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Supersymmetric Resolvent-Based Fourier Transform

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Abstract

We calculate in a numerically friendly way the Fourier transform $\mathcal{F}$ of a non-integrable function, such as $\varphi(x)=1$, by replacing $\mathcal{F}$ with $\mathcal{R}^{-1}\mathcal{F}\mathcal{R}$, where $\mathcal{R}$ represents the resolvent for harmonic oscillator Hamiltonian. As contrasted with the non-analyticity of $\left(x^2+a^2\right)^{-1}\varphi(x)$ at $x=\pm ia$ in the case of a simple replacement of $\mathcal{F}$ by $\left(\hat{p}^2+a^2\right)\mathcal{F}\left(\hat{q}^2+a^2\right)^{-1}$, where $\hat{p}$ and $\hat{q}$ represent the momentum and position operators, respectively, the $\mathcal{R}\varphi$ turns out to be an entire function. In calculating the resolvent kernel, the sampling theorem is of great use. The resolvent based Fourier transform can be made supersymmetric (SUSY), which not only makes manifest the usefulness of the even-odd decomposition of φ in a more natural way, but also leads to a natural definition of SUSY Fourier transform through the commutativity with the SUSY resolvent.

Keywords

Resolvent, Fourier Transform, Supersymmetry, Harmonic Oscillator Hamiltonian, Sampling Theorem

1. Introduction

Fourier transform (FT) $\mathcal{F}: L^2(\mathbb{R})\to L^2(\mathbb{R})$ by

$$(\mathcal{F}\varphi)(x)=\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}\mathrm{e}^{ikx}\varphi(k)\mathrm{d}k,$$

which is a unitary operator, is a fundamental method in function analysis and is applied to many fields in physics. The corresponding self-adjoint operator is given by the harmonic oscillator Hamiltonian $\mathcal{H}: L^2(\mathbb{R})\to L^2(\mathbb{R})$ by

$$(\mathcal{H}\varphi)(x)=\frac{1}{2}\left(\hat{p}^2+\hat{q}^2-1\right)\varphi(x), \tag{1}$$

where $(\hat{p}\varphi)(x)=\frac{1}{\mathrm{i}}\frac{\mathrm{d}}{\mathrm{d}x}\varphi(x)$ and $(\hat{q}\varphi)(x)=x\varphi(x)$, through the relation

$$\mathcal{F}=\mathrm{e}^{\mathrm{i}\theta\mathcal{H}},\ \theta=\frac{\pi}{2}. \tag{2}$$

The validity of (2) is verified by noticing that the Hermite polynomial $H_n(x)$ (multipled by $\mathrm{e}^{-x^2/2}$) is a simultaneous eigenfunction of $\mathcal{F}$ and $\mathcal{H}$, with their eigenvalues given by i^n and n, respectively.

If a function $\varphi:\mathbb{R}\to\mathbb{C}$ is integrable, its FT is well defined. However, if the function φ is not integrable, for example $\varphi(x)=1$, its FT should be regarded as a generalized function. To calculate the FT of $\varphi(x)=1$ in a numerically friendly way, one of the methods is to replace $\mathcal{F}$ by $\mathcal{G}^{-1}\mathcal{F}\mathcal{G}$ such that $[\mathcal{F},\mathcal{G}]=0$, and to choose $\mathcal{G}$ as the resolvent for $\mathcal{H}$, that is [1]

$$\mathcal{G}=(\mathcal{H}-a)^{-1} \qquad (a\notin\operatorname{spect}\mathcal{H}).$$

Considering that $\mathcal{H}$ includes the term proportional to $\hat{q}^2$, we find that $(\mathcal{G}\varphi)(x)$ behaves like x^{-2} for $|x|\to\infty$. Thus $\mathcal{G}\varphi$ can be Fourier transformed.

To make $\varphi(x)=1$ square integrable, it is sufficient to reduce the order of $\varphi(x)$ (for $|x|\to\infty$) by one, not necessarily by two. This implies that it is sufficient to choose $\mathcal{G}\simeq\mathcal{H}^{-1/2}$, not necessarily $\mathcal{G}\simeq\mathcal{H}^{-1/2}$, as given above. However, the square root of the operator $\mathcal{H}$, in general, is somewhat complicated to deal with, so we adopt an alternative approach, supersymmetrization. The supersymmetry (SUSY) can be realized by adding $\mathcal{H}$ in (1) to $f^\dagger f$ [2], where $f^\dagger, f:\mathbb{C}^2\to\mathbb{C}^2$, representing the fermionic creation and annihilation operators, respectively, satisfying $f^2=0$, $(f^\dagger)^2=0$, and $\{f,f^\dagger\}=I$, with $\{A,B\}=AB+BA$. The modified Hamiltonian $\mathcal{H}'=\mathcal{H}I+f^\dagger f$ can be decomposed into $\mathcal{H}=\mathcal{Q}^2$, where $\mathcal{Q}$ is called a supercharge. Under the modification $\mathcal{H}\to\mathcal{H}'$, it is natural to transform $\mathcal{F}$ to $\mathcal{F}'=\mathrm{e}^{\mathrm{i}\theta\mathcal{H}'}$, as is analogous to (2).

The aim of this paper is replace $\mathcal{F}'$ by

$$\mathcal{F}'=(\mathcal{Q}+\alpha I)\mathcal{F}'(\mathcal{Q}+\alpha I)^{-1},$$

with $\alpha\in\mathbb{C}$ chosen in an appropriate way, to finally find that the introduction of SUSY clarifies the availability of the even-odd decomposition of φ in a more natural way. In Section 2, we generalize the resolvent kernel for $\mathcal{H}$, where $\mathcal{H}$ can be regarded as the specialization of the Hamiltonian $\mathcal{H}^{(\alpha,\beta)}$ whose eigenfunction is given by the Jacobi polynomial. In calculating the resolvent kernel, the sampling theorem [3] is fully employed. In Section 3, we first reexamine the FT of $\varphi(x)=1$, based on the resolvent for $\mathcal{Q}$. Then we compare the resolvent based method with other methods, to find that the former has some merits of being numerical calculation friendly and free of singularity for $(\mathcal{H}-a)^{-1}\varphi$, even after analytic continuation. Analytic property is significant for calculating, for example, path integral in Minkowski space (Wick roration), and the Shannon entropy in the limit of the Rényi entropy (replica trick). We give conclusion in Section 4.

2. Methods

In this section, we first obtain the resolvent kernel for the Hamiltonian whose eigenfunction is given by the Jacobi polynomial. Then we calculate the resolvent kernel for $\mathcal{H}$ as a specialization of the former.

