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Padé Approximation Modelling of an Advertising-Sales Relationship*

M. C. Gil-Fariña, C. Gonzalez-Concepcion, C. Pestano-Gabino

Department of Applied Economics, Faculty of Economics and Business Administration, University of La Laguna (ULL), Campus of Guajara, Tenerife, Spain.
Email: {mgil, cogonzal, cpestano}@ull.es

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ABSTRACT

Forecasting reliable estimates on the future evolution of relevant variables is a main concern if decision makers in a variety of fields are to act with greater assurances. This paper considers a time series modelling method to predict relevant variables taking VARMA and Transfer Function models as its starting point. We make use of the rational Padé-Laurent Approximation, a relevant type of rational approximation in function theory that allows the decision maker to take part in the building of estimates by providing the available information and expectations for the decision variables. This method enhances the study of the dynamic relationship between variables in non-causal terms and allows for an ex ante sensibility analysis, an interesting matter in applied studies. The alternative proposed, however, must adhere to a type of model whose properties are of an asymptotic nature, meaning large chronological data series are required for its efficient application. The method is illustrated through the well-known data series on advertising and sales for the Lydia Pinkham Medicine Company, which has been used by various authors to illustrate their own proposals.

Keywords: *Time Series Modelling, Expectations, Economics, Numerical Methods, Padé Approximation*

1. Introduction

The possibility of forecasting reliable estimates for the future behaviour of relevant variables is main concern to decision making in numerous fields (including business, industry, energy, environmental, government agencies and medical and social network fields). Consequently, from a scientific standpoint, it is necessary to investigate alternative methods that can provide estimates while also introducing them into a technological system that allows the decision maker himself to participate in the composition of said predictions. Many researchers have attempted to satisfy this requirement from different perspectives, such that the prediction problem is always present in any generic data-mining task. And yet, the technique selected for use depends on the availability and type of data in relation to the hypotheses of the methods that sustain the desired technique and which yield different degrees of accuracy, time horizons and different computational and social costs. The use of a relevant technique within the scope of rational approximation, namely the Padé Approximation (PA), has had a gratifyingly enriching and stimulating effect on the study of the dynamic relationship structure between variables, especially within the context of identifying univariate and multivariate ARMA models (see, among others, [1–5]).

In particular, within the scope of rational model formulation in causal terms, this technique acquires a special relevance for characterizing simple models at a computational level with which to specify the deterministic part in certain time series models. At the same time, it provides reliable initial estimates for obtaining a definitive model by using more efficient, iterative methods once the random component has been identified.

The choice of a particular VARMA model, namely the Transfer Function [6] model which constitutes one of the most widely used representations in the input-output context of dynamic stochastic systems through the use of polynomial rational expressions, serves to highlight the influence that the expectations or forecasts for an explanatory variable (input) exerts on the model under consideration.

In this sense, the formulation of the time series analysis instruments that encompass the available sampling

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information and the establishment of non-causal models into which the forecasts are incorporated, that is, the knowledge that by way of *ex ante* or *ex post* expectations is provided by economic theory or by the empirical evidence, if any, about the model's explanatory variable(s), provide a basic framework for tackling the study of the rational identification of time series models from a more general context.

The method that allows for a sustained study of the time series identification from an evolutionary, but not necessarily causal, standpoint of the variables involved is based on a generalization of the PA concept to the study of formal Laurent series [7]. The use of this approach allows for the study of new dynamic identification procedures by means of a single model that simultaneously approximates both time directions in a formal Laurent series while encompassing the classical case as a specific one when the expectations are not included (see, for example, [8,9]).

The consideration of this broader dynamic framework, however, into which future information on the model's variables is incorporated, favours a continuous feedback process through which the formulated expectations are replaced period by period by updated information as the empirical evidence modifies or confirms the predictions made.

It is worth noting that the models are in no way insensitive to changes in the way the expectations are made; on the contrary, these changes lead to different dynamic behaviours insofar as it is the precise nature of the expectations that determines the explicit dynamics of the forecasted variables.

In this sense, the possibility of offering different models for a single data series by simply modifying the *ex ante* expectations made by economic agents allows us to obtain future knowledge about their influence on the model and therefore to contrast and compare different dynamic specifications so as to yield an optimum model.

