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Phantom and Quintessence Fields Coupled to Scalar Curvature in General *f*(*R*) Gravity Theory

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Abstract

This paper reviews the development of *f*(*R*) gravity theory and Phantom and Quintessence fields. Specifically, we present a new general action of *f*(*R*) gravity and Phantom and Quintessence fields coupled to scalar curvature. Then, we deduce Euler-Lagrange Equations of different fields, matter tensor and effective matter tensor. Additionally, this paper obtains the general pressure, density and speed sound of the new general field action, and investigates different cosmological evolutions with inflation. Further, this paper investigates a general *f*(*R*) gravity theory with a general matter action and obtains the different field equations, general matter tensor and effective matter tensor. Besides, this paper obtains the effective Strong Energy Condition (SEC) and effective Null Energy Condition (NEC). Then, we prove that when *f*(*R*) approaches to *R*, the effective SEC and the effective NEC approach to the usual SEC and the usual NEC, respectively. Finally, this paper presents a general action of *f*(*R*) gravity, Quintessence and Phantom fields and their applications.

Keywords

Theory of *f*(*R*) Gravity, Dark Energy, Sound Speed, Quintessence Field, Phantom Field

1. Introduction

Researchers began to question the theory of gravity after the advent of the theory of general relativity (GR). Weyl (1919) and Eddington (1923) considered modifications to the theory by including higher-order invariants in the action [1]-[8]. In the 1960s, the more complex gravitational action with many advantages appeared. GR was not renormalizable at that time and cannot be conventionally quantized. In 1962, Utiyama and De Witt showed that renormalization at one loop demands the Einstein Hilbert action be supplemented by high-order curvature [9]. Then, Stelle showed that higher-order actions are renormalizable [10]. There is higher-order curvature action for the effective low-energy gravitational action, when quantum corrections or string theory are considered [11] [12] [13].

GR correction is not an easy task. First, there are many naive GR corrections, which are unrealistic [14]-[21]. The best-known example is an alternative to GR, *i.e.* scalar-tensor theory. There are also many methods of gravity correction [22]-[27]. The typical examples are Dvali-Gabadadze-Porrati gravity [28], brane-world gravity [29], vector-scalar tensor theory [30] and Einstein-Aether theory [31]. However, there are many different $f(R)$ gravity theories. These theories are the summary of the Einstein-Hilbert action,

$$S_{EH}=\int \frac{1}{2k}\mathrm{d}^4x\sqrt{-g}R \tag{1}$$

where $k=8\pi G$, G is the gravitational constant, g is the determinant of the metric and R is the Ricci scalar ($c=\hbar=1$). With the general function of R, the following is obtained.

$$S=\int \frac{1}{2k}\mathrm{d}^4x\sqrt{-g}f(R) \tag{2}$$

There are two motivations for GR correction: (1) adding higher-order gravitational action in high-energy physics, and (2) applying the GR correction to cosmology and astrophysics.

However, there are two problems. The first problem is why specifically $f(R)$ actions and not more general ones, which include other higher-order invariants, such as $R_{\mu\nu}R^{\mu\nu}$. The answer is twofold. First, $f(R)$ actions are sufficiently general to encapsulate some of the basic characteristics of higher-order gravity, and at the same time they are simple enough to be easily handle. For instance, viewing $f(R)$ as a series expansion, *i.e.*,

$$f(R)=\sum_{n=-\infty}^{n=\infty}\alpha_n R^n \tag{3}$$

When the α_n coefficients have certain values, the action shows some interesting phenomenology. In sum, there are some advantages of $f(R)$ theory in gravity correction. It can help the high-order gravitational theory to avoid the fatal Ostrogradski instability [32] [33].

The second problem is that it is related to a possible loophole. First, how can high-energy corrections of the gravitational action have nothing to do with late-time cosmological phenomenology? Would not effective field theory considerations require that the coefficients in Equation (3) be such, as to make any modifications to the standard Einstein-Hilbert term important only near the Planck scale? [32] [33]

The observed late-time acceleration of the Universe poses one challenge to theoretical physics. In principle, this phenomenon may be the result of unknown physical processes. For instance, it involves either the correction of gravitational theory or the existence of new fields in high-energy physics. Although the latter one is most commonly used, the correction of gravitational theory is an attractive and complementary approach to explain this phenomenon, known as $f(R)$ gravity [34]. Some researchers added the weight of Ricci scalar in the Einstein-Hilbert Lagrangian for the GR correction [35]-[44].

In [45], a gravitational theory of a scalar field ϕ with non-minimal derivative coupling to curvature is considered. The results show that a cosmological model with non-minimal derivative coupling is able to explain in a unique manner both a quasi-de Sitter phase and an exit from it without any fine-tuned potential. In [46], the authors approached the problem of testing dark energy and alternative gravity models to general relativity by cosmography. The results show that degeneration among parameters can be removed by accurate data analysis of large data samples and also present the examples.

Several different forms for $f(R)$ have been suggested in the literatures [47]-[57]. These different $f(R)$-gravity theories have also been discussed in the stability conditions [58]-[62], inflationary epoch [63], compatibility with solar-system tests and galactic data [64]-[72] and the late-time cosmological evolution [73]-[80]. Additional constraints to $f(R)$ theories may be caused by imposing the so-called energy conditions [81] [82] [83], for example, the phantom field potentials [84], expansion history of the Universe [85] [86] [87] [88] [89], as well as evolution of the deceleration parameter and their confrontation with supernovae observations [90] [91]. If $f(R)$ gravity is considered as a step toward a more complicated theory, that which generalization would be more straightforward will depend on the chosen representation (see also Sotiriou *et al.*, 2008 for a discussion) [92]-[100].

In this paper, we introduce a new action, the new action effective amount of inclusion of Quintessence and Phantom can solve the problem. Thus, this model is a more general model of dark energy. We obtained the general sound speed in the evolutions of the Universe, and give the exact expressions for the exact energy-momentum tensor, pressure, energy, and different $f(R)$ gravity theories.

The rest of this paper is organized as follows. Section 2 investigates a general action of $f(R)$ gravity and Phantom and Quintessence fields coupled to scalar curvature. Section 3 studies the exact energy-momentum tensor and sound speed of the new general single field action of $f(R)$ gravity. Sect.4 shows different cosmological evolutions with single field inflation. Section 5 studies general $f(R)$ gravity theory with general matter action and its applications. Section 6 gives general $f(R)$ gravity theory with many general Phantom and Quintessence fields. Section 7 presents summary and conclusions.

2. General Action of *f*(*R*) Gravity and Phantom and Quintessence Fields Coupled to the Scalar Curvature

The first idea is to combine the actions for Phantom and Quintessence fields into one action, by adding a parameter a into the general action of $f(R)$ gravity.

Thus, we have a new action as follows:

$$A_0 = \int \alpha f(R)\sqrt{-g}\,\mathrm{d}^4x + \int \left[\alpha_1 g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi)\right]\sqrt{-g}\,\mathrm{d}^4x, \tag{4}$$

(i) When $\alpha_1 = -\frac{1}{2}$, Equation (4) is just a general action of $f(R)$ gravity and Phantom field;

(ii) When $\alpha_1 = \frac{1}{2}$, Equation (4) is a general action of $f(R)$ gravity and Quintessence field.

Further, the scalar curvature R can be coupled to the Phantom and Quintessence fields. Thus, the second idea is to add scalar curvature R into Equation (4)

$$A = \int \alpha f(R)\sqrt{-g}\,\mathrm{d}^4x + \int \left[\left(\alpha_1 g^{\mu\nu} + \alpha_2 R g^{\mu\nu}\right)\partial_\mu\phi\partial_\nu\phi - V(\phi)\right]\sqrt{-g}\,\mathrm{d}^4x, \tag{5}$$

where $\alpha_2 R g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi$ is related to the scalar curvature. Because general characteristics of the scalar curvature are coupled to Phantom and Quintessence fields, the scalar and curvature are naturally coupled. Therefore, Equation (5) is a new action with $f(R)$ gravity, Phantom and Quintessence fields and scalar curvature R. Compared with our previous researches, this action provides the possibility to explore the energy-momentum tensor and sound speed of the single field action of $f(R)$ gravity. A general function of scalar curvature (*i.e.* $f(R)$) is considered for Phantom and Quintessence fields. It is very natural to consider coupling scalar curvature R to matter field ϕ, e.g., Phantom and Quintessence fields. Because Equation (5) satisfies the invariance of the general coordinate transformation (specially coupling of R to scalar field is the constant Einstein-Hilbert action, which has all invariant properties of Einstein's RG. Its coupling strength can be fitted by coupling parameter α_2 from physical experiments), it means that the whole laws remain effective in the whole spacetime and are important in field theories. Thus, Equation (5) is consistent.