2.1. Jacobi Polynomial

Let $\mathcal{H}^{(\alpha,\beta)}: L^2(\Omega)\to L^2(\Omega)$ (where $\Omega=[-1,1]\subset\mathbb{R}$) be the Hamiltonian

$$\left(\mathcal{H}^{(\alpha,\beta)}\varphi\right)(x)=\left[\frac{1}{\sqrt{w(x)}}\frac{\mathrm{d}}{\mathrm{d}x}\left(1-x^2\right)w(x)\frac{\mathrm{d}}{\mathrm{d}x}\frac{1}{\sqrt{x}}\right]\varphi(x),\quad w(x)=(1-x)^{\alpha}(1+x)^{\beta}.$$

The (normalized) eigenfunction for $\mathcal{H}^{(\alpha,\beta)}$ is given by

$$\phi_n^{(\alpha,\beta)}(x)=\sqrt{\frac{w(x)}{N_n^{(\alpha,\beta)}}}P_n^{(\alpha,\beta)}(x)\qquad(\text{for }\alpha,\beta>-1),$$

where $P_n^{(\alpha,\beta)}(x)$ and $N_n^{(\alpha,\beta)}$ represent the Jacobi polynomial and its normalization constant as

$$P_n^{(\alpha,\beta)}(x)=\frac{\Gamma(n+\alpha+1)}{\Gamma(n+1)\Gamma(\alpha+1)}{}_2F_1\left(n+\alpha+\beta+1,-n;\alpha+1;\frac{1-x}{2}\right),\tag{3}$$

$$N_n^{(\alpha,\beta)}=\int_\Omega w(x)\left(P_n^{(\alpha,\beta)}(x)\right)^2\mathrm{d}x=\frac{2^{\alpha+\beta+1}}{2n+\alpha+\beta+1}\frac{\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{\Gamma(n+1)\Gamma(n+\alpha+\beta+1)},$$

with $\Gamma(z)$ and ${}_2F_1(a,b;c;z)$ the Gamma function and the hypergeometric function, respectively. The corresponding eigenvalue is given by

$$\left[\mathcal{H}^{(\alpha,\beta)}+n(n+\alpha+\beta+1)\right]\phi_n^{(\alpha,\beta)}(x)=0.\tag{4}$$

The resolvent kernel for $\mathcal{H}^{(\alpha,\beta)}$ (denoted by $K_\nu^{(\alpha,\beta)}(x,y)$) can be expanded using the eigenfunctions $\phi_n^{(\alpha,\beta)}(x)$'s (for $n\in\mathbb{N}$) as

$$\begin{aligned}K_\nu^{(\alpha,\beta)}(x,y)&=\left\langle x\left|\left[\nu(\nu+\alpha+\beta+1)+\mathcal{H}^{(\alpha,\beta)}\right]^{-1}\right|y\right\rangle\\&=\sum_{n=0}^{\infty}\left\langle x\left|\left[\nu(\nu+\alpha+\beta+1)+\mathcal{H}^{(\alpha,\beta)}\right]^{-1}\right|\phi_n^{(\alpha,\beta)}\right\rangle\left\langle\phi_n^{(\alpha,\beta)}\middle|y\right\rangle\\&=\sum_{n=0}^{\infty}\left[\nu(\nu+\alpha+\beta+1)-n(n+\alpha+\beta+1)\right]^{-1}\phi_n^{(\alpha,\beta)}(x)\phi_n^{(\alpha,\beta)}(y),\end{aligned}\tag{5}$$

where in the second and third equalities, use has been made of the completeness for $\left\{\phi_n^{(\alpha,\beta)}(x)\right\}_{n\in\mathbb{N}}$ and (4), respectively.

There seems to be no such formula as the series sum of (5) for general parameters α and β. However, it will be found that the sum can be represented as the product of two hypergeometric functions as follows. The starting point would be the following formula, which corresponds to the particular case of $(\alpha,\beta)=(0,0)$ as [4] [5]

$$\frac{\sin\pi\nu}{\pi}K_\nu^{(0,0)}(-x,y)=\frac{1}{2\nu+1}\phi_\nu^{(0,0)}(x)\phi_\nu^{(0,0)}(y)\qquad(\text{for }(x,y)\in D),\tag{6}$$

where $D:=\left\{(x,y)\in\mathbb{R}^2\middle|x+y>0;x,y<1\right\}$. Notice that $\phi_\nu^{(0,0)}(x)$ is given by the Legendre function $P_\nu(x)$ as

$$\phi_\nu^{(0,0)}(x)=\sqrt{\frac{2\nu+1}{2}}P_\nu(x),$$

where $P_\nu^{(\alpha,\beta)}(x)$ is defined by replacing n in (3) with ν. Before proceeding further, we discuss the validity of (6). By applying $\left[\mathcal{H}^{(0,0)}+\nu(\nu+1)\right]$ to (6) from the left, it is found that both sides of (6) satisfy the same second order differential equation for $x+y\neq 0$, due to the completeness relation of $\sum_{n=0}^{\infty}\phi_n^{(0,0)}(-x)\phi_n^{(0,0)}(y)=\delta(x+y)$. The reason of restricting (x,y) to $x+y>0$ is as follows. To avoid the singularity of $\delta(x+y)$ at $x+y=0$, (x,y) should be restricted to either $x+y>0$ or $x+y<0$. Moreover, to avoid the singularity of $P_\nu^{(0,0)}(x)$ (for $\nu\notin\mathbb{N}$) at $x=-1$, the region of $x+y<0$ is not allowed.

Furthermore, it should be noted that the left-hand side of (6) turns out to be $\frac{1}{2}\sum_{n=-\infty}^{\infty}\frac{\sin\pi(\nu-n)}{\pi(\nu-n)}P_n(x)P_n(y)$, due to the relation

$$P_{-n-1}(x)=P_n(x)=(-1)^n P_n(-x)\qquad (\text{for } all\ n\in\mathbb{N}).$$

Thus the relation of (6) can be rewritten as

$$\sum_{n=-\infty}^{\infty}\operatorname{sinc}(\nu-n)P_n(x)P_n(y)=P_\nu(x)P_\nu(y)\qquad (\text{for } (x,y)\in D),\tag{7}$$

where $\operatorname{sinc}(z):=\frac{\sin\pi z}{\pi z}$, so that the sampling theorem [3] can be applied to $P_\nu(x)P_\nu(y)$. The sampling theorem states that for $f:\mathbb{R}\to\mathbb{C}$

$$\left(\operatorname{supp}\mathcal{F}f\subseteq[-\pi,\pi]\right)\Rightarrow\left(f(\nu)=\sum_{n=-\infty}^{\infty}\operatorname{sinc}(\nu-n)f(n)\right),\tag{8}$$

where $\operatorname{supp}(\cdot)$ represents the support. Hence the validity of (7) is guaranteed by showing that $\operatorname{supp}\mathcal{F}f\subseteq[-\pi,\pi]$ for $f(\nu)=P_\nu(x)P_\nu(y)$ (with $(x,y)\in D$). To show it, it is convenient to use the integral representation for $P_\nu(\cos\theta)$ as [6]

$$P_\nu(\cos\theta)=\frac{\sqrt{2}}{\pi}\int_0^{|\theta|}(\cos\phi-\cos\theta)^{-1/2}\cos\left[\left(\nu+\frac{1}{2}\right)\phi\right]\mathrm{d}\phi\qquad (\text{for } |\theta|<\pi),$$

from which it is found that $\int_{\mathbb{R}}\mathrm{e}^{\mathrm{i}t\nu}P_\nu(\cos\theta)P_\nu(\cos\psi)\mathrm{d}\nu$ is vanishing for $|t|>\pi$ under the conditions of $|\theta+\psi|<\pi$ and $|\theta-\psi|<\pi$. Here, we have used the integral representation for the Dirac delta as $\delta(t-t')\propto\int_{\mathbb{R}}\mathrm{e}^{\mathrm{i}(t-t')\nu}\mathrm{d}\nu$. Noticing further that

$$\begin{cases}|\theta+\psi|<\pi\\ |\theta-\psi|<\pi\end{cases}\Leftrightarrow\quad(\cos\theta,\cos\psi)\in D,$$

we can eventually prove the relation of (7) by employing the sampling theorem.

