

# Correlations and Hyper-Correlations

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## Abstract

Recent developments in quantum information allow for a new understanding of quantum correlations. The aim of this paper is to physically explain why quantum mechanics obeys a stronger bond than the non-signaling requirement or alternatively why it obeys a principle of information causality. It is shown that a physical theory violating the quantum bond allows for correlations between settings while quantum mechanics only allows for correlations between possible outcomes. In fact, correlations between settings would violate the protocols used in quantum cryptography. The conclusion is that information codification is a local operation and quantum mechanics sets the general conditions for information exchanging in our universe since it satisfies and saturates the bond that is imposed by the principle of information causality, and in so doing it also sets specific constraints on both the possible interdependencies and the possible interactions (also causal interconnections) in our universe.

**Keywords:** Entanglement, Non-Locality, Tsirelson Inequality, Principle of Information Causality

## 1. Introduction

One of the biggest mysteries of quantum mechanics is represented by the concept of entanglement, originally proposed by Schrödinger [1] as a solution of the EPR paradox [2] and since then widely tested and fully accepted by the scientific community [3]. Although many issues have been clarified in the last years, a lot of questions are still open.

As is well known, in 1964 Bell was able to derive an inequality that should be satisfied by any classical theory not admitting the existence of any kinds of non-local interdependencies between systems [4]. Often, scholars in this field make use of a reformulation of Bell's inequality that is known as CHSH inequality [5]:

$$|\langle a,b \rangle + \langle a,b' \rangle + \langle a',b \rangle - \langle a',b' \rangle| \leq 2, \quad (1)$$

where  $a'$  is a setting alternative to  $a$  (traditionally a spin direction) as well as  $b'$  to  $b$ . From a quantum-mechanical point of view the expression  $\langle a,b \rangle$  (and analogues) is a short-hand for the mean value  $\langle (\hat{\sigma}_1 \cdot a)(\hat{\sigma}_2 \cdot b) \rangle$  of the spin observables along the directions  $a, b$  for the particles 1 and 2 computed on a singlet state. Conventional partners are called Alice and Bob and we may think that Alice can choose to use either the setting  $a$  or  $a'$  whilst Bob can choose between  $b$  and  $b'$ .

It is largely proven that this inequality is violated by quantum systems, which therefore display some non-local interdependencies. It was however unknown to date which kind of interdependencies the quantum-mechanical ones are and whether or not quantum mechanics obeys some bond.

## 2. A Quantum-Mechanical Bond

In 1980 Tsirelson was able to prove that quantum mechanics is bonded by the value  $2\sqrt{2}$  [6]. Indeed, let  $\hat{O}^a, \hat{O}^{a'}, \hat{O}^b, \hat{O}^{b'}$  be arbitrary Hermitian operators on a Hilbert space  $H$  satisfying the condition  $[\hat{O}^a, \hat{O}^b] = 0$  and so on for the other couples  $(a, b'), (a', b), (a', b')$ . Moreover, each operator has eigenvalues 1 and  $-1$ . We now define the Bell operator [7], which is clearly related to the CHSH inequality:

$$\hat{B} = \hat{O}^a \hat{O}^b + \hat{O}^{a'} \hat{O}^b + \hat{O}^a \hat{O}^{b'} - \hat{O}^{a'} \hat{O}^{b'}. \quad (2)$$

Since the square of each operator is equal to the identity, this implies

$$\begin{aligned} 2\sqrt{2} - \hat{B} &= \frac{1}{\sqrt{2}} \left[ (\hat{O}^a)^2 + (\hat{O}^{a'})^2 + (\hat{O}^b)^2 + (\hat{O}^{b'})^2 \right] - \hat{B} \\ &= \frac{1}{\sqrt{2}} \left[ \left( \hat{O}^a - \frac{\hat{O}^b + \hat{O}^{b'}}{\sqrt{2}} \right)^2 + \left( \hat{O}^{a'} - \frac{\hat{O}^b - \hat{O}^{b'}}{\sqrt{2}} \right)^2 \right] = \hat{A}. \end{aligned} \quad (3)$$

Since the sum or difference between Hermitian operators is itself a Hermitian operator, the operator  $\hat{A}$  is Hermitian and being the sum of squares of Hermitian operators has a non-negative expectation value

$$\langle \hat{A} \rangle \geq 0, \quad (4)$$

which implies

$$\left| \langle \hat{B} \rangle \right| \leq 2\sqrt{2}. \quad (5)$$

This important result sets a clear bond for quantum-mechanical systems but does not physically justify it and as a consequence does not fully resolve the problem of the kind of correlations we deal with in quantum mechanics.