The variance of Equation (5) is

$$\begin{aligned}
\delta A &= \delta\int \alpha f(R)\sqrt{-g}\,\mathrm{d}^4x + \delta\int \left[\left(\alpha_1 g^{\mu\nu} + \alpha_2 R g^{\mu\nu}\right)\partial_\mu\phi\partial_\nu\phi - V(\phi)\right]\sqrt{-g}\,\mathrm{d}^4x \\
&= \int \left(\alpha\delta f(R)\sqrt{-g} + \alpha f(R)\delta\sqrt{-g}\right)\mathrm{d}^4x \\
&\quad + \delta_{g^{\mu\nu}}\int \left\{\left[\left(\alpha_1 g^{\mu\nu} + \alpha_2 R g^{\mu\nu}\right)\partial_\mu\phi\partial_\nu\phi - V(\phi)\right]\sqrt{-g}\right\}\mathrm{d}^4x \\
&\quad + \delta_\phi\int \left\{\left[\left(\alpha_1 g^{\mu\nu} + \alpha_2 R g^{\mu\nu}\right)\partial_\mu\phi\partial_\nu\phi - V(\phi)\right]\sqrt{-g}\right\}\mathrm{d}^4x \\
&= \int \left[\alpha f_R(R)\sqrt{-g}\,\delta R + \alpha f(R)\left(-\frac{\sqrt{-g}}{2}g_{\mu\nu}\delta g^{\mu\nu}\right)\right]\mathrm{d}^4x \\
&\quad + \int \Big\{\left(\alpha_1\delta g^{\mu\nu} + \alpha_2\delta R g^{\mu\nu} + \alpha_2 R\delta g^{\mu\nu}\right)\partial_\mu\phi\partial_\nu\phi\sqrt{-g} \\
&\quad + \left[\left(\alpha_1 g^{\mu'\nu'} + \alpha_2 R g^{\mu'\nu'}\right)\partial_{\mu'}\phi\partial_{\nu'}\phi - V(\phi)\right]\left(-\frac{\sqrt{-g}}{2}g_{\mu\nu}\delta g^{\mu\nu}\right)\Big\}\mathrm{d}^4x \\
&\quad + \int \Big\{-\partial_\mu\left[\left(\alpha_1 g^{\mu\nu} + \alpha_2 R g^{\mu\nu}\right)2\partial_\nu\phi\sqrt{-g}\right]\delta\phi \\
&\quad + \partial_\mu\left[\left(\alpha_1 g^{\mu\nu} + \alpha_2 R g^{\mu\nu}\right)2\partial_\nu\phi\sqrt{-g}\,\delta\phi\right] - \sqrt{-g}\,\frac{\delta V(\phi)}{\delta\phi}\delta\phi\Big\}\mathrm{d}^4x,
\end{aligned} \tag{6}$$

For the deducing details see Appendix A.

To study Equation (6), we discuss δR generally as follows [1]

$$\delta R = \delta g^{\mu\nu} R_{\mu\nu} + g^{\mu\nu}\delta R_{\mu\nu}, \tag{7}$$

and

$$g^{\mu\nu}\delta R_{\mu\nu} = g^{\mu\nu}\left(\nabla_\lambda \delta\Gamma^\lambda_{\mu\nu} - \nabla_\nu \delta\Gamma^\lambda_{\mu\lambda}\right). \tag{8}$$

We have

$$\begin{aligned}\delta\Gamma^\lambda_{\mu\nu} &= \delta\left[\frac{1}{2}g^{\lambda\alpha}\left(g_{\alpha\nu,\mu}+g_{\alpha\mu,\nu}-g_{\mu\nu,\alpha}\right)\right]\\ &=\frac{1}{2}\delta g^{\lambda\alpha}\left(g_{\alpha\nu,\mu}+g_{\alpha\mu,\nu}-g_{\mu\nu,\alpha}\right)+\frac{1}{2}g^{\lambda\alpha}\left(\delta\partial_\mu g_{\alpha\nu}+\delta\partial_\nu g_{\alpha\mu}-\delta\partial_\alpha g_{\mu\nu}\right).\end{aligned} \tag{9}$$

Using

$$\nabla_\alpha g_{\mu\nu} = \partial_\alpha g_{\mu\nu} - \Gamma^\beta_{\mu\alpha} g_{\beta\nu} - \Gamma^\beta_{\nu\alpha} g_{\mu\beta} = 0. \tag{10}$$

We rewrite Equation (9) as

$$\begin{aligned}\delta\Gamma^\lambda_{\mu\nu} &= \frac{1}{2}\delta g^{\lambda\alpha}\left(\Gamma^\beta_{\alpha\mu}g_{\beta\nu}+\Gamma^\beta_{\nu\mu}g_{\alpha\beta}+\Gamma^\beta_{\alpha\nu}g_{\beta\mu}+\Gamma^\beta_{\mu\nu}g_{\alpha\beta}-\Gamma^\beta_{\mu\alpha}g_{\beta\nu}-\Gamma^\beta_{\nu\alpha}g_{\mu\beta}\right)\\ &\quad+\frac{1}{2}g^{\lambda\alpha}\left(\delta\partial_\mu g_{\alpha\nu}+\delta\partial_\nu g_{\alpha\mu}-\delta\partial_\alpha g_{\mu\nu}\right)\\ &=\frac{1}{2}g^{\lambda\alpha}\left(-\Gamma^\beta_{\alpha\mu}\delta g_{\beta\nu}-\Gamma^\beta_{\nu\mu}\delta g_{\alpha\beta}-\Gamma^\beta_{\alpha\nu}\delta g_{\beta\mu}-\Gamma^\beta_{\mu\nu}\delta g_{\alpha\beta}+\Gamma^\beta_{\mu\alpha}\delta g_{\beta\nu}+\Gamma^\beta_{\nu\alpha}\delta g_{\mu\beta}\right)\\ &\quad+\frac{1}{2}g^{\lambda\alpha}\left(\delta\partial_\mu g_{\alpha\nu}+\delta\partial_\nu g_{\alpha\mu}-\delta\partial_\alpha g_{\mu\nu}\right)\\ &=\frac{1}{2}g^{\lambda\alpha}\left(\nabla_\mu\delta g_{\nu\alpha}+\nabla_\nu\delta g_{\alpha\mu}-\nabla_\alpha\delta g_{\mu\nu}\right).\end{aligned} \tag{11}$$

Substituting Equation (11) into Equation (8), we have

$$g^{\mu\nu}\delta R_{\mu\nu} = g_{\mu\nu}\nabla_\lambda\nabla^\lambda\delta g^{\mu\nu} - \nabla_\mu\nabla_\nu\delta g^{\mu\nu}, \tag{12}$$

The deducing details of Equation (12) are in Appendix B.

Substituting Equation (12) into Equation (7), we obtain

$$\delta R = R_{\mu\nu}\delta g^{\mu\nu} + g_{\mu\nu}\Box\delta g^{\mu\nu} - \nabla_\mu\nabla_\nu\delta g^{\mu\nu}. \tag{13}$$

Substituting Equation (13) into Equation (6), we have

$$\begin{aligned}\delta A &= \int\left[\alpha f_R(R)\left(R_{\mu\nu}\delta g^{\mu\nu}+g_{\mu\nu}\Box\delta g^{\mu\nu}-\nabla_\mu\nabla_\nu\delta g^{\mu\nu}\right)\sqrt{-g}+\alpha f(R)\left(-\frac{\sqrt{-g}}{2}g_{\mu\nu}\delta g^{\mu\nu}\right)\right]\mathrm{d}^4x\\ &\quad+\int\left\{\left[\alpha_1\delta g^{\mu\nu}+\alpha_2 g^{\mu\nu}\left(R_{\alpha\beta}\delta g^{\alpha\beta}+g_{\alpha\beta}\Box\delta g^{\alpha\beta}-\nabla_\alpha\nabla_\beta\delta g^{\alpha\beta}\right)+\alpha_2 R\delta g^{\mu\nu}\right]\partial_\mu\phi\partial_\nu\phi\sqrt{-g}\right.\\ &\quad\left.+\left[\left(\alpha_1 g^{\mu'\nu'}+\alpha_2 R g^{\mu'\nu'}\right)\partial_{\mu'}\phi\partial_{\nu'}\phi-V(\phi)\right]\left(-\frac{\sqrt{-g}}{2}g_{\mu\nu}\delta g^{\mu\nu}\right)\right\}\mathrm{d}^4x\\ &\quad+\int\left\{-\partial_\mu\left[\left(\alpha_1 g^{\mu\nu}+\alpha_2 R g^{\mu\nu}\right)2\partial_\nu\phi\sqrt{-g}\right]\delta\phi\right.\\ &\quad\left.+\partial_\mu\left[\left(\alpha_1 g^{\mu\nu}+\alpha_2 R g^{\mu\nu}\right)2\partial_\nu\phi\sqrt{-g}\delta\phi\right]-\sqrt{-g}\frac{\delta V(\phi)}{\delta\phi}\delta\phi\right\}\mathrm{d}^4x.\end{aligned} \tag{14}$$

Considering a general partial integration, we have

$$\begin{aligned}&\int F\sqrt{-g}\left(g_{\mu\nu}\Box\delta g^{\mu\nu}-\nabla_\mu\nabla_\nu\delta g^{\mu\nu}\right)\mathrm{d}^4x\\&=\int F\left(\nabla_\alpha\nabla^\alpha g_{\mu\nu}\delta g^{\mu\nu}-\nabla_\mu\nabla_\nu\delta g^{\mu\nu}\right)\sqrt{-g}\mathrm{d}^4x\\&=\int\left[\left(\nabla^\alpha\nabla_\alpha F\right)g_{\mu\nu}\delta g^{\mu\nu}-\nabla_\mu\nabla_\nu F\delta g^{\mu\nu}\right]\sqrt{-g}\mathrm{d}^4x,\end{aligned}\tag{15}$$

The deducing details of Equation (15) are in Appendix C. Thus, we have

$$\begin{aligned}&\int F\sqrt{-g}\left(g_{\mu\nu}\Box\delta g^{\mu\nu}-\nabla_\mu\nabla_\nu\delta g^{\mu\nu}\right)\mathrm{d}^4x\\&=\int\left[\left(\nabla^\alpha\nabla_\alpha F\right)g_{\mu\nu}\delta g^{\mu\nu}-\nabla_\mu\nabla_\nu F\delta g^{\mu\nu}\right]\sqrt{-g}\mathrm{d}^4x.\end{aligned}\tag{16}$$