In short, the performance of a sensibility analysis will permit for a contrast of the extent to which the forecasts of a variable's future behaviour affect the mode's predictions and, as a consequence, its adequacy to the empirical data.

So as to illustrate the rational Padé Approximation in modelling time series, we develop an application for the study of data from the advertising-sales series involving the activity of the Lydia E. Pinkham Medicine Company for the period 1907–1960 [10]. This series, which has been analyzed by numerous authors to illustrate their proposed methods, presents, as noted in [11,12], various characteristics which make it the ideal example for studying the relationship between both variables. Some of the reasons that justify the prominent role of this series in studies conducted to date are, among others, the importance of advertising as the company's sole marketing instrument, as well as the elevated advertising costs to sales ratio (above 40%) which, in some instances, even exceeded 80%.

This paper is structured into three sections. The first two present certain theoretical foundations for the properties of the PA that allow for the identification of the orders in VARMA models and TF models with expectations. We note the last section, which is devoted to the empirical results of the study on the aforementioned advertising-sales series. We conclude the work with the more relevant conclusions and the main references.

2. VARMA Models

A non-deterministic, k-dimensional centred process can be expressed as a vector autoregressive moving average model (VARMA(*p*, *q*)) if $A_p(B)X_t = B_q(B)a_t$, where

$$A_p(B) = I - A_1 B - \ldots - A_p B^p$$

and

$$B_q(B) = I - B_1 B - \ldots - B_q B^q$$

are *k*x*k* polynomial matrices in the lag operator *B*, that is, the coefficients A_i and B_j $(i = 1, \ldots, p;\ j = 1, \ldots, q)$ are *k*x*k* matrices.

The *k* vector a_t is a series of i.i.d. random variables with a zero mean multivariate norm and covariance matrix Σ.

The series X_t is said to be stationary when the zeros of $\left|A_p(B)\right|$ are outside the unit circle and invertible when the same can be said for those of $\left|B_q(B)\right|$.

This type of model is useful for understanding the dynamic relationships between the components of the series X_t. In this sense, one series can cause another or there may be a feedback relationship or they may be contemporaneously related.

In the case of the Lydia Pinkham advertising-sales series, the joint modelling of these effects by means of the procedures described in [13] allows the type of dynamic relationship existing between both variables to be determined [14].

A consideration of the PA matrix method yields the following theorem, which can be used in the first steps of the VARMA model identification.

Theorem 1 [15].- Let X_t be a second-order stationary *k*-dimensional process. Let $R(h) = Cov(X_t\ X_{t-h})$ be the covariance matrices of the process. Let

$$M1(i, j) = ((R(i - j + 1 + k + h))_{k,h=0}^{j-1}.$$

Then,

$$X_t \sim \text{VARMA}\,(p, q) \Leftrightarrow \text{rank}\, M1(i, j) =$$

$$\text{rank}\ \ M1(i+1, j+1)\ \ \forall i,j / i \ge q,\ j \ge p$$

3. Transfer Function Models with Expectations

Based on the PA definitions for a formal power series [16] and its extension to the study of formal Laurent series [7], we can provide a characterization for the identification of a TF model with expectations by means of the Toeplitz determinants

$$T_{f,g}(c_i) = det\left[(c_{f+k-j})_{k,j=1}^{g}\right]$$

Given two stationary time series y_t, x_t, let us assume the existence of a unidirectional causal dynamic relationship $x_t \rightarrow y_t$ given by the combination of simultaneous and shifted effects of the input variables (including the presence of expected values that may or may not follow the same distribution as the data), and let us consider a generalized TF model of the form:

$$y_t = v(B)x_t + N_t; \quad v(B) = \sum_{i=-\infty}^{\infty} v_i B^i$$

in which the x_{t-i} refers to the present and past of the input series (data) if $i \geq 0$ and to the expected values of the input series (expectations) if $i < 0$, and such that the exogenous variables represented by x_t satisfy a VARMA model.