Before proceeding further, we try to rewrite the summation relation in the right-hand side of (8) in terms of the Dirac notation as

$$|f\rangle=\sum_{n\in\mathbb{Z}}|n\rangle\langle n|f\rangle,\tag{9}$$

where $\langle\nu|f\rangle=f(\nu)$ and

$$\langle\nu|n\rangle=\operatorname{sinc}(\nu-n)\qquad(\text{for all } \nu\in\mathbb{R}),$$

from which we obtain the orthonormality relation $\langle m|n\rangle=\delta_{mn}$ for all $m,n\in\mathbb{Z}$. The relation of (9) implies that the completeness relation $\sum_{n=-\infty}^{\infty}|n\rangle\langle n|=1$ holds, provided it is applied to $|f\rangle$ such that $\operatorname{supp}\mathcal{F}f\subseteq[-\pi,\pi]$. Moreover, interpreting $\mathcal{F}|f\rangle$ and

$\mathcal{F}|n\rangle$ as $|\mathcal{F}f\rangle$ and $|\mathcal{F}n\rangle$, respectively, we can formally obtain from (9)

$$\langle x|\mathcal{F}n\rangle = \frac{1}{\sqrt{2\pi}}\mathrm{e}^{\mathrm{i}nx}W(x) \qquad (\text{for } x\in\mathbb{R}, n\in\mathbb{Z}), \tag{10}$$

where $W:\mathbb{R}\to\mathbb{R}$ represents the window function as

$$W(x) = \begin{cases} 1 & (|x|<\pi), \\ 0 & (|x|>\pi). \end{cases}$$

The relation of (10) should be compared with

$$\langle x|\mathcal{F}k\rangle = \frac{1}{\sqrt{2\pi}}\mathrm{e}^{\mathrm{i}kx} \qquad (\text{for } x,k\in\mathbb{R}). \tag{11}$$

[In the usual Dirac notation, k is reserved for a Fourier transformed variable, so that $|\mathcal{F}k\rangle$ may be simply written as $|k\rangle$. Actually, if we formally write $\langle f|\mathcal{F}k\rangle$ as $\langle f|k\rangle \left(=\int_{\mathbb{R}}\langle f|x\rangle\langle x|k\rangle \mathrm{d}x\right)$, it is found that $\langle x|k\rangle = \frac{1}{\sqrt{2\pi}}\mathrm{e}^{\mathrm{i}kx}$, because

$$\langle f|\mathcal{F}k\rangle = \langle \mathcal{F}^{-1}f|k\rangle = \langle k|\mathcal{F}^{-1}f\rangle^{*} = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} f^{*}(x)\mathrm{e}^{\mathrm{i}kx}\,\mathrm{d}x$$

where use has been made of the unitarity of $\mathcal{F}$ as $\mathcal{F}^{\dagger}=\mathcal{F}^{-1}$. In this sense, $\langle x|\mathcal{F}k\rangle$ can be simply written as $\langle x|k\rangle$.] Notice that (10) cannot be derived from (11) by formally setting $k\in\mathbb{R}$ to $n\in\mathbb{Z}$. This is because $\langle n|$ in (10) can be applied only to $|f\rangle$ such that $\operatorname{supp}\mathcal{F}f\subseteq[-\pi,\pi]$. Notice further that the following relation can be derived from (10):

$$\sum_{n\in\mathbb{Z}}\langle x|\mathcal{F}n\rangle\langle \mathcal{F}n|y\rangle = \delta(x-y)W(x) \qquad (\text{for } x,y\in\mathbb{R}), \tag{12}$$

where we have used $\sum_{n\in\mathbb{Z}}\mathrm{e}^{\mathrm{i}nx} = 2\pi\sum_{n\in\mathbb{Z}}\delta(x-2n\pi)$. The relation of (12) indicates that the completeness relation $\sum_{n\in\mathbb{Z}}|\mathcal{F}n\rangle\langle\mathcal{F}n|=1$ holds, if it is applied to $|f\rangle$ such that $\operatorname{supp} f\subseteq[-\pi,\pi]$, so that $f(x)=f(x)W(x)$. These completeness relations, along with the orthogonal relations, are recapitulated in Table 1, while some examples of $f(\nu)$ satisfying (9) are listed in Table 2.

Now we go back to generalize the relation of (6). Using the integral representation for $\hat{C}_{\nu}^{\lambda}(x) := {}_2F_1\left(\nu+2\lambda,-\nu;\lambda+\frac{1}{2};\frac{1-x}{2}\right)$ (notice that $P_{\nu}(x)=\hat{C}_{\nu}^{1/2}(x)$) as [7]

$$\hat{C}_{\nu}^{\lambda}(\cos\theta) = \frac{c(\lambda)}{\sin^{2\lambda-1}\theta}\int_0^{\theta}(\cos\phi-\cos\theta)^{\lambda-1}\cos\left[(\nu+\lambda)\phi\right]\mathrm{d}\phi \quad (\text{for } 0<\theta<\pi),$$

Table 1. Orthogonal relation and completeness relation, where $x\in\mathbb{R}$.

Variables	$\langle x\|y\rangle$	$\langle x\|\mathcal{F}y\rangle$	$\langle \mathcal{F}y\|\mathcal{F}z\rangle$	Completeness relations
$y,z\in\mathbb{R}$	$\delta(x-y)$	$\frac{1}{\sqrt{2\pi}}\mathrm{e}^{\mathrm{i}xy}$	$\delta(y-z)$	$\int_{\mathbb{R}}\|x\rangle\langle x\|\mathrm{d}x = 1 = \int_{\mathbb{R}}\|\mathcal{F}x\rangle\langle\mathcal{F}x\|\mathrm{d}x$
$y=n, z=m\in\mathbb{Z}$	$\operatorname{sinc}(x-n)$	$\frac{1}{\sqrt{2\pi}}\mathrm{e}^{\mathrm{i}nx}W(x)$	δ_{nm}	$\sum_{n\in\mathbb{Z}}\|n\rangle\langle n\| \overset{a}{=} 1 \overset{b}{=} \sum_{n\in\mathbb{Z}}\|\mathcal{F}n\rangle\langle\mathcal{F}n\|$

[a] $\sum_{n\in\mathbb{Z}}|n\rangle\langle n|=1$ can be applied to $|f\rangle$ such that $\operatorname{supp}\mathcal{F}f\subseteq[-\pi,\pi]$. [b] $\sum_{n\in\mathbb{Z}}|\mathcal{F}n\rangle\langle\mathcal{F}n|=1$ can be applied to $|f\rangle$ such that $\operatorname{supp} f\subseteq[-\pi,\pi]$.