Substituting Equation (16) into Equation (14), we have

$$\begin{aligned}\delta A=&\int\left\{\alpha\left(f_R(R)R_{\mu\nu}-\frac{g_{\mu\nu}}{2}f(R)\right)\delta g^{\mu\nu}+\alpha\left[g_{\mu\nu}\Box f_R(R)-\nabla_\mu\nabla_\nu f_R(R)\right]\delta g^{\mu\nu}\right\}\sqrt{-g}\mathrm{d}^4x\\&+\int\left\{(\alpha_1+\alpha_2R)\left(\partial_\mu\phi\partial_\nu\phi\right)\delta g^{\mu\nu}+\alpha_2\left[g^{\mu\nu}\left(R_{\alpha\beta}\delta g^{\alpha\beta}\right)\left(\partial_\mu\phi\partial_\nu\phi\right)\right.\right.\\&+\left.\left(g_{\mu\nu}\Box-\nabla_\mu\nabla_\nu\right)\left(g^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi\right)\delta g^{\mu\nu}\right]\sqrt{-g}\\&\left.-\frac{1}{2}\left[\left(\alpha_1 g^{\mu'\nu'}+\alpha_2Rg^{\mu'\nu'}\right)\partial_{\mu'}\phi\partial_{\nu'}\phi-V(\phi)\right]\left(g_{\mu\nu}\delta g^{\mu\nu}\sqrt{-g}\right)\right\}\mathrm{d}^4x\\&+\int\left\{-\partial_\mu\left[\left(\alpha_1g^{\mu\nu}+\alpha_2Rg^{\mu\nu}\right)2\partial_\nu\phi\sqrt{-g}\right]\delta\phi\right.\\&\left.+\partial_\mu\left[\left(\alpha_1g^{\mu\nu}+\alpha_2Rg^{\mu\nu}\right)2\partial_\nu\phi\sqrt{-g}\delta\phi\right]-\sqrt{-g}\frac{\delta V(\phi)}{\delta\phi}\delta\phi\right\}\mathrm{d}^4x.\end{aligned}\tag{17}$$

Using Equation (17), we deduce Euler-Lagrange Equations

$$\begin{aligned}&f_R(R)R_{\mu\nu}-\frac{g_{\mu\nu}}{2}f(R)+\left(g_{\mu\nu}\Box f_R(R)-\nabla_\mu\nabla_\nu f_R(R)\right)\\&=-k\left\{(\alpha_1+\alpha_2R)\left(\partial_\mu\phi\partial_\nu\phi\right)+\alpha_2\left[R_{\mu\nu}+\left(g_{\mu\nu}\Box-\nabla_\mu\nabla_\nu\right)\right]\left(g^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi\right)\right.\\&\left.-\frac{1}{2}\left[(\alpha_1+\alpha_2R)g^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi-V(\phi)\right]g_{\mu\nu}\right\},\end{aligned}\tag{18}$$

$$\frac{1}{\sqrt{-g}}\partial_\mu\left[\left(\alpha_1g^{\mu\nu}+\alpha_2Rg^{\mu\nu}\right)\sqrt{-g}\,2\partial_\nu\phi\right]+\frac{\delta V(\phi)}{\delta\phi}=0,\tag{19}$$

where $k=\frac{1}{\alpha}$ and $f_R=\frac{\partial f}{\partial R}$. we can rewrite Equation (18) as

$$\left(R_{\mu\nu}+g_{\mu\nu}\Box-\nabla_\mu\nabla_\nu\right)f_R(R)-\frac{g_{\mu\nu}}{2}f(R)=-kT_{\mu\nu},\tag{20}$$

i.e.,

$$f(R)\left[\left(R_{\mu\nu}+g_{\mu\nu}\Box-\nabla_\mu\nabla_\nu\right)\frac{\mathrm{d}\ln f(R)}{\mathrm{d}R}-\frac{g_{\mu\nu}}{2}\right]=-k\left(T^m_{\mu\nu}+T^{effect}_{uv}\right),\tag{21}$$

where

$$T^m_{\mu\nu}=\alpha_1\partial_\mu\phi\partial_\nu\phi-\frac{1}{2}\left[\alpha_1g^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi-V(\phi)\right]g_{\mu\nu},\tag{22}$$

$$T_{uv}^{effect}=\alpha_2 R\partial_\mu\phi\partial_\nu\phi+\alpha_2\left[R_{\mu\nu}-\frac{g_{\mu\nu}}{2}R+\left(g_{\mu\nu}\Box-\nabla_\mu\nabla_\nu\right)\right]\left(g^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi\right). \quad (23)$$

Therefore, a general action of $f(R)$ gravity and Phantom and Quintessence fields is generally presented. We generalize Equation (4) to a general action of $f(R)$ gravity and Phantom and Quintessence fields coupled to scalar curvature by making variance of the general Lagrangian. We further deduce Euler-Lagrange Equations of different fields, matter tensor, effective matter tensor, etc.

3. Exact Energy-Momentum Tensor and Sound Speed of the New General Single Field Action of *f*(*R*) Gravity

From Equation (5) and Equations (20)-(23), we can obtain the exact energy-momentum tensor.

$$\begin{aligned}T_{\mu\nu}&=\alpha_1\partial_\mu\phi\partial_\nu\phi-\frac{1}{2}\left[\alpha_1 g^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi-V(\phi)\right]g_{\mu\nu}+\alpha_2 R\partial_\mu\phi\partial_\nu\phi\\&\quad+\alpha_2\left[R_{\mu\nu}-\frac{g_{\mu\nu}}{2}R+\left(g_{\mu\nu}\Box-\nabla_\mu\nabla_\nu\right)\right]\left(g^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi\right)\\&=\left(\alpha_1+\alpha_2 R\right)\partial_\mu\phi\partial_\nu\phi-\frac{1}{2}\left[\left(\alpha_1+\alpha_2 R-2\alpha_2\Box\right)g^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi-V(\phi)\right]g_{\mu\nu}\\&\quad-\alpha_2\nabla_\mu\nabla_\nu g^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi+\alpha_2 R_{\mu\nu}g^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi.\end{aligned} \quad (24)$$

To simplify the exact energy-momentum tensor, we can rewrite Equation (24) as

$$\begin{aligned}T_{\mu\nu}&=\left(\alpha_1+\alpha_2 R\right)\partial_\mu\phi\partial_\nu\phi+\left[\left(\alpha_1+\alpha_2 R-2\alpha_2\Box\right)X+\frac{1}{2}V(\phi)\right]g_{\mu\nu}\\&\quad+2\left(\alpha_2 R_{\mu\nu}-\alpha_2\nabla_\mu\nabla_\nu\right)X,\end{aligned} \quad (25)$$

where $X=-\frac{1}{2}g^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi$. Similar to [101], we can generally define

$$\gamma=-\left(\alpha_1+\alpha_2 R\right). \quad (26)$$

Substituting Equation (26) into Equation (25), we obtain a general energy-momentum tensor

$$T_{\mu\nu}=-\gamma\partial_\mu\phi\partial_\nu\phi-\left[\left(\gamma+2\alpha_2\Box\right)X-\frac{1}{2}V(\phi)\right]g_{\mu\nu}-2\left(\alpha_2 R_{\mu\nu}-\alpha_2\nabla_\mu\nabla_\nu\right)X, \quad (27)$$

where γ is different from the past expression.

In general, using $T_{\mu\nu}=(P+\rho)U_\mu U_\nu-Pg_{\mu\nu}$, $U_\mu=\{1,0,0,0\}$ and Friedman-Robertson-Walker metric

$$g_{\mu\nu}=diag\left(1,-\frac{R^2(t)}{1-kr^2},-r^2R^2(t),-r^2R^2(t)\sin^2\theta\right)$$

we have

$$\rho=T_{00}=-\gamma\partial_0\phi\partial_0\phi-\left[\left(\gamma+2\alpha_2\Box\right)X-\frac{1}{2}V(\phi)\right]g_{00}+2\left(\alpha_2 R_{00}-\alpha_2\nabla_0\nabla_0\right)X$$

i.e.,

$$\rho = T_{00} = -\gamma\partial_0\phi\partial_0\phi - \left[\left(\gamma + 2\alpha_2\Box\right) X - \frac{1}{2}V\left(\phi\right) \right] + 2\left(\alpha_2 R_{00} - \alpha_2\nabla_0\nabla_0\right) X, \quad (28)$$

and

$$\begin{aligned} T^{\alpha}{}_{\nu} &= g^{\alpha\mu}T_{\mu\nu} = -g^{\alpha\mu}\gamma\partial_\mu\phi\partial_\nu\phi \\ &\quad - \left[\left(\gamma + 2\alpha_2\Box\right) X - \frac{1}{2}V\left(\phi\right) \right] g^{\alpha\mu}g_{\mu\nu} - 2g^{\alpha\mu}\left(\alpha_2 R_{\mu\nu} - \alpha_2\nabla_\mu\nabla_\nu\right) X \\ &= -\gamma\partial^\alpha\phi\partial_\nu\phi - \left[\left(\gamma + 2\alpha_2\Box\right) X - \frac{1}{2}V\left(\phi\right) \right]\delta^{\alpha}{}_{\nu} - 2\left(\alpha_2 R^{\alpha}{}_{\nu} - \alpha_2\nabla^\alpha\nabla_\nu\right) X, \end{aligned} \quad (29)$$

i.e.,

$$P = -\frac{1}{3}T_j^j = \frac{1}{3}\gamma\partial^j\phi\partial_j\phi - \left[\left(\gamma + 2\alpha_2\Box\right) X - \frac{1}{2}V\left(\phi\right) \right] - \frac{2}{3}\left(\alpha_2 R_{jj} - \alpha_2\nabla_j\nabla_j\right) X. \quad (30)$$