### 3. The Non-Locality Requirement and Quantum Mechanics

An important step was when Popescu and Rohrlich [8] showed which is the bond set on any theory satisfying the relativistic requirement that no superluminal signals can be sent (in short, *non-signaling requirement*). This requirement implies that the operations that one can perform locally are not influenced by operations that one performs elsewhere, which implies in particular that the probability to obtain a certain outcome (say 1) when choosing the direction  $a$  is independent from the outcomes (either +1 or -1) when elsewhere one chooses a direction  $b$  or  $b'$ , that is,

$$\wp_{a,b}(1,1) + \wp_{a,b}(1,-1) = \wp_{a,b'}(1,1) + \wp_{a,b'}(1,-1). \quad (6)$$

Similar considerations hold for any direction. All the four different expectation values on the LHS of inequality (1) can be formulated in terms of the above probabilities:

$$\begin{aligned} \langle a, b \rangle &= \wp_{a,b}(1,1) + \wp_{a,b}(-1,-1) \\ &\quad - \wp_{a,b}(1,-1) - \wp_{a,b}(-1,1), \end{aligned} \quad (7)$$

which corresponds to the classical case in which there is no correlation (bond = 2). However, the non-signaling requirement allows us also to build the set of probabilities

$$\begin{aligned} \wp_{a,b}(1,1) &= \wp_{a,b}(-1,-1) = \frac{1}{2} \\ \wp_{a,b'}(1,1) &= \wp_{a,b'}(-1,-1) = \frac{1}{2} \\ \wp_{a',b}(1,1) &= \wp_{a',b}(-1,-1) = \frac{1}{2} \\ \wp_{a',b'}(1,-1) &= \wp_{a',b'}(-1,1) = \frac{1}{2}, \end{aligned} \quad (8)$$

while all other probabilities are zero and where I remark

that only the  $\wp_{a',b'}$  probabilities show anti-correlation. In this case, two out the four terms in the expectation value (7)—and similar ones—vanish. Since each of those mean values in Equation (1) lies now between -1 and +1, the natural upper bound for the entire expression is +4:

$$\left| \langle a, b \rangle + \langle a, b' \rangle + \langle a', b \rangle - \langle a', b' \rangle \right| \leq 4, \quad (9)$$

Given this result, we are left with two important questions:

- Why quantum mechanics satisfy a stricter bond than the non-signaling requirement? To answer this question we need to deal with the nature of information transmission.
- Which kind of physical entities are correlated according to quantum mechanics? We shall see that to give a specific answer to this question we need to consider which kind of physics would be the one violating the quantum-mechanical bond  $2\sqrt{2}$  but still satisfying the non-signaling bond 4.

The answer to the first question was recently provided in [9] and turns out to be really surprising: Using all his local resources (which may be correlated with her resources) and allowing classical communication from Alice to Bob, the amount of information that the latter can recover is bounded by the information volume of the communication. Namely, if Alice classically communicates  $n$  bits to Bob, the total information obtainable by Bob cannot be greater than  $n$ . This has been called the principle of information causality. To a certain extent, it could appear obvious. What is less obvious is that it is connected with the quantum-mechanical bond discovered by Tsirelson [6].

### 4. Eberhard's Theorem

To address the second question above I would like to write the operator  $\hat{B}$  as a combination of correlations  $C_{jk}$  (where  $j, k = a, b, a', b'$ ) expressed in informational terms, that is, with  $j, k = 1, 0$ . I would also like to express in informational term 1,0 the possible outputs of Alice's and Bob's measurements. In other words, instead of speaking of polarization directions  $a$ ,  $a'$ ,  $b$ , and  $b'$ , or of observables  $\hat{O}^a, \hat{O}^{a'}, \hat{O}^b, \hat{O}^{b'}$ , I would like to introduce generic inputs  $a, b = 0, a', b' = 1$ ; moreover, instead of having possible results -1,1, I introduce information outputs 0, 1. With these assumptions, I rewrite the correlations occurring in the CHSH inequality as [10]