A finite-order representation for the Impulse Response Function (IRF), namely, $v(B) = \sum_{i=-\infty}^{\infty} v_i B^i$ that simultaneously approximates both directions in time and enables the estimation of a finite number of observations will be characterized by the following result:

Theorem 2 [17].- Given the series $v(B) = \sum_{i=-\infty}^{\infty} v_i B^i$, the following conditions are equivalent:

a) $$v(B) = \frac{\sum_{i=H}^{K} a_i B^i}{(1+\sum_{i=1}^{N} q_{-i}B^{-i})(1+\sum_{i=1}^{U} q_i B^i)}$$

b)

$$T_{H,N}(v_i) \neq 0,\ T_{K,U}(v_i) \neq 0;\ T_{J,M}(v_i) = 0\ \forall J < \text{H} \wedge M > \text{N};$$

$$T_{J,M}(v_i) = 0\ \forall J > K \wedge M > U$$

Due to the properties of the lag operator, the following equivalency holds:

$$y_t = \frac{A_{H,K}(B)}{Q_{-N,U}(B)} x_t + N_t \equiv \frac{A_{I,L}(B)}{Q_M(B)} x_t + N_t$$

where

$I = H + N,\ L = K + N,\ M = N + U, A_{I,L}(B)$ y $Q_M(B)$ are polynomials of the form:

$$A_{I,L}(B) = \sum_{i=I}^{L} a_i B^i \qquad Q_M(B) = \sum_{i=0}^{M} q_i B^i$$

Assuming the non-existence of common roots and the conditions for ensuring the stability of the model hold, the problem will consist of determining, in keeping with the sample information available and by means of a direct estimation of the IRF, the best orders I, L, M that describe $v(B)$.

With this in mind, the steps to follow in studying the dynamic identification for a generalized formulation of the TF model are:

1) Obtain the estimates $\hat{v}_i$ for the weights v_i of the IRF $v(B)$, approximating $v(B)$ by a finite number of terms, that is, $v(B) \cong \sum_{i=k}^{k'} \hat{v}_i B^i$ with $k, k' \in Z$, normally $k < 0, k' > 0$.

2) Calculate the Toeplitz determinants associated with the series of estimated relative weights $\hat{\eta}_i = \frac{\hat{v}_i}{\max_{k \leq i \leq k} |\hat{v}_i|}$ and arrange them in a tabular form (T-table) as a generalization of the C-Table for the classic case.

3) Estimate the model parameters.

4. Empirical Results

The first model for the Lydia Pinkham advertising-sales for the period 1907–1960 involving the TF models method was proposed by [10], which assumed a unidirectional causal dynamic relationship from advertising to sales. The possible existence of a relationship in the other direction has, however, been mentioned by other authors. Various subsequent papers, including [14,18,19], have illustrated the application of this method using the analyzed series as reference. This assumption of unidirectional causality has been questioned on several occasions, however, as some maintain there is a feedback relationship [11,14,19,20]. That is why, in what follows, we analyze two cases, one involving the identification of a VARMA model, and another in which, assuming the existence of a unidirectional causal dynamic relationship from advertising to sales, we present various TF models and analyze the influence on the sales trends of different behaviour schemes or *ex ante* forecasts for the variable input, which in this case is advertising.

In any case, and given that both series exhibit non-stationarity, we present the models by taking first differences.

4.1 VARMA Models

Once the ranks for $M1(i, j)$, $0 \leq i \leq 5,\ 0 \leq j \leq 5$ are obtained, applying Theorem 1 to the aforementioned data provides the results shown in Tables 1 and 2, depending on the sample size used in the estimates. This indicates, as per the above theorem, that the differentiated data follow a VARMA (0, 1) model exchangeable with (1, 0), according to Table 1, and a VARMA (0, 2) interchangeable with (2, 0), according to Table 2.

Note how in Table 1, even though square (1, 1) does not have the same value as squares (i, i), i≥2 due to rounding errors in the calculations, using theoretical Padé Matrix Approximation results we know that these values have to coincide.

Once the model orders are calculated, the maximization of the likelihood function allows for a determination of efficient estimators. Using the Time Series Processor (TSP) software package [21], the results for the estimated VARMA (1, 0) and VARMA (2, 0) models yield:

$$X_t - \begin{pmatrix} -0.20 & 0.40 \\ -0.05 & 0.45 \end{pmatrix} X_{t-1} = \varepsilon_t$$

$$\Sigma = \begin{pmatrix} 44132 & \\ 23893 & 46790 \end{pmatrix}$$

and

$$X_t - \begin{pmatrix} -0.25 & 0.55 \\ -0.09 & 0.51 \end{pmatrix} X_{t-1} - \begin{pmatrix} -0.49 & -0.04 \\ -0.32 & 0.08 \end{pmatrix} X_{t-2} = \varepsilon_t$$

$$\Sigma = \begin{pmatrix} 33326 & \\ 18249 & 45672 \end{pmatrix}$$

Table 1. Orders of VARMA models. Alternative 1

	0	1	2	3	4	5
0	0	0	0	0	0	0
1	2	1	0	0	0	0
2	4	2	2	0	0	0
3	5	4	2	2	0	0
4	6	5	4	2	2	0
5	7	6	5	4	2	2