Due to $\hat{T}^{\alpha}_{\mu} = g^{\alpha\nu}\hat{T}_{\mu\nu} = diag\left(\hat{\rho}, -\hat{p}, -\hat{p}, -\hat{p}\right)$, we have $T = g^{\alpha\nu}T_{\alpha\nu} = \rho - 3P$. Further, using Equations (29) and (32), we obtain a new general density ρ related to P, V, X, γ as follows

$$\begin{aligned} \rho &= 3P + T \\ &= 3P + 2\gamma X - \left[\left(\gamma + 2\alpha_2\Box\right) X - \frac{1}{2}V\left(\phi\right) \right] 4 - 2\left(\alpha_2 R^{\alpha}{}_{\alpha} - \alpha_2\nabla^\alpha\nabla_\alpha\right) X. \end{aligned} \quad (31)$$

Furthermore, we can obtain the general sound speed

$$C_s^2 = \frac{P_{,X}}{\rho_{,X}}.$$

Using Equations (30) and (31), we obtain

$$C_s^2 = \frac{P_{,X}}{3P_{,X} - \left\{ \gamma\partial^\alpha\phi\partial_\alpha\phi - 4\left[\left(\gamma + 2\alpha_2\Box\right) X - \frac{1}{2}V\left(\phi\right) \right] - 2\left(\alpha_2 R^{\alpha}{}_{\alpha} - \alpha_2\nabla^\alpha\nabla_\alpha\right) X \right\}_{,X}}. \quad (32)$$

Thus, we can rewrite Equations (33) and (34) as

$$\begin{aligned} \rho &= 3P + 2\gamma X - \left[\left(\gamma + 2\alpha_2\Box\right) X - \frac{1}{2}V\left(\phi\right) \right] 4 - 2\left(\alpha_2 R^{\alpha}{}_{\alpha} - \alpha_2\nabla^\alpha\nabla_\alpha\right) X \\ &= 3P - 2\left[\gamma X + \alpha_2 RX - V\left(\phi\right) \right] - 6\alpha_2\Box X, \end{aligned} \quad (33)$$

where $R^{\alpha}{}_{\alpha} = g^{\alpha\beta}R_{\alpha\beta} = R$. The deducing details are in Appendix D. And then we have

$$\begin{aligned} C_s^2 &= \frac{P_{,X}}{3P_{,X} - \left[2\left(\gamma X + \alpha_2 RX - V\left(\phi\right)\right) + 6\alpha_2\Box X \right]_{,X}} \\ &= \frac{P_{,X}}{3P_{,X} - 2\left(\gamma + \alpha_2 R\right) - 6\alpha_2\left(\Box X\right)_{,X}}, \end{aligned} \quad (34)$$

where $\partial^\alpha\phi\partial_\alpha\phi = g^{\alpha\beta}\partial_\beta\phi\partial_\alpha\phi = -2X$.

In conclusion, we obtain the general energy-momentum tensor, pressure, density and speed sound of the new general single field action of $f\left(R\right)$ gravity and Phantom and Quintessence fields coupled to scalar curvature.

4. Different Cosmological Evolutions with Single Field Inflation

The most hopeful models for the different evolutions of the Universe are the cosmological models of the initial evolution and subsequent development. They are supported by the most comprehensive and accurate explanations based on the current scientific evidences and observations. According to the observations on the current Universe, the dark matter accounts for 24% of the mass-energy density of the observable Universe, the dark energy amounts for nearly 72% and the ordinary matter only accounts for about 4%.

The equation of state (EOS) is a powerful way to describe the matter and the evolutions of the Universe. In cosmology, the EOS of a perfect fluid is characterized by a dimensionless number that is equal to the ratio of its pressure to its energy density. It is closely related to the thermodynamic EOS and ideal gas law.

Therefore, with EOSs of matter, dark energy and dark matter, the new general action can be used to explain the different cosmological evolutions: (I) Big Rip Universe; (II) De Sitter Universe; (III) Harmonic Universe.

In the case of cosmological inflation, using Equations (28), (30) and (5), we deduce the EOS

$$\omega=\frac{P}{\rho}=\frac{\frac{1}{3}\gamma\partial^{j}\phi\partial_{j}\phi+\left[\left(\gamma+2\alpha_{2}\Box\right)X-\frac{1}{2}V\left(\phi\right)\right]+\frac{2}{3}\left(\alpha_{2}R_{j}^{j}-\alpha_{2}\nabla^{j}\nabla_{j}\right)X}{3P-2\left[\gamma X+\alpha_{2}RX-V\left(\phi\right)\right]-6\alpha_{2}\Box X}. \tag{35}$$

Substituting γ into Equation (35), it follows that

$$\omega=\frac{P}{\rho}=\frac{-\frac{1}{3}\left(\alpha_{1}+\alpha_{2}R\right)\partial^{j}\phi\partial_{j}\phi+\left[\left(2\alpha_{2}\Box-\left(\alpha_{1}+\alpha_{2}R\right)\right)X-\frac{1}{2}V\left(\phi\right)\right]+\frac{2}{3}\left(\alpha_{2}R_{j}^{j}-\alpha_{2}\nabla^{j}\nabla_{j}\right)X}{3P+2\left[\left(\alpha_{1}+\alpha_{2}R\right)X-\alpha_{2}RX+V\left(\phi\right)\right]-6\alpha_{2}\Box X}. \tag{36}$$

To discuss the different evolutions of the Universe, we use the Friedman equations [102]

$$\rho=3M_{pl}^{2}H^{2}, \tag{37}$$

$$-3H\left(\rho+P\right)=\dot{\rho}, \tag{38}$$

and then we have

$$-\left(\rho+P\right)=2M_{pl}^{2}\dot{H}. \tag{39}$$

Now accelerating expansion $\left(\ddot{a}>0\right)$ requires smallness of the variation of the Hubble parameter $H\equiv\partial_{t}Ina$, as defined by the parameter [103]

$$\varepsilon\equiv-\frac{\dot{H}}{H^{2}}=\frac{3}{2}\left(1+\omega\right)<1, \tag{40}$$

thus

$$\omega<-\frac{1}{3}. \tag{41}$$

Using Equation (33), we rewrite Equation (35) as

$$\omega=\frac{P}{\rho}=\frac{P}{3P-2\left[\gamma X+\alpha_{2}RX-V\left(\phi\right)\right]-6\alpha_{2}\Box X}, \tag{42}$$

Equation (42) can describe the different evolution characteristics of the Universe:

(i) Big Rip Universe, *i.e.*, the case where $\omega < -1$, the inflation and the accelerating expansion of the Universe can be characterized by the EOS of dark energy. Generally, the expansion of the Universe is accelerating when $\omega < -\frac{1}{3}$. When EOS for Phantom energy is $\omega < -1$, the Big Rip will occur. According to Equation (42), the total pressure of our Universe is negative, and then the relation between pressure P and γ is deduced as follows

$$2P < 2\left[\gamma X + \alpha_2 RX - V(\phi)\right] + 6\alpha_2 \Box X. \tag{43}$$

(ii) De Sitter Universe, *i.e.* the case where $\omega \sim -1$, the relation between pressure P and γ is

$$2P \sim 2\left[\gamma X + \alpha_2 RX - V(\phi)\right] + 6\alpha_2 \Box X. \tag{44}$$

The de Sitter Universe is a solution to Einstein's field equations of General Relativity, which is named after Willem de Sitter. When one considers the Universe as spatially flat and neglects ordinary matter, the dynamics of the Universe would be dominated by the cosmological constant, corresponding to dark energy. If the current acceleration of our Universe is due to a cosmological constant, then the universe would continue to expand. All the matter and radiation will be diluted. Eventually, there will be almost nothing left except the cosmological constant, and the Universe will become a de Sitter Universe.

(iii) Harmonic Universe, *i.e.*, the case where $\omega \sim 0$, the relation between pressure P and γ is

$$P = 0. \tag{45}$$

The EOS for ordinary non-relativistic matter is $\omega = 0$, which means that it is diluted as $\rho \propto a^{-3} \propto V^{-1}$. This is natural for ordinary non-relativistic matter.

The EOS of radiation and matter is $\omega = \frac{1}{3}$ in the very early Universe, and then the Universe is diluted as $\rho \propto a^{-4}$. In the expanding universe, the energy density decreases more quickly than the volume expansion.