$$\begin{aligned} C_{00} &= \wp(11|00) + \wp(00|00), \\ C_{10} &= \wp(11|10) + \wp(00|10), \\ C_{01} &= \wp(11|01) + \wp(00|01), \\ C_{11} &= \wp(10|11) + \wp(01|11), \end{aligned} \quad (10)$$

where the inputs follow the vertical lines and the outputs precede them. Therefore, it is quite natural that we interpret the inputs as the chosen settings and the outputs as the measurement outcomes (representing the steps of premeasurement and measurement, respectively). Indeed, the above correlations allow Bob to guess which is the outcome of Alice (whether 0 or 1) if she communicates to him which is the setting she has chosen (whether 0 or 1). In other words, quantum entanglement is a non-local interdependency between *possible outcomes*. This is a quantum-information resource that is commonly used in quantum cryptography [11,12]. However, the above conditional probabilities tell us nothing about the kind of dependence we deal with. Abstractly speaking, they also allow for a sort of Bayesian inversion, that is, they allow Bob to guess the setting of Alice (whether 0 or 1) if she communicates to him her measurement outcome (whether 0 or 1). Are these two possibilities mutually exclusive? If the only requirement is the non-signaling one, this is not the case. Then, we would have a situation in which there is not only correlation between possible outcomes but also correlation between *distant settings*. This is however not the case for quantum mechanics.

An often forgotten result of P. Eberhard [13] can help us to understand this point. Let us prove this theorem in its full generality and let  $\hat{O}_1$  and  $\hat{O}_2$  be two observables on subsystems  $S_1$  and  $S_2$  of a system  $S$ , respectively, and  $\wp(o_a, a; o_b, b)$  be the probability that the results of a measurement of  $\hat{O}_1$  on  $S_1$  and  $\hat{O}_2$  on  $S_2$  yield  $o_a$  and  $o_b$  when certain settings of the measurement apparatus are  $a$  and  $b$ , respectively. According to Eberhard, the probability distribution of  $\hat{O}_1$  (or  $\hat{O}_2$ ), independently of the measurement operations on  $\hat{O}_2$  (or  $\hat{O}_1$ ), obtained by integrating or summing the probabilities  $\wp(o_a, a; o_b, b)$  over the possible outcomes  $o_b$  (or  $o_a$ ), needs to be independent of the other setting  $b$  (or  $a$ ), that is, the two probabilities must depend on local settings only:

$$\sum_{o_b} \wp(o_a, a; o_b, b) = \wp(o_a, a); \quad \sum_{o_a} \wp(o_a, a; o_b, b) = \wp(o_b, b). \tag{11}$$

Indeed, the joint probability of obtaining the two results  $o_a$  and  $o_b$  given the settings  $a$  and  $b$ , is given by

$$\begin{aligned} \wp(o_a, a; o_b, b) &= \wp(o_a, a) \wp'(o_b, b | o_a, a) \\ &= \wp(o_a, a) \frac{\text{Tr} \left[ \hat{P}_{o_b, b} \hat{P}_{o_a, a} \hat{\rho} \hat{P}_{o_a, a} \right]}{\wp(o_a, a)} \tag{12} \\ &= \text{Tr} \left[ \hat{P}_{o_b, b} \hat{P}_{o_a, a} \hat{\rho} \hat{P}_{o_a, a} \right], \end{aligned}$$

where the density matrix  $\hat{\rho}$  describes the state of the compound system. Given these assumptions, by making

use of the properties of projectors and of the cyclic property of the trace, we can obtain the following result that is in accordance with Equation (11):

$$\begin{aligned} \sum_{o_a} \wp(o_a, a; o_b, b) &= \text{Tr} \sum_{o_a} \left[ \hat{P}_{o_b, b} \hat{P}_{o_a, a} \hat{\rho} \hat{P}_{o_a, a} \right] \\ &= \text{Tr} \left[ \hat{P}_{o_b, b} \hat{\rho} \right] = \wp(o_b, b), \end{aligned} \tag{13}$$

and similarly for  $\wp(o_a, a)$ . According to Eberhard, if the requirement (11) were not satisfied, we would have a *causal non-local interdependence* between the two subsystems (violating in this way relativistic locality), because, by changing the setting  $a$  (or  $b$ ), we would be able to act on the result of the other measurement, and hence, if we performed experiments on subsystems that are space-like separated, we would be able to transmit a message with superluminal or even infinite speed. Actually, Eberhard's interpretation of his own result is not fully accurate. Indeed, what happens is that his result is a stronger requirement than that imposed by locality (non-signaling). This is clear if we reformulate Equation (11) in analogy with Equation (6) as

$$\begin{aligned} \wp_{a,b}(1,1) + \wp_{a,b}(1,-1) &= p_a(1) \text{ and} \\ \wp_{a,b}(1,1) + \wp_{a,b}(-1,1) &= \wp_a(1), \end{aligned} \tag{14}$$

and similarly for the other outcomes.