Table 2. Orders of VARMA models. Alternative 2

	0	1	2	3	4	5
0	0	0	0	0	0	0
1	2	2	1	0	0	0
2	3	3	3	1	0	0
3	4	3	3	3	1	0
4	5	4	3	3	3	1
5	6	5	4	3	3	3

where $X_t = (X_{1t}, X_{2t})^t$ and X_{1t} and X_{2t} are the first differences of the advertising and sales data, respectively.

As for the relationship between variables, we can conclude that a) the value of element (1, 2) in coefficient A_1 in both models, namely 0.40 and 0.55, indicates the existence of a relationship between sales and future advertising in a period, b) the negligible value of element (2,1) in coefficient A_1, namely -0.05 and -0.09, indicates a weak relationship between advertising and future sales, and c) the estimated correlation between residuals $\hat{\varepsilon}_{1t}$ and $\hat{\varepsilon}_{2t}$ indicates that advertising and sales are contemporaneously related.

Figures 1 and **2** show both models.

4.2 FT Models with Expectations

Taking into account the relationship between sales and future advertising noted in the above VARMA models, we now build transfer function models associated with three specific cases, depending on whether the advertising expectations follow a increasing or decreasing trend or whether they respect the predictions of the ARMA model for the input series (advertising), that is,

$$(1 + 0.269B^2 - 0.325B^4)\Delta x_t = N_t .$$

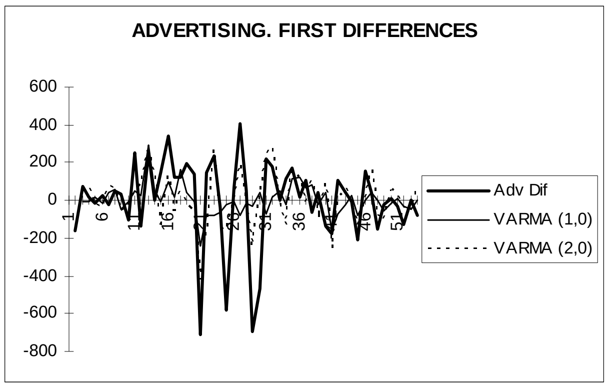

Figure 1. Advertising. First differences

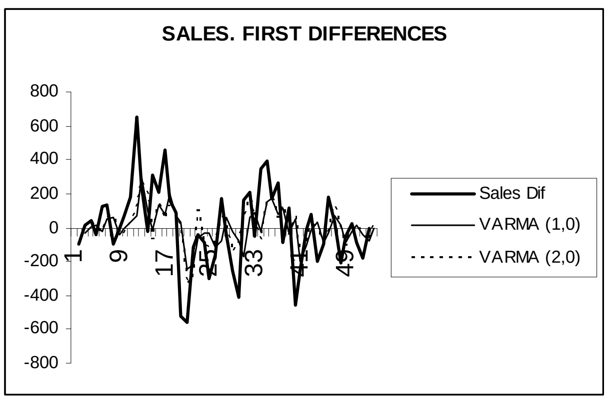

Figure 2. Sales. First differences

Starting from a generalized TF model for the advertising (x_t) and sales (y_t) variables, and keeping in mind that the series are first-order integrable, we formulate the model

$$\Delta y_t = v(B)\Delta x_t + a_t$$

in which $v(B)$ is approximated by a finite number of terms from k to k', that is, $v(B) \cong \sum_{i=k}^{k'} \hat{v}_i B^i$, normally $k < 0, k' > 0$.

Following the steps suggested by the method proposed yields the following results:

Case a: The expectations follow an increasing trend

After estimating the weights $\hat{v}_i$, the resulting T-table for the series of estimated relative weights is as shown in Table 3.