Substituting Equation (26) into Equation (42), we can further deduce a concrete expression for Equation (42)

$$\omega = \frac{P}{\rho} = \frac{P}{3P + 2\left[(\alpha_1 + \alpha_2 R)X - \alpha_2 RX + V(\phi)\right] - 6\alpha_2 \Box X}, \tag{46}$$

which shows their concrete evolution details. For example, we focus on the case of different potentials as follows:

(a) When $V(\phi) = \frac{1}{2} m_\phi^2 \phi^2$, the EOS is

$$\omega = \frac{P}{3P + 2\left[(\alpha_1 + \alpha_2 R)X - \alpha_2 RX + \frac{1}{2} m_\phi^2 \phi^2\right] - 6\alpha_2 \Box X}. \tag{47}$$

(b) When $V(\phi) = V_\phi e^{-\lambda k \phi}$, the EOS is

$$\omega = \frac{P}{3P + 2\left[\left(\alpha_1 + \alpha_2 R\right) X - \alpha_2 RX + V_\phi e^{-\lambda k\phi}\right] - 6\alpha_2 \Box X}. \tag{48}$$

(c) When $V(\phi) = V_\phi e^{-bk^2\phi^2}$, the EOS is

$$\omega = \frac{P}{3P + 2\left[\left(\alpha_1 + \alpha_2 R\right) X - \alpha_2 RX + V_\phi e^{-bk^2\phi^2}\right] - 6\alpha_2 \Box X}. \tag{49}$$

Thus, we deduce the EOS, *i.e.*, Equations (35), (36), (42), (47)-(49), of $f(R)$ gravity and Phantom and Quintessence fields coupled to scalar curvature. We investigate different cosmological evolutions with single field inflation including the Big Rip Universe, De Sitter Universe and Harmonic Universe. We further study the cases of different potentials for different EOSs.

5. General *f*(*R*) Gravity Theory with General Matter Action and Its Applications

We begin with a general action of $f(R)$ gravity plus general matter action

$$S = \int \mathrm{d}^4 x \sqrt{-g} f(R) + S_m \tag{50}$$

where g is the determinant of the metric $g_{\mu\nu}$, R is the Ricci scalar and S_m is a general matter action. Varying this action with respect to the metric, we obtain the field equation [104]-[112]

$$T_{\mu\nu} = -\frac{1}{k}\left[f' R_{\mu\nu} - \frac{f}{2} g_{\mu\nu} - \left(\nabla_\mu \nabla_\nu - g_{\mu\nu}\Box\right) f'\right], \tag{51}$$

where a prime denotes differentiation with respect to R and $\Box \equiv g^{\alpha\beta}\nabla_\alpha \nabla_\beta$. To use an approach to the null energy condition (NEC) and strong energy condition (SEC) similar to that in general relativity (GR) context, we note that Equation (51) can be rewritten as

$$f' R_{\mu\nu} = -kT_{\mu\nu} + \frac{f}{2} g_{\mu\nu} + \left(\nabla_\mu \nabla_\nu - g_{\mu\nu}\Box\right) f', \tag{52}$$

i.e.,

$$R_{\mu\nu} = \left[-kT_{\mu\nu} + \frac{f}{2} g_{\mu\nu} + \left(\nabla_\mu \nabla_\nu - g_{\mu\nu}\Box\right) f'\right] \Big/ f'. \tag{53}$$

Multiplying Equation (53) by $g^{\mu\nu}$, we have

$$g^{\mu\nu} R_{\mu\nu} = \left[-kg^{\mu\nu} T_{\mu\nu} + \frac{f}{2} g^{\mu\nu} g_{\mu\nu} + g^{\mu\nu}\left(\nabla_\mu \nabla_\nu - g_{\mu\nu}\Box\right) f'\right] \Big/ f', \tag{54}$$

i.e.,

$$R = \left[-kT + \frac{f}{2} 4 + \left(\Box - 4\Box\right) f'\right] \Big/ f' = \left[-kT + 2f - 3\Box f'\right] / f' = kT_{eff}. \tag{55}$$

Then, we get

$$T_{eff} = \frac{2f - kT - 3\Box f'}{kf'}. \tag{56}$$

In addition, we can have

$$R_{\mu\nu} = -k\left(T_{eff\,\mu\nu} - \frac{T_{eff}}{2} g_{\mu\nu}\right). \tag{57}$$

Using Equations (56) and (57), we deduce

$$\begin{aligned} T_{eff\,\mu\nu} &= \frac{T_{eff}}{2} g_{\mu\nu} - R_{\mu\nu}/k = \frac{\left[2f - kT - 3\Box f'\right]/\left(kf'\right)}{2} g_{\mu\nu} - R_{\mu\nu}/k \\ &= \frac{2f - kT - 3\Box f'}{2kf'} g_{\mu\nu} - R_{\mu\nu}/k, \end{aligned} \tag{58}$$

Combining Equation (56) with Equation (58), we have

$$\begin{aligned} T_{eff\,\mu\nu} &= \frac{T_{eff}}{2} g_{\mu\nu} - R_{\mu\nu}/k \\ &= \frac{1}{kf'}\left[kT_{\mu\nu} - \nabla_\mu \nabla_\nu f' + \frac{\left(-kT + f - \Box f'\right)}{2} g_{\mu\nu}\right]. \end{aligned} \tag{59}$$

The deducing details are in Appendix E.

Equations (56) and (59) are consistent. This is because by multiplying Equation (59) with $g^{\mu\nu}$ we have

$$\begin{aligned} g^{\mu\nu} T_{eff\,\mu\nu} &= \left\{kT_{\mu\nu} g^{\mu\nu} - g^{\mu\nu}\nabla_\mu \nabla_\nu f' + \frac{\left(-kT + f - \Box f'\right)}{2} g^{\mu\nu} g_{\mu\nu}\right\} \Big/ f'k \\ &= \frac{2f - kT - 3\Box f'}{kf'}. \end{aligned} \tag{60}$$

The deducing details are in Appendix F. Equation (60) is just Equation (56), so Equations (56) and (59) are consistent.

In addition, we have

$$T_{eff\,\nu}^{\ \ \mu} = diag\left\{\rho_{eff}, -P_{eff}, -P_{eff}, -P_{eff}\right\}. \tag{61}$$

When $R_{\mu\nu} u^\mu u^\nu \geq 0$, the virtue of Einstein's Equation (57) implies

$$R_{\mu\nu} u^\mu u^\nu = -k\left(T_{eff\,\mu\nu} - \frac{T_{eff}}{2} g_{\mu\nu}\right) u^\mu u^\nu \geq 0, \tag{62}$$

where $T_{eff\,\mu\nu}$ is an effective Equation (59).

Using Equation (61), we similarly have

$$T_{eff\,\mu\nu} = \left(\rho_{eff} + P_{eff}\right) u_\mu u_\nu - g_{\mu\nu} P_{eff}, \tag{63}$$

where $u_\mu = (1,0,0,0)$.

For the homogeneous and isotropic Friedman-Lemaitre-Robertson-Walker (FLRW) metric with scale factor $a(t)$ and for a perfect fluid $T_{\mu\nu}$, $R_{\mu\nu} k^\mu k^\nu \geq 0$ and $R_{\mu\nu} u^\mu u^\nu \geq 0$ [3]. Substituting Equation (61) into Equation (62), we have

$$R_{\mu\nu} u^\mu u^\nu = -k\left(T_{eff\,\mu\nu} - \frac{T_{eff}}{2} g_{\mu\nu}\right) u^\mu u^\nu = -k\frac{1}{2}\left[\left(\rho_{eff} + 3P_{eff}\right)\right], \tag{64}$$

where $T_{eff} = \rho_{eff} - 3P_{eff}$ and $k < 0$. (For the deducing details see Appendix G). Therefore, we obtain an effective SEC

$$\rho_{eff} + 3P_{eff} \geq 0. \tag{65}$$

The evolution equation for the expansion of a null geodesic congruence is defined by a vector field k^{μ}, which has the same form as the Raychaudhuri equation, with $-R_{\mu\nu}k^{\mu}k^{\nu}$ in the last term. In this case, the condition for the convergence (geodesic focusing) of hyper-surface orthogonal congruences of null geodesics along with Einsteins's equation implies

$$R_{\mu\nu}k^{\mu}k^{\nu} = -k\left(T_{eff\,\mu\nu} - \frac{T_{eff}}{2}g_{\mu\nu}\right)k^{\mu}k^{\nu} \geq 0 = -k\left[\left(\rho_{eff} + P_{eff}\right)\right] \geq 0, \tag{66}$$

where $k < 0$. The deducing details are in Appendix H. Therefore, we obtain an effective NEC

$$\rho_{eff} + P_{eff} \geq 0\,. \tag{67}$$

Using Equation (59), we concretely have

$$\begin{aligned}\rho_{eff} = T_{eff\,00} &= \frac{1}{kf'}\left[kT_{00} - \nabla_0\nabla_0 f' + \frac{\left(-kT + f - \Box f'\right)}{2}g_{00}\right] \\ &= \frac{1}{kf'}\left[k\rho - \nabla_0\nabla_0 f' + \frac{\left(-kT + f - \Box f'\right)}{2}\right],\end{aligned} \tag{68}$$

and

$$\begin{aligned}P_{eff} &= -\frac{1}{3}g^{ij}T_{eff\,ij} = -\frac{1}{3}T_{eff\,j}^{\;j} = -\frac{1}{3}\frac{1}{kf'}\left[kT_j^j - \nabla^j\nabla_j f' + \frac{\left(-kT + f - \Box f'\right)}{2}\delta_j^j\right] \\ &= \frac{1}{kf'}\left[k\left(-\frac{1}{3}T_j^j\right) - -\frac{1}{3}\nabla^j\nabla_j f' - \frac{1}{3}\frac{\left(-kT + f - \Box f'\right)}{2}\delta_j^j\right] \\ &= \frac{1}{kf'}\left[kP + \frac{1}{3}\nabla^j\nabla_j f' - \frac{\left(-kT + f - \Box f'\right)}{2}\right].\end{aligned} \tag{69}$$

Substituting Equations (68) and (69) into Equations (65) and (67), we deduce inequalities

$$\begin{aligned}&\rho_{eff} + 3P_{eff} \geq 0 \\ &= \frac{1}{kf'}\left[2k\rho + 2\nabla^j\nabla_j f' - f\right] \geq 0.\end{aligned} \tag{70}$$

For the deducing details see Appendix I.