Table 2. Examples of $f(\nu)$ satisfying (9), where $P_\nu(x)\left(=P_\nu^{(0,0)}(x,y)\right)$ and $H_\nu(x)$ represent the Legendre and Hermite functions, respectively. Here, $g_i:\mathbb{R}^N\to\mathbb{R}$ (for $i=1,2,\cdots,2^{N-1}$) is given by $g_1(x_1,\cdots,x_N)=\sum_{k=1}^N x_k$, $g_j(x_1,\cdots,x_N)=-x_j+\sum_{k\neq j}x_k$ (for $j=2,\cdots,N$); and so on. It should be remarked that $f(\nu)$ can be chosen as a more generalized function where $P_\nu(x)$ is replaced by ${}_2F_1\left(\nu+\alpha+\beta+1,-\nu;\alpha+1;\frac{1-x}{2}\right)\propto P_\nu^{(\alpha,\beta)}(x)$. For the case where $f(\nu)$ is given by $H_\nu(x)$, see Section 2.2 below.

$f(\nu)$	Parameters	References
$P_{k\nu+\sigma}(\cos\theta)$	$\sigma=0,1,\cdots,k-1;\ \lvert\theta\rvert<\frac{\pi}{k}$ (for $k=1,2,\cdots$)	[4] [5] (for $k=1$)
$\prod_{i=1}^N P_\nu(\cos\theta_i)$	$\lvert g_i(\theta_1,\cdots,\theta_N)\rvert<\pi$ (for $i=1,\cdots,2^{N-1}$)	[4] (for $N=2$)
$\frac{1}{2^{2\nu}\Gamma(\nu+1)}H_{2\nu+\epsilon}(x)$	$\epsilon=0,1;\ x>0$	[10]
$\frac{1}{2^{\nu}\Gamma(\nu+1)}H_\nu(x)H_\nu(y)$	$x+y>0$	[1]

where $c(\lambda)=\frac{1}{\sqrt{\pi}}\frac{2^\lambda\Gamma\left(\lambda+\frac{1}{2}\right)}{\Gamma(\lambda)}$, we find that $P_\nu(x)$ in **Table 2** can be generalized to $\hat{C}_\nu^\lambda(x)$, and more generally to $\nu^k\hat{C}_\nu^\lambda(x)$ (for $k\in\mathbb{N}$, due to $\delta^{(k)}(t)\propto\int_{\mathbb{R}}\nu^k e^{it\nu}d\nu$). As a special case of $f(\nu)$ in (9), we obtain

$$f_\nu^\lambda(x,y)=\sum_{n=-\infty}^{\infty}\operatorname{sinc}(\nu-n)f_n^\lambda(x,y)\qquad\left(\text{for }(x,y)\in D;\lambda\in\mathbb{N}+\frac{1}{2},\mathbb{N}+1\right),\tag{13}$$

where $f_\nu^\lambda(x,y)=N_\nu^\lambda\hat{C}_\nu^\lambda(x)\hat{C}_\nu^\lambda(y)$, with $N_\nu^\lambda=\frac{\Gamma(\nu+2\lambda)}{\Gamma(\nu+1)\Gamma(2\lambda)}$ (notice that for $\lambda\in\mathbb{N}+\frac{1}{2},\mathbb{N}+1$, it turns out that N_ν^λ is given by a polynomial with respect to ν). For $C_\nu^\lambda(x):=N_\nu^\lambda\hat{C}_\nu^\lambda(x)$, representing the Gegenbauer function, we have the following relations:

$$\begin{cases}C_n^{m+\frac{1}{2}}(x)=+C_{-n-2m-1}^{m+\frac{1}{2}}(x)\\ C_n^{m+1}(x)=-C_{-n-2m-2}^{m+1}(x)\end{cases}\qquad(\text{for }n,m\in\mathbb{N}),$$

and

$$\begin{cases}C_{-n}^{m+\frac{3}{2}}(x)=0 & (n=1,2,\cdots,2m+2)\\ C_{-n}^{m+1}(x)=0 & (n=1,2,\cdots,2m+1)\end{cases}\qquad(\text{for }m\in\mathbb{N}).$$

Then it is found that the sum over $n\in\mathbb{Z}$ in the right-hand side of (13) can be replaced by the sum over $n\in\mathbb{N}$ as

$$\sum_{n=-\infty}^{\infty}\operatorname{sinc}(\nu-n)f_n^\lambda(x,y)=\frac{\sin\pi\nu}{\pi}\sum_{n=0}^{\infty}(-1)^n\left(\frac{1}{\nu-n}-\frac{1}{\nu+n+2\lambda}\right)f_n^\lambda(x,y),\tag{14}$$

where use has been made of $\sin\pi(\nu-n)=(-1)^n\sin\pi\nu$ for all $n\in\mathbb{Z}$. Once we have replaced the right-hand side of (13) by that of (14), it is not necessary to restrict the parameter λ to either $\mathbb{N}+\frac{1}{2}$ or $\mathbb{N}+1$. This is because $f_\nu^\lambda(x,y)$ and the right-hand side of (14) satisfy the same second order differential equation for $x+y\neq 0$, despite the value of λ. By re-parameterizing λ in the right-hand side of (14) as $\alpha+\frac{1}{2}$, the relation of (6) is generalized to

$$\frac{\sin\pi\nu}{\pi}K_\nu^{(\alpha,\alpha)}(-x,y)=\frac{1}{2\nu+2\alpha+1}\phi_\nu^{(\alpha,\alpha)}(x)\phi_\nu^{(\alpha,\alpha)}(y)\qquad(\text{for }(x,y)\in D),\tag{15}$$

where use has been made of $C_n^\lambda(-x)=(-1)^n C_n^\lambda(x)$ for all $n\in\mathbb{N}$.

The relation of (15) can be further generalized. Recall that $P_\nu(x)$ in Table 2 can be generalized to $\hat{P}_\nu^{(\alpha,\beta)}(x):={}_2F_1\left(\nu+\alpha+\beta+1,-\nu;\alpha+1;\frac{1-x}{2}\right)$, which is proportional to the Jacobi function. Following an analogous procedure for manipulating the Gegenbauer function $C_\nu^\lambda(x)$ above, we finally obtain [1]

$$\frac{\sin\pi\nu}{\pi}K_\nu^{(\alpha,\beta)}(-x,y)=\frac{1}{2\nu+\alpha+\beta+1}\phi_\nu^{(\beta,\alpha)}(x)\phi_\nu^{(\alpha,\beta)}(y)\qquad(\text{for }(x,y)\in D),$$

where use has been made of the relation

$$\phi_n^{(\alpha,\beta)}(-x)=(-1)^n\phi_n^{(\beta,\alpha)}(x)\qquad(\text{for all }n\in\mathbb{N}).$$

Notice the the superscripts α,β in the left-hand and right-hand sides are exchanged.

