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An Algorithm for Traffic Equilibrium Flow with Capacity Constraints of Arcs

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Abstract

In the traffic equilibrium problem, we introduce capacity constraints of arcs, extend Beckmann's formula to include these constraints, and give an algorithm for traffic equilibrium flows with capacity constraints on arcs. Using an example, we illustrate the application of the algorithm and show that Beckmann's formula is a sufficient condition only, not a necessary condition, for traffic equilibrium with capacity constraints of arcs.

Keywords

The Traffic Equilibrium Problem with Capacity Constraints of Arcs, Equilibrium Flow, Algorithm, Capacity of Arc, Saturated Path

1. Introduction

In [1], Wardrop introduced the traffic equilibrium problem and proposed a scalar equilibrium principle. In [2], Beckmann *et al.* gave a mathematical programming problem that was equivalent to Wardrop's traffic equilibrium problem. Using Beckmann's work, it is possible to find the traffic equilibrium flow if the cost function is a scalar. In [3], Chen and Yen generalized Wardrop's equilibrium principle to a (weak) vector equilibrium principle. In [4] [5], we extended the vector equilibrium principle to capacity constraints along arcs and derived existence and stability results for (weak) vector equilibrium flows. In this paper, we introduce the traffic equilibrium problem with capacity constraints of arcs (TEPCCA), extend Beckmann's transformation to cover capacity constraints along arcs, and give an algorithm for traffic equilibrium flows with capacity constraints of arcs for scalar cost functions. As an example, we illustrate the algorithm and show that Beckmann's transformation is a sufficient condition only, not a necessary condition, for traffic equilibria with capacity constraints of arcs. For other results with respect to traffic equilibrium with capacity constraints of arcs, we refer to [6], and for other results with respect to algorithms of equilibrium flows, we refer to [7]-[9] and the references therein.

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For a traffic network, let V denote the set of nodes, E the set of directed arcs, and W the set of origin-destination O-D pairs. For each $\omega \in W$, let P_ω denote the set of available paths joining O-D pair ω and denote by $K = \bigcup_{\omega\in W} P_\omega$, $m = |K|$. Let $D = (d_\omega)_{\omega\in W}$ denote the demand vector, with $d_\omega (>0)$ denoting the traffic demand on O-D pair ω. For each $a \in E$, the arc flow $f_a \in R_+ = \{z \in R : z \ge 0\}$. For each $\omega \in W$ and $k \in P_\omega$, let $f_k (\ge 0)$ denote the traffic flow on path k. $f = (f_k)^{\mathrm{T}}_{k\in K} \in R^m_+ = \{(z_1, z_2, \cdots, z_m) \in R^m : z_i \ge 0, i = 1, 2, \cdots, m\}$ is said to be a path flow (flow). Clearly, for $a \in E$, $f_a = \sum_{\omega\in W}\sum_{k\in P_\omega} \delta_{ak} f_k$, where $\delta_{ak} = 1$ if arc a belongs to path k, otherwise $\delta_{ak} = 0$, thus $f_a = f_a(f)$. Let $C = (c_a)_{a\in E}$ denote the capacity vector, where $c_a (>0)$ denotes the capacity of flow on arc a. A traffic network is usually denoted by $\aleph = \{V, E, W, D, C\}$. For each arc $a \in E$, the flow on arc a needs to satisfy the capacity constraint: $c_a \ge f_a \ge 0$, and for each $\omega \in W$, the flow f needs to satisfy the demand constraint: $\sum_{k\in P_\omega} f_k = d_\omega$. A flow f satisfying the demand and capacity constraints is called a feasible path flow (a feasible flow for short). Let

$$A = \left\{ f \in R^m_+ : \forall \omega \in W, \sum_{k\in P_\omega} f_k = d_\omega \text{ and } \forall a \in E, c_a \ge f_a \ge 0 \right\}.$$

In this paper, we assume that for each $\omega \in W$, the demand d_ω is fixed and $A \ne \varnothing$. It is easy to verify that A is convex and compact. For each $a \in E$, let $t_a = t_a(f_a) = t_a(f) \in R_+$ be the cost on arc a, and for each $\omega \in W, k \in P_w$, the cost t_k along path k is assumed to be the sum of all arc costs along k, *i.e.*, $t_k(f) = \sum_{a\in E} \delta_{ak} t_a(f)$.