Thus, we have

$$\begin{aligned}&\rho_{eff} + P_{eff} \geq 0 \\ &= \frac{1}{kf'}\left[k\rho - \nabla_0\nabla_0 f' + \frac{\left(-kT + f - \Box f'\right)}{2}\right] \\ &\quad + \frac{1}{kf'}\left[kP + \frac{1}{3}\nabla^j\nabla_j f' - \frac{\left(-kT + f - \Box f'\right)}{2}\right] \\ &= \frac{1}{kf'}\left\{k\rho - \nabla_0\nabla_0 f' + \frac{\left(-kT + f - \Box f'\right)}{2} + kP + \frac{1}{3}\nabla^j\nabla_j f' - \frac{\left(-kT + f - \Box f'\right)}{2}\right\} \\ &= \frac{1}{kf'}\left[k\left(\rho + P\right) - \nabla_0\nabla_0 f' + \frac{1}{3}\nabla^j\nabla_j f'\right] \geq 0,\end{aligned} \tag{71}$$

where $g^{00} = 1$.

For $f = R$, Equations (70) and (71) give

$$\begin{aligned}&\rho_{eff}+3P_{eff}\geq 0\\&\frac{1}{kf'}\left[2k\rho+2\nabla^j\nabla_j f'-f\right]=\frac{1}{k}\left[2k\rho-R\right]\geq 0\\&=\frac{1}{k}\left[2k\rho-kT\right]=\frac{1}{k}\left[2k\rho-k\left(\rho-3P\right)\right]=\rho+3P\geq 0.\end{aligned} \tag{72}$$

and

$$\begin{aligned}&\rho_{eff}+P_{eff}\geq 0\\&=\frac{1}{kf'}\left[k\left(\rho+P\right)-\nabla_0\nabla_0 f'+\frac{1}{3}\nabla^j\nabla_j f'\right]\geq 0\\&=\frac{1}{k1}\left[k\left(\rho+P\right)\right]\geq 0\\&=\rho+P\geq 0,\end{aligned} \tag{73}$$

where $\nabla_j f'=\partial_j f'=\partial_j 1=0$. Therefore, Equations (68) and (69) can return to the well-known forms of the SEC and NEC in GR [85] [86] [87] [88] [89]. Therefore, the above investigations are consistent.

In sum, we investigate a general $f(R)$ gravity with general matter action and obtain the different field equations, general matter tensor and effective matter tensor. Further, we obtain an effective SEC and an effective NEC. When $f(R)$ approaches to R, the effective SEC and the effective NEC approaches, respectively, to the usual SEC and the usual NEC. Thus, these studies are consistent.

6. General *f*(*R*) Gravity Theory with General Scalar Fields

To investigate more general cases and extend the applications of the new action proposed in Section 2, the single scalar field $V(\phi)$ is changed into a more general form $V\left(\phi^1,\phi^2,\cdots,\phi^n\right)$.

We now generalize Equation (4) to a general action of $f(R)$ gravity, and many general scalar fields are coupled to scalar curvature as follows.

$$\begin{aligned}A&=\int\alpha f(R)\sqrt{-g}\,\mathrm{d}^4x+\int\left[\left(\alpha_1 g^{\mu\nu}+\alpha_2 Rg^{\mu\nu}\right)\partial_\mu\boldsymbol{\phi}\cdot\partial_\nu\boldsymbol{\phi}-V(\boldsymbol{\phi})\right]\sqrt{-g}\,\mathrm{d}^4x\\&=\int\alpha f(R)\sqrt{-g}\,\mathrm{d}^4x\\&\quad+\int\left[\left(\alpha_1 g^{\mu\nu}+\alpha_2 Rg^{\mu\nu}\right)\left(\partial_\mu\phi^1\partial_\nu\phi^1+\partial_\mu\phi^2\partial_\nu\phi^2+\cdots+\partial_\mu\phi^n\partial_\nu\phi^n\right)\right.\\&\quad\left.-V\left(\phi^1,\phi^2,\cdots,\phi^n\right)\right]\sqrt{-g}\,\mathrm{d}^4x\end{aligned} \tag{74}$$

When $\alpha_1=\frac{1}{2}$ and $\alpha_2=0$, Equation (74) is a generic action of $f(R)$ gravity and Quintessence fields as follows.

$$\begin{aligned}A=&\int\alpha f(R)\sqrt{-g}\,\mathrm{d}^4x+\int\left[\frac{1}{2}g^{\mu\nu}\left(\partial_\mu\phi^1\partial_\nu\phi^1+\partial_\mu\phi^2\partial_\nu\phi^2+\cdots+\partial_\mu\phi^n\partial_\nu\phi^n\right)\right.\\&\left.-V\left(\phi^1,\phi^2,\cdots,\phi^n\right)\right]\sqrt{-g}\,\mathrm{d}^4x,\end{aligned} \tag{75}$$

i.e.,

$$L=\sqrt{-g}\left[\alpha f(R)+\frac{1}{2}g^{\mu\nu}\left(\partial_\mu\phi^1\partial_\nu\phi^1+\partial_\mu\phi^2\partial_\nu\phi^2+\cdots+\partial_\mu\phi^n\partial_\nu\phi^n\right)-V\left(\phi^1,\phi^2,\cdots,\phi^n\right)\right], \tag{76}$$

Equation (76) is a generic Lagrangian of $f(R)$ gravity and Quintessence fields.

When $\alpha_1 = -\frac{1}{2}$ and $\alpha_2 = 0$, Equation (74) is a generic action of $f(R)$ gravity and Phantom fields as follows

$$A = \int \alpha f(R)\sqrt{-g}\,\mathrm{d}^4x + \int\Big[-\frac{1}{2}g^{\mu\nu}\left(\partial_\mu\phi^1\partial_\nu\phi^1 + \partial_\mu\phi^2\partial_\nu\phi^2 + \cdots + \partial_\mu\phi^n\partial_\nu\phi^n\right) - V\left(\phi^1,\phi^2,\cdots,\phi^n\right)\Big]\sqrt{-g}\,\mathrm{d}^4x, \tag{77}$$

i.e.,

$$L = \sqrt{-g}\left[\alpha f(R) - \frac{1}{2}g^{\mu\nu}\left(\partial_\mu\phi^1\partial_\nu\phi^1 + \partial_\mu\phi^2\partial_\nu\phi^2 + \cdots + \partial_\mu\phi^n\partial_\nu\phi^n\right) - V\left(\phi^1,\phi^2,\cdots,\phi^n\right)\right], \tag{78}$$

Equation (78) is a generic Lagrangian of $f(R)$ gravity and Phantom fields.

Using Equation (74) or Equation (77), we can calculate and obtain all corresponding results similar to all investigations above Equation (74).

7. Summary and Conclusions

This paper introduces the development process of $f(R)$ gravity theory, which is based on the Quintessence, Phantom and $f(R)$ theories [1]. We generally present a general action of $f(R)$ gravity, Phantom and Quintessence fields and generalizes the action to a new general action of $f(R)$ gravity, Phantom and Quintessence fields coupled to scalar curvature. Further, we deduce Euler-Lagrange Equations of different fields, and give matter tensor and effective matter tensor.

Then, this paper obtains the general energy-momentum tensor, pressure, density and speed sound of the new general single field action of $f(R)$ gravity, Phantom and Quintessence fields coupled to scalar curvature and so on.

Further, this paper deduces the equations of states of $f(R)$ gravity, Phantom and Quintessence fields coupled to scalar curvature. We also investigate different cosmological evolutions with single field inflation including the Big Rip Universe, De Sitter Universe and Harmonic Universe. Besides, we study the cases of different potentials for different equations of states.

In addition, this paper investigates a general $f(R)$ gravity theory with general matter action and obtains the different field equations, general matter tensor and effective matter tensor. We further obtain an effective Strong Energy Condition and an effective Null Energy Condition. We prove that when *f(R)* approaches to *R*, the effective Strong Energy Condition and the effective Null Energy Condition approach, respectively, to the usual Strong Energy Condition and the usual Null Energy Condition. Thus, these investigations are consistent.

The Hawking-Penrose singularity theorems invoke the weak and strong energy conditions, whereas the proof of the second law of black hole thermodynamics requires the null energy condition. More recently, several researchers used the classical energy conditions of GR to investigate cosmological issues.

In the cosmology, these theories provide an alternative way to explain the cosmic speed-up. The freedom in building different functional forms of $f(R)$ causes the problem of how to constrain these many possible $f(R)$ gravities from theoretical or observational aspects. Recently, this possibility has been explored by testing the cosmological viability of some specific forms of $f(R)$ gravities. Specially, all calculations with appendixes in this paper are helpful for beginners in this field.

Finally, we generalize Equation (4) to a general action of $f(R)$ gravity and many scalar fields, and obtain a general action of $f(R)$ gravity, Quintessence field and Phantom field. Then more applications can be done. For $f(R)$ levels and multi-field coupling, we will carry out further study [113].