### 5. Discussion

Eberhard's theorem clearly shows that quantum mechanics requires a full independence of the settings (here expressed e.g. by the orientation  $a$ ), which need to be local operations performed in complete separation from other operations that could be performed elsewhere. Now, in a world in which settings (and not only outcomes) were shared, this would imply that also information codification were shared as well. Indeed, information codification deals with the choice of a basis (the code) which in a measurement context is the choice of a particular setting. In other words, in a world showing hyper-correlations based on the sharing of settings, information codification would be no longer a local procedure. In such a case, the principle of information causality could be violated: communicating any strings of bit, in appropriate conditions, could allow somebody to guess the code used by the sender, and this would represent also a violation of the principles of quantum cryptography. Indeed, in the Bennett and Brassard's protocol [11], Alice and Bob are connected by a quantum communication channel which allows the transmission of quantum states. In addition, they may communicate via a public classical channel. In particular, if the quantum channel is represented by the transmission of photons (e.g. in a optical

**Table 1. An example of sequence transmission in the Bennett and Brassard's protocol for quantum key distribution.**

Alice's random bits	0	1	1	1	0	0	1	0
Alice's random chosen basis	+	+	×	×	×	+	+	×
Photon polarization sent by Alice	$ \rightarrow\rangle$	$ \uparrow\rangle$	$ \nearrow\rangle$	$ \nearrow\rangle$	$ \searrow\rangle$	$ \rightarrow\rangle$	$ \uparrow\rangle$	$ \searrow\rangle$
Bob's chosen basis	+	×	+	×	×	×	+	+
Bob's measurement result	$ \rightarrow\rangle$	$ \searrow\rangle$	$ \uparrow\rangle$	$ \nearrow\rangle$	$ \searrow\rangle$	$ \nearrow\rangle$	$ \uparrow\rangle$	$ \rightarrow\rangle$
Shared secret key	0	-	-	1	0	-	1	-

fiber or in free space), they can make use of two possible bases of polarization,

$$+ = \{|\uparrow\rangle, |\rightarrow\rangle\} \quad \text{and} \quad \times = \{|\nearrow\rangle, |\searrow\rangle\}, \quad (15)$$

representing, respectively, a vertical--horizontal polarization and a  $45^\circ$  -  $135^\circ$  polarization states. First, Alice establishes a one-to-one correspondence between the classical bits (0, 1) that she desires to communicate with each state of the two bases, for instance

$$0 \leftrightarrow |\rightarrow\rangle, |\searrow\rangle, \quad (16a)$$

$$1 \leftrightarrow |\uparrow\rangle, |\nearrow\rangle. \quad (16b)$$

Then, she sends several bits of information by choosing at random one or the other basis. Bob will measure the photons choosing again at random one of the two bases. After this exchange and measurement, they publicly tell each other which basis (setting) they have used. They will immediately discard photon transmissions where the two bases do not match (on the average 50% of the transmitted bits). The bits for which Alice and Bob chose the same basis constitute the shared key. In other words, there are two pieces of information that are necessary here in order to constitute a common code:

- The information about the settings: This is communicated through a classical channel;
- The information about the outcomes in order to correctly pair Alice's code and Bob's code: This is provided by the quantum channel, *i.e.* the entanglement between the photons.

The situation is schematically shown in **Table 1**. Instead, hypercorrelations would allow for sharing information about the settings without any classical communication at all.

## 6. Conclusions

The fact that quantum mechanics forbids setting-sharing justifies quantum information as a general theory of information since

- It satisfies and saturates the (Tsirelson) bond that is imposed by the principle of information causality, and in so doing
- It also sets specific constraints on both the possible interdependencies and the possible interactions (also

causal interconnections) in our universe.

## 7. References

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