2. Preliminaries

For the following definitions, see [4] [5].

Definition 2.1. Assume that a flow $f \in A$.

1) for $a \in E$, if $f_a = c_a$, then a is said to be a saturated arc of flow f, otherwise a nonsaturated arc of flow f.

2) for $k \in \bigcup_{\omega\in W} P_\omega$, if there exists a saturated arc a of flow f such that a belongs to path k, then k is said to be a saturated path of flow f, otherwise a nonsaturated path of flow f.

Definition 2.2. (Equilibrium principle with capacity constraints of arcs). A flow $f \in A$ is said to be in equilibrium if,

$$\forall \omega \in W, \forall k, j \in P_\omega, t_k(f) - t_j(f) > 0$$

$$\Rightarrow f_k = 0 \text{ or path } j \text{ is a saturated path of flow } f.$$

f is said to be an equilibrium flow or solution of the TEPCCA. A TEPCCA is usually denoted by $\Gamma = \{\aleph, A, t\}$.

3. A Generalization of Beckmann's Formula

For the TEPCCA $\Gamma = \{\aleph, A, t\}$, construct the following mathematical programming problem Q:

$$\operatorname{Min} z(f) = \sum_{a\in E} \int_0^{f_a} t_a(x)\,\mathrm{d}x$$

$$s.t. \begin{cases} \sum_k f_k = d_\omega, & \forall \omega \in W, k \in P_\omega \\ f_a = \sum_\omega \sum_k f_k \delta_{ak} \le c_a, & \forall a \in E, \omega \in W, k \in P_\omega \\ f_k \ge 0, & \forall \omega \in W, k \in P_\omega. \end{cases}$$

The above formula is a generalization of Beckmann's formula. The next theorem shows that each solution of the generalization of Beckmann's formula is an equilibrium flow for Γ.

Theorem 3.1. Consider the TEPCCA. Assume that for each $a \in E$, $t_a(f)$ is continuous on R^m_+, then the flow $f \in A$ is in equilibrium if f solves the mathematical programming problem Q.

Proof. Set $h_\omega = \sum_k f_k - d_\omega$ and $g_a = c_a - f_a$. The Kuhn-Tucker conditions for the problem Q are:

$$\begin{cases} \frac{\partial z[f]}{\partial f_k} - \sum_{\omega} \rho_{\omega} \frac{\partial h_{\omega}}{\partial f_k} - \sum_{a} \lambda_a \frac{\partial g_a}{\partial f_k} - \beta_k = 0, & \forall \omega \in W, k \in P_{\omega} \\ \lambda_a (c_a - f_a) = 0, & \forall a \in E \\ \beta_k f_k \geq 0, & \forall \omega \in W, k \in P_{\omega} \\ \rho_{\omega} \geq 0, \lambda_a \geq 0, \beta_k \geq 0, & \forall \omega \in W, a \in E, k \in P_{\omega} \end{cases}$$

where ρ_{ω}, λ_a and β_k are Lagrange multipliers. Since for each $a \in E$, $t_a(f)$ is continuous on R_+^m, we have

$$\frac{\partial z[f]}{\partial f_k} = \frac{\partial}{\partial f_k}\left(\sum_a \int_0^{f_a} t_a(x)\mathrm{d}x\right) = \sum_a \frac{\partial}{\partial f_a}\int_0^{f_a} t_a(x)\mathrm{d}x \cdot \frac{\partial f_a}{\partial f_k} = \sum_a t_a(f)\delta_{ak} = t_k, \sum_{\omega} \rho_{\omega} \frac{\partial h_{\omega}}{\partial f_k} = \rho_{\omega}.$$

When path k is a nonsaturated path of flow f, for each $a \in k$, we have $c_a - f_a > 0$. Note that $\lambda_a(c_a - f_a) = 0$, we have $\lambda_a = 0$. Thus,

$$\sum_a \lambda_a \frac{\partial g_a}{\partial f_k} = \sum_{a \in k} \lambda_a \frac{\partial g_a}{\partial f_k} = -\sum_{a \in k} \lambda_a \begin{cases} = 0 & \text{if path } k \text{ is a nonsaturated path of flow } f \\ \leq 0 & \text{otherwise.} \end{cases}$$