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Appendix A

$$
\begin{aligned}
\delta L &= \delta\int \alpha f(R)\sqrt{-g}\,\mathrm{d}^4x + \delta\int\left[\left(\alpha_1 g^{\mu\nu} + \alpha_2 R g^{\mu\nu}\right)\partial_\mu\phi\partial_\nu\phi - V(\phi)\right]\sqrt{-g}\,\mathrm{d}^4x \\
&= \int\left(\alpha\delta f(R)\sqrt{-g} + \alpha f(R)\delta\sqrt{-g}\right)\mathrm{d}^4x \\
&\quad + \delta_{g^{\mu\nu}}\int\left\{\left[\left(\alpha_1 g^{\mu\nu} + \alpha_2 R g^{\mu\nu}\right)\partial_\mu\phi\partial_\nu\phi - V(\phi)\right]\sqrt{-g}\right\}\mathrm{d}^4x \\
&\quad + \delta_\phi\int\left\{\left[\left(\alpha_1 g^{\mu\nu} + \alpha_2 R g^{\mu\nu}\right)\partial_\mu\phi\partial_\nu\phi - V(\phi)\right]\sqrt{-g}\right\}\mathrm{d}^4x \\
&= \int\left[\alpha f_R(R)\sqrt{-g}\,\delta R + \alpha f(R)\left(-\frac{\sqrt{-g}}{2}g_{\mu\nu}\delta g^{\mu\nu}\right)\right]\mathrm{d}^4x \\
&\quad + \int\left\{\left(\alpha_1\delta g^{\mu\nu} + \alpha_2\delta R g^{\mu\nu} + \alpha_2 R\delta g^{\mu\nu}\right)\partial_\mu\phi\partial_\nu\phi\sqrt{-g}\right. \\
&\quad \left. + \left[\left(\alpha_1 g^{\mu'\nu'} + \alpha_2 R g^{\mu'\nu'}\right)\partial_{\mu'}\phi\partial_{\nu'}\phi - V(\phi)\right]\left(\delta\sqrt{-g}\right)\right\}\mathrm{d}^4x \\
&\quad + \int\left[\left(\alpha_1 g^{\mu\nu} + \alpha_2 R g^{\mu\nu}\right)\frac{\delta\left(\partial_\mu\phi\partial_\nu\phi\right)\sqrt{-g}}{\delta\left(\partial_\alpha\phi\right)}\delta\left(\partial_\alpha\phi\right) - \sqrt{-g}\,\frac{\delta V(\phi)}{\delta\phi}\delta\phi\right]\mathrm{d}^4x \\
&= \int\left[\alpha f_R(R)\sqrt{-g}\,\delta R + \alpha f(R)\left(-\frac{\sqrt{-g}}{2}g_{\mu\nu}\delta g^{\mu\nu}\right)\right]\mathrm{d}^4x \\
&\quad + \int\left\{\left(\alpha_1\delta g^{\mu\nu} + \alpha_2\delta R g^{\mu\nu} + \alpha_2 R\delta g^{\mu\nu}\right)\partial_\mu\phi\partial_\nu\phi\sqrt{-g}\right. \\
&\quad \left. + \left[\left(\alpha_1 g^{\mu'\nu'} + \alpha_2 R g^{\mu'\nu'}\right)\partial_{\mu'}\phi\partial_{\nu'}\phi - V(\phi)\right]\left(-\frac{\sqrt{-g}}{2}g_{\mu\nu}\delta g^{\mu\nu}\right)\right\}\mathrm{d}^4x \\
&\quad + \int\left[\left(\alpha_1 g^{\mu\nu} + \alpha_2 R g^{\mu\nu}\right)\left(2\partial_\nu\phi\right)\sqrt{-g}\,\partial_\mu\delta\phi - \sqrt{-g}\,\frac{\delta V(\phi)}{\delta\phi}\delta\phi\right]\mathrm{d}^4x \\
&= \int\left[\alpha f_R(R)\sqrt{-g}\,\delta R + \alpha f(R)\left(-\frac{\sqrt{-g}}{2}g_{\mu\nu}\delta g^{\mu\nu}\right)\right]\mathrm{d}^4x \\
&\quad + \int\left\{\left(\alpha_1\delta g^{\mu\nu} + \alpha_2\delta R g^{\mu\nu} + \alpha_2 R\delta g^{\mu\nu}\right)\partial_\mu\phi\partial_\nu\phi\sqrt{-g}\right. \\
&\quad \left. + \left[\left(\alpha_1 g^{\mu'\nu'} + \alpha_2 R g^{\mu'\nu'}\right)\partial_{\mu'}\phi\partial_{\nu'}\phi - V(\phi)\right]\left(-\frac{\sqrt{-g}}{2}g_{\mu\nu}\delta g^{\mu\nu}\right)\right\}\mathrm{d}^4x \\
&\quad + \int\left\{-\partial_\mu\left[\left(\alpha_1 g^{\mu\nu} + \alpha_2 R g^{\mu\nu}\right)2\partial_\nu\phi\sqrt{-g}\right]\delta\phi\right. \\
&\quad \left. + \partial_\mu\left[\left(\alpha_1 g^{\mu\nu} + \alpha_2 R g^{\mu\nu}\right)2\partial_\nu\phi\sqrt{-g}\,\delta\phi\right] - \sqrt{-g}\,\frac{\delta V(\phi)}{\delta\phi}\delta\phi\right\}\mathrm{d}^4x.
\end{aligned}
\tag{79}
$$

Appendix B

$$
\begin{aligned}
g^{\mu\nu}\delta R_{\mu\nu} &= g^{\mu\nu}\nabla_\lambda\frac{1}{2}g^{\alpha\lambda}\left(\nabla_\mu\delta g_{\nu\alpha} + \nabla_\nu\delta g_{\alpha\mu} - \nabla_\alpha\delta g_{\mu\nu}\right) - \nabla_\nu\left[\frac{g^{\alpha\lambda}}{2}\left(\nabla_\lambda\delta g_{\alpha\mu} + \nabla_\nu\delta g_{\alpha\lambda} - \nabla_\alpha\delta g_{\mu\lambda}\right)\right] \\
&= \frac{1}{2}g^{\mu\nu}\left\{g^{\lambda\alpha}\left(\nabla_\lambda\nabla_\mu\delta g_{\nu\alpha} + \nabla_\lambda\nabla_\nu\delta g_{\alpha\mu} - \nabla_\lambda\nabla_\alpha\delta g_{\mu\nu}\right) - g^{\lambda\alpha}\left(\nabla_\nu\nabla_\lambda\delta g_{\alpha\mu} + \nabla_\nu\nabla_\mu\delta g_{\alpha\lambda} - \nabla_\nu\nabla_\alpha\delta g_{\mu\lambda}\right)\right\} \\
&= \frac{1}{2}\left[g^{\mu\nu}\nabla_\lambda\nabla_\mu\left(-\delta g^{\lambda\alpha}g_{\nu\alpha}\right) - g^{\mu\nu}\nabla_\lambda\nabla_\nu\delta g^{\lambda\alpha}g_{\alpha\mu} - \nabla_\lambda\nabla^\lambda g^{\mu\nu}\delta g_{\mu\nu}\right. \\
&\quad \left. - g^{\mu\nu}\left(-\nabla_\nu\nabla_\lambda\delta g^{\lambda\alpha}g_{\alpha\mu}\right) - \nabla^\mu\nabla_\mu g^{\lambda\alpha}\delta g_{\alpha\lambda} + g^{\mu\nu}\nabla_\nu\nabla_\alpha\left(-\delta g^{\lambda\alpha}g_{\mu\lambda}\right)\right] \\
&= \frac{1}{2}\left[-\nabla_\lambda\nabla_\mu\delta g^{\lambda\mu} - \nabla_\lambda\nabla_\nu\delta g^{\lambda\nu} + \nabla_\lambda\nabla^\lambda g^{\mu\nu}\delta g_{\mu\nu} + \nabla_\nu\nabla_\lambda\delta g^{\lambda\nu} + \nabla^\mu\nabla_\mu\delta g^{\lambda\alpha}g_{\lambda\alpha} - \nabla_\nu\nabla_\alpha\delta g^{\nu\alpha}\right] \\
&= g_{\mu\nu}\nabla_\lambda\nabla^\lambda\delta g^{\mu\nu} - \nabla_\mu\nabla_\nu\delta g^{\mu\nu}
\end{aligned}
\tag{80}
$$