2.2. Hermite Polynomial

In this subsection, we obtain the resolvent kernel for $\mathcal{H}$, whose eigenfunction is given by the Hermite polynomial $H_n(x)$. Considering that $H_n(x)$ can be given by the specialization of the Gegenbauer polynomial $C_n^\lambda(x)$ as [8]

$$H_n(x)=n!\lim_{\lambda\to\infty}\left[\lambda^{-n/2}C_n^\lambda\left(\frac{x}{\sqrt{\lambda}}\right)\right],\tag{16}$$

then we obtain from (15), together with the asymptotic expansion as $\frac{\Gamma(\nu+2\lambda)}{\Gamma(2\lambda)}\to(2\lambda)^\nu$ (for $\lambda\to\infty$), the following formula:

$$\sum_{n=0}^{\infty}\operatorname{sinc}(\nu-n)\frac{1}{N_n}H_n(x)H_n(y)=\frac{1}{N_\nu}H_\nu(x)H_\nu(y)\qquad(\text{for }x+y>0),\tag{17}$$

where $N_\nu=\sqrt{\pi}2^\nu\Gamma(\nu+1)$ (N_n amounts to the normalization constant as $\int_{\mathbb{R}}H_n(x)H_m(x)e^{-x^2}dx=N_n\delta_{nm}$). Here, $H_\nu(x)$, which is formally given by $n\to\nu$ in (16), is related to the parabolic cylinder function $D_\nu(x)$ as

$$D_\nu(x)=2^{-\nu/2}e^{-x^2/4}H_\nu\left(\frac{x}{\sqrt{2}}\right)$$

where

$$D_\nu(x)=2^{\nu/2}\mathrm{e}^{-x^2/4}\left[\frac{\sqrt{\pi}}{\Gamma\left(\frac{1-\nu}{2}\right)}\Phi\left(-\frac{\nu}{2},\frac{1}{2};\frac{x^2}{2}\right)-\frac{\sqrt{2\pi}}{\Gamma\left(-\frac{\nu}{2}\right)}x\Phi\left(\frac{1-\nu}{2},\frac{3}{2};\frac{x^2}{2}\right)\right],$$

with $\Phi(a,c;z)={}_1F_1(a;c;z)$, the confluent hypergeometric function. Considering that $\frac{1}{N_{-n}}=0$ (for $n=1,2,\cdots$) due to $|\Gamma(-n)|=\infty$, and that

$$\begin{cases}H_{-1}(x)=\frac{\sqrt{\pi}}{2}\mathrm{e}^{x^2}\operatorname{erfc}(x),\\ H_{-(n+1)}(x)=-\frac{1}{2n}\frac{\mathrm{d}}{\mathrm{d}x}H_{-n}(x)\quad(\text{for } n=1,2,\cdots),\end{cases}$$

where $\operatorname{erfc}(x)=\frac{2}{\sqrt{\pi}}\int_x^\infty \mathrm{e}^{-t^2}\mathrm{d}t$, we find that the sum over n in the left-hand side of (17) can be formally extended to all $n\in\mathbb{Z}$. Thus, $f(\nu)=\frac{1}{N_\nu}H_\nu(x)H_\nu(y)$ satisfies the relation of (9) for $x+y>0$ (listed in the fourth row in **Table 2**).

For later convenience, we divide the left-hand side of (17) into even and odd parts as

$$h_\nu^{(\varepsilon)}(x,y):=\sum_{n\in2\mathbb{N}+\varepsilon}\frac{1}{\nu-n}\frac{1}{2^n n!}H_n(x)H_n(y)\qquad(\text{for } \varepsilon=0,1).$$

Recalling that $H_n(-x)=(-1)^n H_n(x)$ for all $n\in\mathbb{N}$, we obtain from (17)

$$h_\nu^{(\varepsilon)}(x,y)=-\frac{1}{2^{\nu+1}}\Gamma\left(\frac{\varepsilon-\nu}{2}\right)(2x)^\varepsilon\Phi\left(\frac{\varepsilon-\nu}{2},\frac{1}{2}+\varepsilon;x^2\right)H_\nu(x)\qquad(\text{for } y>|x|),\quad(18)$$

where use has been made of the following formulae:

$$\begin{cases}\frac{\pi}{\sin\pi z}=-\Gamma(-z)\Gamma(z+1),\\ \Gamma(2z)=\frac{2^{2z-1}}{\sqrt{\pi}}\Gamma(z)\Gamma\left(z+\frac{1}{2}\right).\end{cases}$$

The condition of $y>|x|$ comes from the intersection of $x+y>0$ and $(-x)+y>0$. To obtain $h_\nu^{(\varepsilon)}(x,y)$ for $(-y)>|x|$ (complementary to $y>|x|$), it may be convenient to rewrite $H_\nu(y)$ using another confluent hypergeometric function $\Psi(a,c;z)=\frac{\Gamma(1-c)}{\Gamma(1+a-c)}\Phi(a,c,z)+\frac{\Gamma(c-1)}{\Gamma(a)}z^{1-c}\Phi(1+a-c,2-c;z)$ as

$$2^{-\nu}H_\nu(y)=\Psi\left(\frac{-\nu}{2},\frac{1}{2};y^2\right)=y\Psi\left(\frac{1-\nu}{2},\frac{3}{2};y^2\right)\qquad(\text{for } y>0).\quad(19)$$

Substituting (19) into (18), and using $H_n(-y)=(-1)^n H(y)$ again, we obtain the relation that is valid not only for $y>|x|$ but also for $(-y)>|x|$ in the form

$$h_\nu^{(\varepsilon)}(x,y)=-\frac{1}{2}\Gamma\left(\frac{\varepsilon-\nu}{2}\right)(2xy)^\varepsilon\Phi\left(\frac{\varepsilon-\nu}{2},\frac{1}{2}+\varepsilon;x^2\right)\Psi\left(\frac{\varepsilon-\nu}{2},\frac{1}{2}+\varepsilon;y^2\right)\quad(20)$$
$$(\text{for } |y|>|x|),$$

which was derived from a somewhat more straightforward approach [1].

In a practical application, it is convenient to choose the parameter ν so that the y-dependence of $h_\nu^{(\varepsilon)}(x,y)$ may be written as simply as possible. Considering that $H_n(y)$ is given by a polynomial of y of order n, we can choose ν as 0 for $\varepsilon=1$. In the case of $\varepsilon=0$, however, ν cannot be chosen as 0, due to the divergence of $h_\nu^{(\varepsilon)}(x,y)$, but can be chosen as 1. To summarize, we have

$$\begin{cases} h_0^{(1)}(x,y) = -\dfrac{\pi}{2}\operatorname{erfi}(x)\operatorname{sgn}(y) \\ h_1^{(0)}(x,y) = \dfrac{\pi}{2}e^{x^2}\dfrac{\mathrm{d}}{\mathrm{d}x}\left(\mathrm{e}^{-x^2}\operatorname{erfi}(x)\right)|y| \end{cases} \qquad \left(\text{for } |y|>|x|\right),$$

where $\operatorname{erfi}(x)=\dfrac{2}{\sqrt{\pi}}\int_0^x \mathrm{e}^{t^2}\mathrm{d}t$. No such formula as (20) but $h_0^{(1)}(x,y)$ has been listed in Ref. [9].