Hence, when k is a nonsaturated path, we have $f_k(t_k - \rho_{\omega}) = 0$, *i.e.*,

$$\begin{aligned} &\text{if } f_k > 0, \quad t_k = \rho_{\omega} \quad \forall \omega \in W, k \in P_{\omega} \\ &\text{if } f_k = 0, \quad t_k \geq \rho_{\omega} \quad \forall \omega \in W, k \in P_{\omega} \end{aligned}$$

and when k is a saturated path, we have $f_k\left(t_k - \rho_{\omega} + \sum_{a \in k} \lambda_a\right) = 0$, *i.e.*,

$$\begin{aligned} &\text{if } f_k > 0, \quad t_k(\geq 0) = \rho_{\omega} - \sum_{a \in k} \lambda_a \leq \rho_{\omega} \quad \forall \omega \in W, k \in P_{\omega} \\ &\text{if } f_k = 0, \quad t_k \geq 0 \quad \forall \omega \in W, k \in P_{\omega} \end{aligned}$$

In other words, if paths k is a nonsaturated path, then $t_k \geq \rho_{\omega}$, and if paths k such that $f_k > 0$, then $t_k \leq \rho_{\omega}$. Thus, for $\forall \omega \in W, \forall k, j \in P_{\omega}, t_k(f) - t_j(f) > 0$ and j is a nonsaturated path, then $f_k = 0$, otherwise $f_k > 0$, which implies that $t_k(f) \leq \rho_{\omega} \leq t_j(f)$, a contradiction. By Definition 2.2, the proof is finished.

From the generalization of Beckmann's formula, it is easy to construct an algorithm to calculate the equilibrium flow for the TEPCCA $\Gamma = \{\aleph, A, t\}$.

4. An Algorithm for the Traffic Equilibrium Flow with Capacity Constraints of Arcs

For the TEPCCA $\Gamma = \{\aleph, A, t\}$, because there are usually many paths in $K = \bigcup_{\omega \in W} P_{\omega}$, implying that there are many variable in the generalization of Beckmann's formula, it is often difficult to compute its solution. Note that there are many paths for which the flow is zero in an equilibrium flow. If we delete these from K, it does not cause any change in the equilibrium flow. For this season, we construct the following algorithm to compute the equilibrium flow with capacity constraints of arcs. Assume that for each $a \in E$, $t_a(f)$ is continuous on R_+^m.

Step 1. Find a feasible flow $f^0 \in A$ and denote by $H^0 = \{l \in K : f_l^0 > 0\}$. Let $i = 0$.

Step 2. Solve the restricted problem $\bar{Q}^i$:

$$\operatorname{Min} z(f) = \sum_{a \in E} \int_0^{f_a} t_a(x)\mathrm{d}x$$

$$s.t. \begin{cases} \sum_k f_k = d_{\omega}, & \forall \omega \in W, k \in H^i \\ f_a = \sum_{\omega} \sum_k f_k \delta_{ak} \leq c_a, & \forall a \in E \\ f_k \geq 0, & \forall k \in H^i. \end{cases}$$

We obtain solution $f^{i+1} \in A$. For each O-D pair $\omega \in W$, denote by $\tau_{\omega}^{i+1} = \max_{l \in P_{\omega}} \left\{t_l\left(f^{i+1}\right) : f_l^{i+1} > 0\right\}$, where $t_l\left(f^{i+1}\right)$ denotes the cost of path l when flow is f^{i+1} on the network $\aleph$.

Step 3. After deleting all saturated arcs of the flow f^{i+1} in the network $\aleph$, we compute its shortest path for each O-D pair. For each O-D pair $\omega \in W$, let $S_\omega^{i+1} = \{ l \in P_\omega : l$ is a shortest path for ω and $t_l(f^{i+1}) < \tau_\omega^{i+1} \}$.

Step 4. If $S^{i+1} = \bigcup_{\omega \in W} S_\omega^{i+1} = \varnothing$, go to Step 5; otherwise let $H^{i+1} = H^i \cup S^{i+1}, i = i+1$ and go to Step 2.