The behaviour of this table suggests a model of the form $\frac{A_{-3,0}(B)}{Q_{-1,1}(B)} \equiv \frac{\hat{A}_{-2,1}(B)}{\hat{Q}_{0,2}(B)}$

Specifically, once the parameters are estimated and the noise process is analyzed using the SCA software [22], the resulting model is:

$$\Delta y_t = \frac{.4334B^{-1} - .3267B}{1 - 1.5010B + .6518B^2}\Delta x_t + a_t$$

Table 3. T-Table. Orders of TF models with expectations in Case a

	1	2	3	4	5	6	7
-12	-.29	.09	-.02	.01	.00	.00	.00
-11	-.11	.21	-.05	.05	-.02	.01	.00
-10	.68	.45	.32	.24	.14	.09	.05
-9	-.11	-.07	.12	.24	.07	.04	.08
-8	.12	.04	.10	.21	-.03	-.05	.09
-7	.23	.14	.01	.19	.16	.14	.06
-6	-.76	.47	-.28	.16	.04	-.18	.25
-5	.46	.05	.16	.20	.20	.17	.45
-4	-.22	-.15	-.06	.06	.15	.34	.27
-3	.43	.23	.08	.06	.01	.44	.56
-2	.21	-.13	.013	.05	-.17	.57	.12
-1	.40	-.05	.31	.44	.51	.78	.82
0	1.00	.93	.87	.75	.51	.35	-.01
1	.16	-.16	.16	.28	-.01	.17	.02
2	.18	.00	.08	.11	-.09	.08	-.02
3	.22	.09	.04	.07	.01	.03	.03
4	-.26	.08	-.06	.04	.02	.00	-.03
5	-.05	-.09	.01	.04	.01	.02	.02

Case b: The expectations follow a decreasing trend

In this case, the method yields the following model:

$$\Delta y_t = \frac{.4331B^{-1} - .3285B}{1 - 1.5016B + .6514B^2}\Delta x_t + a_t$$

as suggested by Table 4.

Note the analogy between the models for cases a) and b).

Case c: The expectations are generated by the ARMA model of the input series

In this case, the T-table obtained by the series of estimated relative weights is shown in Table 5 and suggests a model of the form $\frac{A_{-3,0}(B)}{Q_{-2,1}(B)} \equiv \frac{\hat{A}_{-1,2}(B)}{\hat{Q}_{0,3}(B)}$ which, once estimated, yields

$$\Delta y_t = \frac{.5413B^{-1} - .1775}{1 - 1.3754B + .6612B^2}\Delta x_t + a_t$$

whose parameters differ significantly from those obtained in the first two cases.

The results for the three cases considered are shown in **Figures 3, 4** and **5**.

As we can see, the incorporation in the model of the *ex ante* advertising forecasts as well as the various formative mechanisms that not only include the information contained in the available historical data but also that derived from the company's desires and the strategies and outlooks devised by the economic agents in the decision-making process, facilitate obtaining valid dynamic formulations from a data fitting perspective. In addition,

Table 4. T-Table. Orders of TF models with expectations in Case b

	1	2	3	4	5	6	7
-12	-.21	.04	-.01	.00	.00	.00	.00
-11	-.14	.19	-.06	.04	-.02	.01	.00
-10	.83	.68	.58	.51	.42	.34	.28
-9	-.14	-.13	.20	.47	.10	.01	.25
-8	.18	.07	.18	.39	.01	-.07	.21
-7	.25	.21	.02	.32	.28	.25	.14
-6	-.82	.56	-.38	.25	.04	-.30	.51
-5	.46	.02	.16	.24	.27	.27	.76
-4	-.23	-.13	-.05	.06	.15	.44	.40
-3	.39	.20	.07	.05	-.02	.50	.69
-2	.22	-.11	.10	.06	-.16	.60	.18
-1	.41	-.05	.26	.41	.50	.78	.83
0	1.00	.91	.83	.71	.47	.33	-.02
1	.21	-.13	.16	.26	-.03	.15	.01
2	.18	-.02	.07	.10	-.08	.07	-.01
3	.23	.10	.04	.06	.01	.02	.03
4	-.23	.08	-.06	.03	.02	.00	-.03
5	-.10	-.07	.01	.03	.01	.02	.03