At the end of this subsection, we deal with the sampling-theorem based summation formula for a single Hermite function of the form

$$\sum_{n=0}^{\infty}\operatorname{sinc}(\nu-n)\gamma_n H_{2n+\varepsilon}(x) = \gamma_\nu H_{2\nu+\varepsilon}(x) \qquad \left(\text{for } \varepsilon=0,1\right),$$

where the coefficient $\gamma_\nu\in\mathbb{C}$ is to be determined in such a way that the sum over n in the left-hand side can be formally extended to all integers, namely, $\gamma_{-n}=0$ (for $n=1,2,\cdots$). Bearing the specialization of (16) in mind, we find that the corresponding summation formula for a single Gegenbauer function is given by

$$\sum_{n=-\infty}^{\infty}\operatorname{sinc}(\nu-n)N_n^{m/2}\hat{C}_{2n+\varepsilon}^m(x) = N_\nu^{m/2}\hat{C}_{2\nu+\varepsilon}^m(x) \quad \left(\text{for } 0<x<1; m=1,2,\cdots\right). \tag{21}$$

Actually, the left-hand side of (21) can be rewritten as

$$\begin{aligned} &\sum_{n=-\infty}^{\infty}\operatorname{sinc}(\nu-n)N_n^{m/2}\hat{C}_{2n+\varepsilon}^m(x) \\ &= \sum_{n=0}^{\infty}N_n^{m/2}\left[\operatorname{sin}c(\nu-n)\hat{C}_{2n+\varepsilon}(x) + (-1)^{m-1}\operatorname{sin}c(\nu+n+m)\hat{C}_{2n-\varepsilon}^m(x)\right], \end{aligned}$$

where use has been made of $\hat{C}_{-2n-2m+\varepsilon}^m(x)=\hat{C}_{2n-\varepsilon}^m(x)$, and $\dfrac{\Gamma(-n)}{\Gamma(-n-m+1)}=(-1)^{m-1}\dfrac{\Gamma(n+m)}{\Gamma(n+1)}$ for $m\in\mathbb{N}$. Under the specialization of (16), we finally obtain from (21)

$$\sum_{n=0}^{\infty}\operatorname{sinc}(\nu-n)f_n(x) = f_\nu(x) \quad \left(\text{for } x>0\right), \tag{22}$$

where $f_\nu(x)=\dfrac{1}{2^{2\nu}\Gamma(\nu+1)}H_{2\nu+\varepsilon}(x)$ for $\varepsilon=0,1$. The condition of $x>0$ in (22) originates from the condition of $0<x<1$ in (21), which is equivalent to $x=\cos\theta$, with $|\theta|<\pi/2$ (corresponding to the case of $k=2$ in the first row in **Table 2**). The relation of (22) is listed in Ref. [10], in which $H_{2\nu+\varepsilon}(x)$ is given by using the parabolic cylinder function $D_{2\nu+\varepsilon}\left(\sqrt{2}x\right)$. [$D_{2\nu+\varepsilon}\left(\dfrac{x}{\sqrt{2}}\right)$ in [10] should be read as $D_{2\nu+\varepsilon}\left(\sqrt{2}x\right)$.]

3. Results and Discussion

In this section, we first deal with the FT based on the resolvent for $\mathcal{Q}$. In a matrix representation of $f, f^\dagger : \mathbb{C}^2 \to \mathbb{C}^2$ as

$$f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \qquad f^\dagger = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},$$

the supercharge $\mathcal{Q} : L^2(\mathbb{R}) \times \mathbb{C}^2 \to L^2(\mathbb{R}) \times \mathbb{C}^2$ can be written as

$$\mathcal{Q} = b f^\dagger + b^\dagger f = \begin{pmatrix} 0 & b \\ b^\dagger & 0 \end{pmatrix}, \tag{23}$$

where $b = \frac{1}{\sqrt{2}}(\hat{q} + \mathrm{i}\hat{p})$ and $b^\dagger = \frac{1}{\sqrt{2}}(\hat{q} - \mathrm{i}\hat{p})$. The corresponding SUSY Hamiltonian $\mathcal{H}'$ is given by

$$\mathcal{H}' = \mathcal{Q}^2 = \begin{pmatrix} \mathcal{H}+1 & 0 \\ 0 & \mathcal{H} \end{pmatrix},$$

which amounts to $\mathcal{H}I + f^\dagger f$, where $I = \{f, f^\dagger\}$ ($\mathcal{H}I$ can be simply denoted by $\mathcal{H}$, because I commutes with all the elements generated by $f, f^\dagger, ff^\dagger$, and $f^\dagger f$). Under the transformation $\mathcal{H} \to \mathcal{H}'$, it is natural to transform FT as

$$\mathcal{F} \to \mathcal{F}' = \mathrm{e}^{\frac{\pi}{2}\mathrm{i}\mathcal{H}'}. \tag{24}$$

In this case, $\mathcal{F}'$ turns out to be unitary due to the self-adjointness of $\mathcal{H}'$, and is related to $\mathcal{F}$ through

$$\mathcal{F}' = \begin{pmatrix} \mathrm{e}^{\frac{\pi}{2}\mathrm{i}(\mathcal{H}+1)} & 0 \\ 0 & \mathrm{e}^{\frac{\pi}{2}\mathrm{i}\mathcal{H}} \end{pmatrix} = \begin{pmatrix} \mathrm{i}\mathcal{F} & 0 \\ 0 & \mathcal{F} \end{pmatrix}. \tag{25}$$

By the commutativity $[\mathcal{H}', \mathcal{Q}] = 0$, so is $[\mathcal{F}', \mathcal{Q}] = 0$, it follows from (23) and (25) that

$$\begin{cases} b\mathcal{F} = \mathrm{i}\mathcal{F}b, \\ \mathcal{F}b^\dagger = \mathrm{i}b^\dagger\mathcal{F}, \end{cases} \tag{26}$$

where the second relation can de derived from the conjugate of the first relation (recall that $\mathcal{F}$ is unitary, so that $\mathcal{F}^\dagger = \mathcal{F}^{-1}$).

The resolvent for $\mathcal{Q}$ can be written using $\mathcal{R}_a = (\mathcal{H} - a)^{-1}$ as

$$(\mathcal{Q} + \alpha I)^{-1} = \begin{pmatrix} -\alpha & b \\ b^\dagger & -\alpha \end{pmatrix} \begin{pmatrix} \mathcal{R}_{\alpha^2-1} & 0 \\ 0 & \mathcal{R}_{\alpha^2} \end{pmatrix} \quad (\text{for } \alpha \in \mathbb{C}). \tag{27}$$

