *wedge* of the partial ordering \geq of E . Given a wedge C in E , the binary relation \geq_C defined as follows:

$$x \geq_C y \Leftrightarrow x - y \in C,$$

is a partial ordering on E , called *partial ordering induced by C on E* . If the partial ordering \geq of the space E is *antisymmetric*, namely if $x \geq y$ and $y \geq x$ implies $x = y$, where $x, y \in E$, then P is a cone.

E' denotes the linear space of all linear functionals of E , called *algebraic dual* while E^* is the norm dual of E , in case where E is a normed linear space.

Suppose that C is a wedge of E . A functional $f \in E'$ is called *positive functional* of C if $f(x) \geq 0$ for any $x \in C$. $f \in E'$ is a *strictly positive functional* of C if $f(x) > 0$ for any $x \in C \setminus \{0\}$. A linear functional $f \in E'$ where E is a normed linear space, is called *uniformly monotonic functional* of C if there is some real number $a > 0$ such that $f(x) \geq a\|x\|$ for any $x \in C$. In case where a uniformly monotonic functional of C exists, C is a cone. $C^0 = \{f \in E^* \mid f(x) \geq 0 \text{ for any } x \in C\}$ is the *dual wedge of C in E^** . Also,

by C^{00} we denote the subset $(C^0)^0$ of E^{**} . It can be easily proved that if C is a closed wedge of a reflexive space, then $C^{00} = C$. If C is a wedge of E^* , then the set $C_0 = \{x \in E \mid \hat{x}(f) \geq 0 \text{ for any } f \in C\}$ is the *dual wedge of C in E* , where $\hat{\cdot} : E \rightarrow E^{**}$ denotes the natural embedding map from E to the second dual space E^{**} of E . Note that if for two wedges K, C of E , $K \subseteq C$ holds, then $C^0 \subseteq K^0$.

If C is a cone, then a set $B \subseteq C$ is called *base* of C if for any $x \in C \setminus \{0\}$ there exists a unique $\lambda_x > 0$ such that $\lambda_x x \in B$. The set $B_f = \{x \in C \mid f(x) = 1\}$ where f is a strictly positive functional of C is the *base of C defined by f* . B_f is bounded if and only if f is uniformly monotonic. If B is a bounded base of C such that $0 \notin \overline{B}$ then C is called *well-based*. If C is well-based, then a bounded base of C defined by a $g \in E^*$ exists. If $E = C - C$ then the wedge C is called *generating*, while if $E = \overline{C - C}$ it is called *almost generating*. If C is generating, then C^0 is a cone of E^* in case where E is a normed linear space. Also, $f \in E^*$ is a uniformly monotonic functional of C if and only if $f \in \text{int}C^0$, where $\text{int}C^0$ denotes the norm-interior of C^0 . If E is partially ordered by C , then any set of the form $f \in [x, y] = \{r \in E \mid y \geq_C r \geq_C x\}$ where $x, y \in C$

is called *order-interval of E* . If E is partially ordered by C and for some $e \in E$, $E = \bigcup_{n=1}^{\infty} [-ne, ne]$ holds, then e is called *order-unit* of E . If E is a normed linear space, then if every interior point of C is an order-unit of E . If E is moreover a Banach space and C is closed, then every order-unit of E is an interior point of C . The partially ordered vector space E is a *vector lattice* if for any $x, y \in E$, the supremum and the infimum of $\{x, y\}$ with respect to the partial ordering defined by P exist in E . In this case $\sup\{x, y\}$ and $\inf\{x, y\}$ are denoted by $x \vee y$, $x \wedge y$ respectively. If so, $|x| = \text{osup}\{x, -x\}$ is the *absolute value* of x and if E is also a normed space such that $\| |x| \| = \|x\|$ for any $x \in E$, then E is called *normed lattice*. If a normed lattice is a Banach space, then it is called *Banach lattice*. A Banach lattice E whose norm has the property $\|x + y\| = \|x\| + \|y\|$, $x, y \in E_+$ is called *AL-space*. A set S in a vector lattice E is called *solid* if $|y| \leq |x|$ and $x \in S$ implies $y \in S$. A solid vector subspace of a vector lattice is called *ideal*. An ideal I is a *sublattice* of E , i.e., a subspace of E such that $x \vee y \in I, x \wedge y \in I$ if $x, y \in I$ respectively. A net $\{x_a\}_{a \in A}$ in a vector lattice E is *order convergent* to x if there is a net $\{y_a\}_{a \in A}$ in E with $y_a \downarrow 0$, such that $|x_a - x| \leq y_a$ for each $a \in A$. This convergence is denoted by $x_a \xrightarrow{o} x$. A set D in E is *order closed* if $\{x_a\}_{a \in A} \subseteq D$ and $x_a \xrightarrow{o} x$, implies $x \in D$. If D is also an ideal, then D is called *band*. A Banach lattice has *order continuous* norm, if for any net $\{x_a\}_{a \in A} \subseteq E$ with $x_a \downarrow 0$, $\|x_a\| \downarrow 0$ holds. A Banach lattice E which is a band in its second dual (in the sense of norm topology)

is called *Kantorovich-Banach space*. If S is a subset of a vector lattice E , then its *disjoint complement* is the set $S^d = \{x \in E : |x| \wedge |y| = 0 \text{ for any } y \in S\}$. If for a vector lattice E a band B satisfies the property $E = B \oplus B^d$, then B is called *projection band*. Finally, if E is a partially ordered Banach space whose positive cone is E_+ , if E has a Schauder basis $(e_n)_{n \in \mathbb{N}}$, this basis is called *positive basis* if and only if $E_+ = \left\{ x = \sum_{n=1}^{\infty} \lambda_n e_n \mid \lambda_n \geq 0, n \in \mathbb{N} \right\}$. For linear lattices and positive bases see in ([20], Ch. 8), and [11], respectively.

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