Step 5. The equilibrium flow is f^{i+1} for the TEPCCA and stop.

The following example shows the calculation process of the algorithm.

Example 4.1. Consider the TEPCCA (see **Figure 1**), where

$$V = \{1,2,3,4,5,6\},\quad E = \{e_1,e_2,e_3,e_4,e_5,e_6,e_7,e_8,e_9,e_{10},e_{11}\},\quad C = (6,7,10,8,7,5,9,7,9,11,7),$$

$$W = \{\omega 1, \omega 2\} = \{(1,4),(3,6)\},\quad D = (d_{\omega 1}, d_{\omega 2}) = (9,8),$$

and

$$t_{e_1}(f_{e_1}) = 4f_{e_1}^2 + 17, t_{e_2}(f_{e_2}) = 3f_{e_2}^2 + 18, t_{e_3}(f_{e_3}) = 30f_{e_3}^2 + 120, t_{e_4}(f_{e_4}) = 2f_{e_4}^2 + 84,$$
$$t_{e_5}(f_{e_5}) = f_{e_5}^2 + 112, t_{e_6}(f_{e_6}) = 2f_{e_6} + 18, t_{e_7}(f_{e_7}) = 8f_{e_7}^2 + 62, t_{e_8}(f_{e_8}) = 6f_{e_8}^2 + 65,$$
$$t_{e_9}(f_{e_9}) = f_{e_9}^2 + 18, t_{e_{10}}(f_{e_{10}}) = f_{e_{10}}^2 + 15, t_{e_{11}}(f_{e_{11}}) = 2f_{e_{11}}^2 + 10.$$

For O-D pair $\omega 1 = (1,4)$: $P_{\omega 1}$ contains paths $l_1 = (e_3)$, $l_2 = (e_4 e_{10})$, $l_3 = (e_1 e_5)$, and $l_4 = (e_1 e_6 e_{10})$, and for O-D pair $\omega 2 = (3,6)$: $P_{\omega 2}$ contains paths $l_5 = (e_9)$, $l_6 = (e_7 e_{11})$, $l_7 = (e_2 e_8)$ and $l_8 = (e_2 e_6 e_{11})$.

Let $f = (f_1, f_2, f_3, f_4, f_5, f_6, f_7, f_8)^{\mathrm{T}} \in R_+^8$, where f_j denotes the flow on path $l_j\ (j = 1,2,3,4,5,6,7,8)$. Thus, we have

$$f_{e_1} = f_3 + f_4, f_{e_2} = f_7 + f_8, f_{e_3} = f_1, f_{e_4} = f_2, f_{e_5} = f_3, f_{e_6} = f_4 + f_8,$$

$$f_{e_7} = f_6, f_{e_8} = f_7, f_{e_9} = f_5, f_{e_{10}} = f_2 + f_4, f_{e_{11}} = f_6 + f_8.$$

Next, we compute the equilibrium flow with capacity constraints of arcs.

1) It is easy to verify that

$$f^0 = (f_1, f_2, f_3, f_4, f_5, f_6, f_7, f_8)^{\mathrm{T}} = (9,0,0,0,8,0,0,0)^{\mathrm{T}} \in A.$$

$H^0 = \{ l \in K : f_l^0 > 0\} = \{l_1, l_5\}$. Let $i = 0$.

2) Solve the restricted problem $\bar{Q}^0$:

$$\operatorname{Min} z(f) = \sum_{a \in E} \int_0^{f_a} t_a(x)\,\mathrm{d}x = 10 f_1^3 + 120 f_1 + \frac{1}{3} f_5^3 + 18 f_5$$

$$s.t. \begin{cases} f_1 = 9 \\ f_5 = 8 \end{cases}$$

We obtain solution $f^1 = f^0 \in A$. For O-D pair $\omega 1 = (1,4)$, $\tau_{\omega 1}^1 = 2550$, and for O-D pair $\omega 2 = (3,6)$, $\tau_{\omega 2}^1 = 82$.