The validity of (27) is verified by $(\mathcal{Q} + \alpha I)(\mathcal{Q} + \alpha I)^{-1} = (\mathcal{Q} + \alpha I)^{-1}(\mathcal{Q} + \alpha I) = I$. Recall that in Section 2, a convenient choice of the resolvent parameter a in $\mathcal{R}_a$ is given by 0 (or 1) for an odd (or even) function. This corresponds to the choice of α in (27) as 1, with $\phi \in L^2(\mathbb{R}) \times \mathbb{C}^2$ to which $(\mathcal{Q} + \alpha I)^{-1}$ is applied being given by

$$\phi(x) = \begin{pmatrix} \varphi_-(x) \\ \varphi_+(x) \end{pmatrix}, \tag{28}$$

where $\varphi_\pm(x) = \frac{1}{2}[\varphi(x) \pm \varphi(-x)]$. It should be noted that the ϕ in (28) is the eigenfunction of $\mathcal{F}'^2 \ (\cong (-\mathcal{P}) \oplus \mathcal{P})$, with its eigenvalue being unity, that is

$$\mathcal{F}'^2\phi=\phi, \tag{29}$$

where $\mathcal{P}\left(=\mathcal{F}^2\right)$ represents the space inversion

$$\mathcal{P}:\varphi(x)\mapsto\varphi(-x).$$

The relation $\mathcal{F}^2=\mathcal{P}$ can be formally derived from $\mathcal{F}^2=\mathrm{e}^{\mathrm{i}\pi\mathcal{H}}$ and $\mathcal{H}H_n(x)\mathrm{e}^{-x^2/2}=nH_n(x)\mathrm{e}^{-x^2/2}$, together with $H_n(-x)=(-1)^n H_n(x)$ for all $n\in\mathbb{N}$.

As a simple application, let us reconsider the FT of $\varphi(x)=1$, in which $\phi(x)=\begin{pmatrix}0\\1\end{pmatrix}$. Although the $\phi(x)$ in this case does not belong to $L^2(\mathbb{R})\times\mathbb{C}^2$, we can formally apply $(\mathcal{Q}+I)^{-1}$ to ϕ, with the result that $(\mathcal{Q}+I)^{-1}\phi$ can be Fourier transformed. A series of calculations yields

$$\phi\xrightarrow{(\mathcal{Q}+I)^{-1}}\phi_1\xrightarrow{\mathcal{F}'}\phi_2\xrightarrow{\mathcal{Q}+I}\phi_3,$$

where the ϕ_k 's (for $k=1,2,3$) are given by

$$\phi_1(x)=\begin{pmatrix}\sqrt{2}D(x)\\2\left[1-xD(x)\right]\end{pmatrix},\quad \phi_2(x)=\begin{pmatrix}-\sqrt{\pi}\,\mathrm{e}^{-x^2/2}\operatorname{sgn}(x)\\\sqrt{2\pi}\,\mathrm{e}^{-x^2/2}|x|\end{pmatrix},\quad \phi_3(x)=\begin{pmatrix}0\\\sqrt{2\pi}\,\delta(x)\end{pmatrix}. \tag{30}$$

For $D(x)$, see Table 3.

Notice that $|\phi_1(x)|\simeq x^{-1}$ for $x\to\pm\infty$, as is expected from the property that $\mathcal{Q}$ behaves like the multiplication by x in the limit of $x\to\pm\infty$. Bearing in mind that we have the relation

$$\mathcal{F}'=(\mathcal{Q}+I)\mathcal{F}'(\mathcal{Q}+I)^{-1}$$

by the commutativity $[\mathcal{Q},\mathcal{F}']=0$, so that $\phi_3=\mathcal{F}'\phi$, then we again obtain

$$\mathcal{F}:1\mapsto\sqrt{2\pi}\,\delta(x). \tag{31}$$

Recalling that $D(x)$ is an odd function of x, we find that the first (second) element in ϕ_k (for $k=1,2,3$) in (30) is given by an odd (even) function. It should be noticed that this property holds for a general $\phi(x)$ in (28), not necessarily for $\phi(x)=\begin{pmatrix}0\\1\end{pmatrix}$. The reason is as follows. From $[\mathcal{Q},\mathcal{F}']=0$, together with (29), it is required that

$$(\Gamma_-\oplus\Gamma_+)\phi_k=\phi_k\qquad(k=1,2,3),$$

Table 3. Calculation of φ_1,φ_2, and φ_3 for $\varphi(x)=1$, where $D(x)=\mathrm{e}^{-x^2/2}\int_0^x\mathrm{e}^{t^2/2}\mathrm{d}t$. In the classical method 1, there is a singularity of $\varphi_1(z)$ at $z=\pm\mathrm{i}$. As compared with other methods, it is hard enough to calculate φ_3 from φ_2 in the classical method 2, due to an infinite number of derivatives in $\mathrm{e}^{\hat{p}^2/2}$.

Method	$\mathcal{G}_1$	$\mathcal{G}_2$	$\varphi_1(x)$	$\varphi_2(x)$	$\frac{\varphi_3(x)}{\sqrt{2\pi}}$	$\varphi_1(z)$ singular.	φ_3 calc.
Classical 1	$(\hat{q}^2+1)^{-1}$	$\hat{p}^2+1$	$(x^2+1)^{-1}$	$\sqrt{\frac{\pi}{2}}\mathrm{e}^{-\lvert x\rvert}$	$\delta(x)$	$z=\pm\mathrm{i}$	Easy
Classical 2	$\mathrm{e}^{-\hat{q}^2/2}$	$\mathrm{e}^{+\hat{q}^2/2}$	$\mathrm{e}^{-x^2/2}$	$\mathrm{e}^{-x^2/2}$	$\delta(x)$	-	Hard
Resolvent	$(\mathcal{H}-1)^{-1}$	$\mathcal{H}-1$	$2\left[xD(x)-1\right]$	$-\sqrt{2\pi}\lvert x\rvert\mathrm{e}^{-x^2/2}$	$\delta(x)$	-	Easy

where $\Gamma_{\pm}=\frac{1}{2}(1\pm\mathcal{P})$, projection on the even or odd parity space. Thus, it is found that the first (second) element in ϕ_k is parity odd (even).

In the latter half of this section, we discuss the FT of $\varphi(x)=1$ in another method. Some may point out that the result of (31) can be derived more efficiently from a method where $\mathcal{F}$ is replaced by

$$\mathcal{F}=\mathcal{G}_2\mathcal{F}\mathcal{G}_1 \text{ such that } \mathcal{G}_1\varphi\in L(\mathbb{R}), \tag{32}$$

which is schematically shown as

$$\begin{array}{ccc} \varphi & \xrightarrow{\mathcal{F}} & \varphi_3 \\ \mathcal{G}_1\downarrow & & \uparrow\mathcal{G}_2 \\ \varphi_1 & \xrightarrow[\mathcal{F}]{} & \varphi_2 \end{array}$$

Rewriting (26) as

$$\begin{cases} \mathcal{F}\hat{q}=\hat{p}\mathcal{F}, \\ \mathcal{F}\hat{p}=-\hat{q}\mathcal{F}, \end{cases} \tag{33}$$

we find that $\mathcal{G}_1$ can be chosen as such that depends on $\hat{q}$ only (so that $\mathcal{G}_2$ depends on $\hat{p}$ only), in order to calculate $\mathcal{G}_1\varphi$ in quite a simple way (we call such a case a classical method). To further simplify the calculation by $\mathcal{G}_2$, the functional form of $\mathcal{G}_2$ is given by a polynomial of $\hat{p}$. Considering the condition of $\mathcal{G}_1\varphi\in L(\mathbb{R})$, we find that the simplest form of $\mathcal{G}_1$ and $\mathcal{G}_2$ can be written as

$$\begin{cases} \mathcal{G}_1=\left(\hat{q}^2+1\right)^{-1} \\ \mathcal{G}_2=\hat{p}^2+1 \end{cases} (\text{classical method } 1).$$

The calculation of $\varphi_1\,(=\mathcal{G}_1\varphi), \varphi_2\,(=\mathcal{F}\varphi_1)$, and $\varphi_3\,(=\mathcal{G}_2\varphi_2)$ is summarized in Table 3, together with the corresponding calculation in another classical (named classical 2, discussed in the next-next paragraph) and the resolvent methods.