Table 5. T-Table. Orders of TF models with expectations in Case c

	1	2	3	4	5	6	7
-12	-.25	.06	-.02	.00	.00	.00	.00
-11	-.11	.21	-.05	.05	-.02	.01	-.01
-10	.79	.61	.50	.43	.33	.26	.20
-9	-.12	-.12	.19	.40	.12	.03	.22
-8	.17	.06	.17	.32	.01	-.10	.22
-7	.27	.21	.04	.25	.26	.27	.15
-6	-.79	.50	-.29	.15	.12	-.34	.53
-5	.49	.08	.18	.23	.26	.20	.69
-4	-.20	-.16	-.04	.06	.15	.39	.38
-3	.41	.21	.07	.04	-.01	.47	.66
-2	.23	-.12	.11	.04	-.14	.58	.15
-1	.41	-.06	.27	.40	.49	.77	.83
0	1.00	.92	.83	.70	.46	.31	-.04
1	.21	-.14	.17	.27	-.01	.15	.02
2	.18	-.02	.08	.11	-.09	.07	-.02
3	.25	.10	.05	.07	.01	.02	.03
4	-.24	.08	-.06	.04	.02	.00	-.03
5	-.07	-.08	.01	.04	.02	. 02	.02

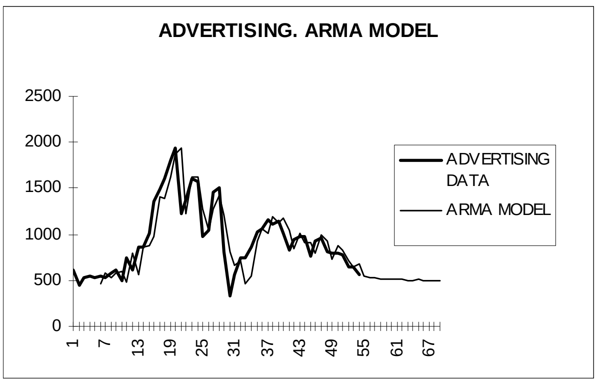

Figure 3. Advertising. ARMA model

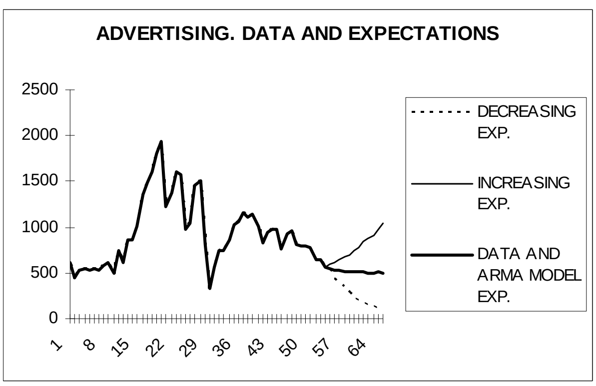

Figure 4. Advertising. Data and expectations

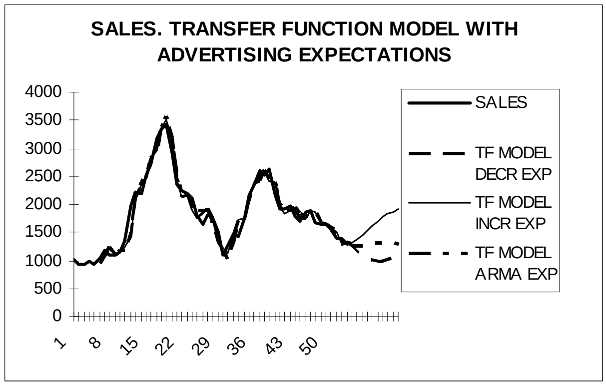

Figure 5. Sales. Transfer function model with advertising expectations

to the extent that a knowledge can be had of the future influence of said forecasts on the relationship under consideration, it is possible to evaluate its sensitivity to alternative dynamic specifications and, as a consequence, determine the optimum model.

5. Conclusions

In this paper we show the usefulness of the Padé-Laurent Approximation to the study of the deterministic part in the dynamic relationship between various variables, since it allows for the introduction of expectations or expected future values for certain variables. We illustrate the technique described by modelling the dynamic relationship between advertising and sales for the available data in the references consulted for the Lydia E. Pinkham Company. Also included is a sensitivity analysis of the estimates as a function of the expected future values or expectations of the decision maker for the advertising variable.

It would be interesting to combine this approach with the use of hybrid models that include a proper combination of linear and/or non-linear models of the type studied in [23].

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