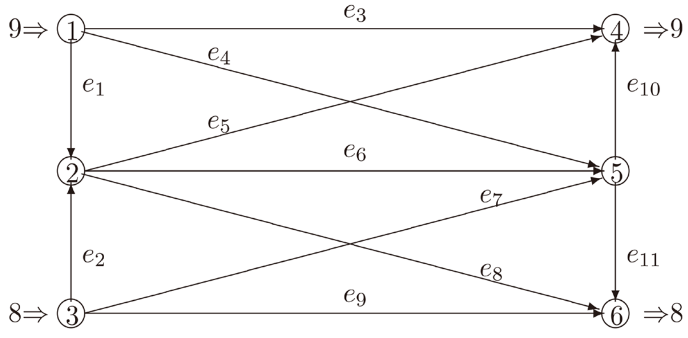

Figure 1. A traffic network.

3) There is no saturated arc of flow f^1 in the network $\aleph$. For O-D pair $\omega 1=(1,4)$, it is easy to verify that the shortest path is l_4, whereas for O-D pair $\omega 2=(3,6)$, the shortest path is l_8. Note that

$$t_{l_4}\left(f^1\right)=50<\tau_{\omega 1}^1=2550, t_{l_8}\left(f^1\right)=46<\tau_{\omega 2}^1=82,$$

thus $S_{\omega 1}^1=\{l_4\}, S_{\omega 2}^1=\{l_8\}$.

4) Since $S^1=S_{\omega 1}^1 \cup S_{\omega 2}^1=\{l_4,l_8\}\neq\varnothing$, let $H^1=H^0\cup S^1=\{l_1,l_4,l_5,l_8\}$ and solve the restricted problem $\bar{Q}^1$:

$$\begin{aligned}\operatorname{Min} z(f)&=\sum\nolimits_{a\in E}\int_0^{f_a} t_a(x)\mathrm{d}x=10 f_1^3+120 f_1+\frac{4}{3} f_4^3+17 f_4+(f_4+f_8)^2+18(f_4+f_8)\\&\quad+\frac{1}{3} f_4^3+15 f_4+\frac{1}{3} f_5^3+18 f_5+f_8^3+18 f_8+\frac{2}{3} f_8^3+10 f_8\\&=10 f_1^3+120 f_1+\frac{5}{3} f_4^3+50 f_4+(f_4+f_8)^2+\frac{1}{3} f_5^3+18 f_5+\frac{5}{3} f_8^3+46 f_8\end{aligned}$$

$$s.t.\begin{cases}f_1+f_4=9\\f_5+f_8=8\\f_4+f_8\le 5\\10\ge f_1\ge 0,6\ge f_4\ge 0,9\ge f_5\ge 0,7\ge f_8\ge 0\end{cases}$$

We obtain solution $f^2=(4,0,0,5,8,0,0,0)^{\mathrm{T}}\in A$. For O-D pair $\omega 1=(1,4)$, $\tau_{\omega 1}^2=600$, and for O-D pair $\omega 2=(3,6)$, $\tau_{\omega 2}^2=82$.

5) After deleting saturated arc e_6 of flow f^2 in the network $\aleph$. For O-D pair $\omega 1=(1,4)$, it is easy to verify that the shortest path is l_2, whereas for O-D pair $\omega 2=(3,6)$, the shortest path is l_5. Note that

$$t_{l_2}\left(f^2\right)=124<\tau_{\omega 1}^2=600, t_{l_5}\left(f^2\right)=82=\tau_{\omega 2}^2=82,$$

thus $S_{\omega 1}^2=\{l_2\}, S_{\omega 2}^2=\varnothing$.

6) Since $S^2=S_{\omega 1}^2\cup S_{\omega 1}^2=\{l_2\}\neq\varnothing$, let $H^2=H^1\cup S^2=\{l_1,l_2,l_4,l_5,l_8\}$ and solve the restricted problem $\bar{Q}^2$:

$$\begin{aligned}\operatorname{Min} z(f)&=\sum\nolimits_{a\in E}\int_0^{f_a} t_a(x)\mathrm{d}x=10 f_1^3+120 f_1+\frac{4}{3} f_4^3+17 f_4+(f_4+f_8)^2+18(f_4+f_8)\\&\quad+\frac{1}{3}(f_2+f_4)^3+15(f_2+f_4)+\frac{2}{3} f_2^3+84 f_2+\frac{1}{3} f_5^3+18 f_5+f_8^3+18 f_8+\frac{2}{3} f_8^3+10 f_8\\&=10 f_1^3+120 f_1+\frac{4}{3} f_4^3+50 f_4+(f_4+f_8)^2+\frac{1}{3}(f_2+f_4)^3\\&\quad+\frac{2}{3} f_2^3+99 f_2+\frac{1}{3} f_5^3+18 f_5+\frac{5}{3} f_8^3+46 f_8\end{aligned}$$