Although all the methods give the same result as (31), there is an essential difference in φ_1 between the classical 1 and resolvent methods from an analytical point of view. While $zD(z)$ is an entire function, $\left(z^2+1\right)^{-1}$ has a pole at $z=\pm\mathrm{i}$. The non-analyticity of φ_1 in the classical method is revealed when the φ_1 is evaluated as φ_1^{Λ} in the limit of $\Lambda\to\infty$:

$$\varphi_1(x)=\lim_{\Lambda\to\infty}\varphi_1^{\Lambda}(x), \qquad \varphi_1^{\Lambda}=\mathcal{F}^{-1}\varphi_2^{\Lambda},$$

where φ_2^{Λ} is given by

$$\varphi_2^{\Lambda}(x)=\varphi_2(x)U_{\Lambda}(x), \quad \text{with } U_{\Lambda}(x)=\begin{cases} 1 & (|x|\le\Lambda), \\ 0 & (|x|>\Lambda). \end{cases}$$

In calculating φ_1 from the inverse FT of φ_2, the limit operation $\Lambda\to\infty$ is necessary, because (inverse) FT is given by an improper integral. After the analytic continuation of $\varphi_1(x)$ and $\varphi_1^{\Lambda}(x)$ from $x\in\mathbb{R}$ to $z\in\mathbb{C}$, it is found that

$$\begin{cases} \forall z\in\mathbb{C}\setminus B, & \lim_{\Lambda\to\infty}\varphi_1^{\Lambda}(z)\neq\varphi_1(z) & (\text{classical method } 1), \\ \forall z\in\mathbb{C}, & \lim_{\Lambda\to\infty}\varphi_1^{\Lambda}(z)=\varphi_1(z) & (\text{resolvent method}), \end{cases} \tag{34}$$

where $B=\{z\in\mathbb{C}|-1\le \operatorname{Im} z\le 1\}$. Actually, for $z=\mathrm{i}y$ $(y\in\mathbb{R})$, for simplicity, we have

$$\varphi_1^{\Lambda}(\mathrm{i}y)-\varphi_1(\mathrm{i}y)=\begin{cases}\dfrac{1}{y^2-1}\mathrm{e}^{-\Lambda}\left(y\sinh\Lambda y+\cosh\Lambda y\right) & (\text{classical method } 1),\\ 2\mathrm{e}^{-\Lambda^2/2}\cosh\Lambda y+\sqrt{\dfrac{\pi}{2}}\,y\mathrm{e}^{y^2/v}\left[\operatorname{erf}\left(\dfrac{\Lambda+y}{\sqrt{2}}\right)-\operatorname{erf}\left(\dfrac{\Lambda-y}{\sqrt{2}}\right)\right] & (\text{resolvent method}),\end{cases}$$

where $\operatorname{erf}(z)=\dfrac{2}{\sqrt{\pi}}\int_0^z \mathrm{e}^{-t^2}\mathrm{d}t$, so that it is confirmed that the relation of (34) holds for $z=\mathrm{i}y$. Notice that $\varphi_1^{\Lambda}(z)$ is an entire function, because $\varphi_2^{\Lambda}(x)$ has a compact support so that its (inverse) FT turns out to be an entire function. Thus it is found that whether or not the relation of $\lim_{\Lambda\to\infty}\varphi_1(z)=\varphi_1(z)$ holds for all $z\in\mathbb{C}$ depends on the property that $\varphi_1(z)$ is an entire function (the identity theorem in complex analysis).

Some may further point out that in the classical method, φ_1 for $\varphi(x)=1$ can be made an entire function by choosing $\mathcal{G}_1$ [hence $\mathcal{G}_2$ by (33)] as

$$\begin{cases}\mathcal{G}_1=\mathrm{e}^{-\hat{q}^2/2}\\ \mathcal{G}_2=\mathrm{e}^{+\hat{p}^2/2}\end{cases}(\text{classical method } 2),$$

in which a series of calculations is summarized in Table 3. Although the $\varphi_1(z)$ is indeed an entire function, it is hard enough to calculate φ_3 from φ_2 (especially in a numerical way), compared to the resolvent method, because $\mathrm{e}^{\hat{p}^2/2}$ includes an infinite number of derivatives. Even if we try to regard $\mathrm{e}^{\hat{p}^2/2}$ as an integral transform, it fails due to the divergence of the corresponding integral kernel $K(x)$. Actually, we obtain from $\mathrm{e}^{\hat{p}^2/2}=\mathcal{F}\mathrm{e}^{\hat{q}^2/2}\mathcal{F}^{-1}$

$$\begin{aligned}\varphi_3(x)&=\left(\mathrm{e}^{\hat{p}^2/2}\varphi_2\right)(x)\\&=\int_{\mathbb{R}}K(x-y)\varphi_2(y)\mathrm{d}y,\qquad K(x)=\int_{\mathbb{R}}\mathrm{e}^{\mathrm{i}kx}\mathrm{e}^{k^2/2}\mathrm{d}k,\end{aligned}$$

which indicates that $K(x)$ (for $x\in\mathbb{R}$) is divergent in a usual sense.

Regarding the analyticity and numerical simplicity in calculating FT of $\varphi(x)=1$, it seems that, based on the above discussion, there is no way other than the resolvent based method.

4. Conclusions

We have obtained, using the resolvent for the harmonic oscillator Hamiltonian $\mathcal{H}$, the FT of a non-integrable function φ, such as $\varphi(x)=1$. As compared with the classical methods in Table 3, the resolvent method has some merits of being numerical calculation friendly and free of singularity for $\varphi_1(z)$. In calculating the resolvent kernel, the sampling theorem is of great use. The introduction of SUSY to $\mathcal{H}$ not only makes transparent the usefulness of the even-odd decomposition of the φ in a more natural way, but also leads to a natural definition of SUSY FT.

For future study, various extensions of the present work are possible. One extension is to deal with other unitary transforms, for example, the Hankel transform, whose eigenfunction is given by the Laguerre polynomials Using the resolvent for the corresponding Hamiltonian, we can obtain an analogous result. Another is to generalize $\varphi:\mathbb{R}\to\mathbb{C}$ to $\varphi:\mathbb{R}^m\to C\ell_{p,q}(\mathbb{R})$, the Clifford algebra over $\mathbb{R}^{p+q}$ [ϕ in (28) corresponds to $C\ell_{0,1}(\mathbb{R})\oplus C\ell_{0,1}(\mathbb{R})$]. Although the Clifford FT, in itself, is defined in various ways [11] [12] [13] [14], mainly due to the non-commutativity of the algebra, the resolvent based calculation will still be of use, despite the non-commutativity.

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