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Appendix C

$$
\begin{aligned}
&\int F\sqrt{-g}\left(g_{\mu\nu}\mathrm{n}\delta g^{\mu\nu}-\nabla_\mu\nabla_\nu\delta g^{\mu\nu}\right)\mathrm{d}^4xa\\
&=\int F\left(\nabla_\alpha\nabla^\alpha g_{\mu\nu}\delta g^{\mu\nu}-\nabla_\mu\nabla_\nu\delta g^{\mu\nu}\right)\sqrt{-g}\mathrm{d}^4x\\
&=\int F\left[\frac{1}{\sqrt{-g}}\partial_\alpha\left(\sqrt{-g}\nabla^\alpha g_{\mu\nu}\delta g^{\mu\nu}\right)-\frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}\nabla_\nu\delta g^{\mu\nu}\right)\right]\sqrt{-g}\mathrm{d}^4x\\
&=\int\left[-\partial_\alpha F\sqrt{-g}\partial^\alpha g_{\mu\nu}\delta g^{\mu\nu}+\partial_\mu F\sqrt{-g}\nabla_\nu\delta g^{\mu\nu}\right]\mathrm{d}^4x\\
&=\int\left[\partial^\alpha\left(\sqrt{-g}\partial_\alpha F\right)g_{\mu\nu}\delta g^{\mu\nu}+\partial_\mu F\sqrt{-g}\left(\partial_\nu\delta g^{\mu\nu}+\Gamma^\mu_{\alpha\nu}\delta g^{\alpha\nu}+\Gamma^\nu_{\alpha'\nu}\delta g^{\mu\alpha'}\right)\right]\mathrm{d}^4x\\
&=\int\left\{\frac{\sqrt{-g}}{\sqrt{-g}}\partial^\alpha\left(\sqrt{-g}\partial_\alpha F\right)g_{\mu\nu}\delta g^{\mu\nu}+\partial_\mu F\sqrt{-g}\left[\frac{1}{\sqrt{-g}}\partial_\nu\left(\sqrt{-g}\delta g^{\mu\nu}\right)+\Gamma^\mu_{\alpha\nu}\delta g^{\alpha\nu}\right]\right\}\mathrm{d}^4x\\
&=\int\left[\left(\nabla^\alpha\nabla_\alpha F\right)g_{\mu\nu}\delta g^{\mu\nu}\sqrt{-g}-\left(\partial_\nu\partial_\mu F\right)\sqrt{-g}\delta g^{\mu\nu}+\partial_\mu F\sqrt{-g}\Gamma^\mu_{\alpha\nu}\delta g^{\alpha\nu}\right]\mathrm{d}^4x\\
&=\int\left\{\left(\nabla^\alpha\nabla_\alpha F\right)g_{\mu\nu}\delta g^{\mu\nu}-\left[\partial_\nu\left(\partial_\mu F\right)-\Gamma^\alpha_{\mu\nu}\partial_\alpha F\right]\delta g^{\mu\nu}\right\}\sqrt{-g}\mathrm{d}^4x\\
&=\int\left[\left(\nabla^\alpha\nabla_\alpha F\right)g_{\mu\nu}\delta g^{\mu\nu}-\left(\nabla_\nu\partial_\mu F\right)\delta g^{\mu\nu}\right]\sqrt{-g}\mathrm{d}^4x\\
&=\int\left[\left(\nabla^\alpha\nabla_\alpha F\right)g_{\mu\nu}\delta g^{\mu\nu}-\nabla_\mu\nabla_\nu F\delta g^{\mu\nu}\right]\sqrt{-g}\mathrm{d}^4x
\end{aligned}
\tag{81}
$$

Appendix D

$$
\begin{aligned}
\rho&=3P+2\gamma X-\left[\left(\gamma+2\alpha_2\Box\right)X-\frac{1}{2}V(\phi)\right]4-2\left(\alpha_2R^\alpha{}_\alpha-\alpha_2\nabla^\alpha\nabla_\alpha\right)X\\
&=3P+2\gamma X-4\gamma X-8\alpha_2\Box X+2V(\phi)-2\alpha_2R^\alpha{}_\alpha X+2\alpha_2\nabla^\alpha\nabla_\alpha X\\
&=3P-2\gamma X+2V(\phi)-8\alpha_2\Box X-2\alpha_2R^\alpha{}_\alpha X+2\alpha_2\Box X\\
&=3P-2\left[\gamma X-V(\phi)\right]-6\alpha_2\Box X-2\alpha_2RX\\
&=3P-2\left[\gamma X+\alpha_2RX-V(\phi)\right]-6\alpha_2\Box X
\end{aligned}
\tag{82}
$$

Appendix E

$$
\begin{aligned}
T_{\mathrm{eff}\,\mu\nu}&=\frac{T_{\mathrm{eff}}}{2}g_{\mu\nu}-R_{\mu\nu}/k=\frac{\left[2f-kT-3\Box f'\right]}{2kf'}g_{\mu\nu}-R_{\mu\nu}/k\\
&=\frac{\left[2f-kT-3\Box f'\right]}{2kf'}g_{\mu\nu}-\left[\frac{f}{2}g_{\mu\nu}-kT_{\mu\nu}+\left(\nabla_\mu\nabla_\nu-g_{\mu\nu}\Box\right)f'\right]\Big/kf'\\
&=\frac{\left[2fg_{\mu\nu}-kTg_{\mu\nu}-3\Box f'g_{\mu\nu}\right]}{2kf'}-\frac{\left[fg_{\mu\nu}-2kT_{\mu\nu}+2\left(\nabla_\mu\nabla_\nu-g_{\mu\nu}\Box\right)f'\right]}{2kf'}\\
&=\frac{1}{kf'}\left[kT_{\mu\nu}-\nabla_\mu\nabla_\nu f'+\frac{\left(-kT+f-\Box f'\right)}{2}g_{\mu\nu}\right]
\end{aligned}
\tag{83}
$$

Appendix F

$$
\begin{aligned}
g^{\mu\nu}T_{\mathrm{eff}\,\mu\nu}&=\left\{kT_{\mu\nu}g^{\mu\nu}-g^{\mu\nu}\nabla_\mu\nabla_\nu f'+\frac{\left[-kT+f-\Box f'\right]}{2}g^{\mu\nu}g_{\mu\nu}\right\}\Big/f'k\\
&=\left\{kT-\Box f'+\frac{\left[-kT+f-\Box f'\right]}{2}4\right\}\Big/f'k\\
&=\left\{kT-\Box f'+\frac{\left[-kT+f-\Box f'\right]}{1}2\right\}\Big/f'k=\frac{2f-kT-3\mathrm{n}f'}{kf'}
\end{aligned}
\tag{84}
$$

Appendix G

$$
\begin{aligned}
R_{\mu\nu}u^\mu u^\nu &= -k\left(T_{eff\,\mu\nu} - \frac{T_{eff}}{2}g_{\mu\nu}\right)u^\mu u^\nu \\
&= -k\left[\left(\rho_{eff} + P_{eff}\right)u_\mu u_\nu - g_{\mu\nu}P_{eff} - \frac{\rho_{eff} - 3P_{eff}}{2}g_{\mu\nu}\right]u^\mu u^\nu \\
&= -k\left[\left(\rho_{eff} + P_{eff}\right)u_\mu u_\nu u^\mu u^\nu - g_{\mu\nu}u^\mu u^\nu P_{eff} - \frac{\rho_{eff} - 3P_{eff}}{2}g_{\mu\nu}u^\mu u^\nu\right] \\
&= -k\left[\left(\rho_{eff} + P_{eff}\right) - P_{eff} - \frac{\rho_{eff} - 3P_{eff}}{2}\right] \\
&= -k\left[\left(\rho_{eff} - \frac{\rho_{eff} - 3P_{eff}}{2}\right)\right] = -k\frac{1}{2}\left[\left(\rho_{eff} + 3P_{eff}\right)\right]
\end{aligned} \tag{85}
$$

Appendix H

$$
\begin{aligned}
R_{\mu\nu}k^\mu k^\nu &= -k\left(T_{eff\,\mu\nu} - \frac{T_{eff}}{2}g_{\mu\nu}\right)k^\mu k^\nu \geq 0 \\
&= -kT_{eff\,\mu\nu}k^\mu k^\nu + k\frac{T_{eff}}{2}g_{\mu\nu}k^\mu k^\nu \geq 0 \\
&= -kT_{eff\,\mu\nu}k^\mu k^\nu + k\frac{T_{eff}}{2}k_\nu k^\nu \geq 0 \\
&= -kT_{eff\,\mu\nu}k^\mu k^\nu \\
&= -k\left[\left(\rho_{eff} + P_{eff}\right)u_\mu u_\nu - g_{\mu\nu}P_{eff}\right]k^\mu k^\nu \\
&= -k\left[\left(\rho_{eff} + P_{eff}\right)u_\mu u_\nu k^\mu k^\nu - g_{\mu\nu}k^\mu k^\nu P_{eff}\right] \\
&= -k\left[\left(\rho_{eff} + P_{eff}\right)k^0 k^0 - k_\nu k^\nu P_{eff}\right] \\
&= -k\left[\left(\rho_{eff} + P_{eff}\right)\right]
\end{aligned} \tag{86}
$$

Appendix I

$$
\begin{aligned}
&\rho_{eff} + 3P_{eff} \geq 0 \\
&= \frac{1}{kf'}\left[k\rho - \nabla_0\nabla_0 f' + \frac{\left(-kT + f - \Box f'\right)}{2}\right] + 3\frac{1}{kf'}\left[kP + \frac{1}{3}\nabla^j\nabla_j f' - \frac{\left(-kT + f - \Box f'\right)}{2}\right] \\
&= \frac{1}{kf'}\left\{k\rho - \nabla_0\nabla_0 f' + \frac{\left(-kT + f - \Box f'\right)}{2} + 3kP + \nabla^j\nabla_j f' - 3\frac{\left(-kT + f - \Box f'\right)}{2}\right\} \\
&= \frac{1}{kf'}\left[k\left(\rho + 3P\right) - \nabla_0\nabla_0 f' + \nabla^j\nabla_j f' - \left(-kT + f - \Box f'\right)\right] \\
&= \frac{1}{kf'}\left[k\left(\rho + 3P\right) - \nabla_0\nabla_0 f' + \nabla^j\nabla_j f' - \left(-kT + f - \nabla_0\nabla_0 f' - \nabla^j\nabla_j f'\right)\right] \\
&= \frac{1}{kf'}\left[k\left(\rho + 3P\right) + 2\nabla^j\nabla_j f' + kT - f\right] \geq 0 \\
&= \frac{1}{kf'}\left[k\left(\rho + 3P\right) + 2\nabla^j\nabla_j f' + k\left(\rho - 3P\right) - f\right] \geq 0 \\
&= \frac{1}{kf'}\left[2k\rho + 2\nabla^j\nabla_j f' - f\right] \geq 0
\end{aligned} \tag{87}
$$