$$s.t.\begin{cases}f_1+f_2+f_4=9\\f_5+f_8=8\\f_2+f_4\le 11\\f_4+f_8\le 5\\10\ge f_1\ge 0,8\ge f_2\ge 0,6\ge f_4\ge 0,9\ge f_5\ge 0,7\ge f_8\ge 0\end{cases}$$

We obtain solution $f^3=(1.44,3.64,0,3.92,6.92,0,0,1.08)^{\mathrm{T}}\in A$. For O-D pair $\omega 1=(1,4)$, $\tau_{\omega 1}^3=182.20$, and for O-D pair $\omega 2=(3,6)$, $\tau_{\omega 2}^3=65.89$.

7) After deleting saturated arc e_6 of flow f^3 in the network $\aleph$. For O-D pair $\omega 1=(1,4)$, it is easy to verify that the shortest path is l_2, whereas for O-D pair $\omega 2=(3,6)$, the shortest path is l_5. Note that $t_{l_2}\left(f^3\right)=182.20=\tau_{\omega 1}^3=182.20, t_{l_5}\left(f^3\right)=65.89=\tau_{\omega 2}^3=65.89$, thus $S_{\omega 1}^3=\varnothing, S_{\omega 2}^3=\varnothing$.

8) Because $S^3 = S^3_{\omega 1} \cup S^3_{\omega 2} = \varnothing$, the equilibrium flow is $f^3 = (1.44, 3.64, 0, 3.92, 6.92, 0, 0, 1.08)^{\mathrm{T}}$, hence stop.

Note that

$$\int_0^{f_{e_1}} t_{e_1}(x)\mathrm{d}x = \int_0^{f_{e_1}} (4x^2+17)\mathrm{d}x = \frac{4}{3} f_{e_1}^3 + 17 f_{e_1} = \frac{4}{3}(f_3+f_4)^3 + 17(f_3+f_4),$$

$$\int_0^{f_{e_2}} t_{e_2}(x)\mathrm{d}x = \int_0^{f_{e_2}} (3x^2+18)\mathrm{d}x = f_{e_2}^3 + 18 f_{e_2} = (f_7+f_8)^3 + 18(f_7+f_8),$$

$$\int_0^{f_{e_3}} t_{e_3}(x)\mathrm{d}x = \int_0^{f_{e_3}} (30x^2+120)\mathrm{d}x = 10 f_{e_3}^3 + 120 f_{e_3} = 10 f_1^3 + 120 f_1,$$

$$\int_0^{f_{e_4}} t_{e_4}(x)\mathrm{d}x = \int_0^{f_{e_4}} (2x^2+84)\mathrm{d}x = \frac{2}{3} f_{e_4}^3 + 84 f_{e_4} = \frac{2}{3} f_2^3 + 84 f_2,$$

$$\int_0^{f_{e_5}} t_{e_5}(x)\mathrm{d}x = \int_0^{f_{e_5}} (x^2+112)\mathrm{d}x = \frac{1}{3} f_{e_5}^3 + 112 f_{e_5} = \frac{1}{3} f_3^3 + 112 f_3,$$

$$\int_0^{f_{e_6}} t_{e_6}(x)\mathrm{d}x = \int_0^{f_{e_6}} (2x+18)\mathrm{d}x = f_{e_6}^2 + 18 f_{e_6} = (f_4+f_8)^2 + 18(f_4+f_8),$$

$$\int_0^{f_{e_7}} t_{e_7}(x)\mathrm{d}x = \int_0^{f_{e_7}} (8x^2+62)\mathrm{d}x = \frac{8}{3} f_{e_7}^3 + 62 f_{e_7} = \frac{8}{3} f_6^3 + 62 f_6,$$

$$\int_0^{f_{e_8}} t_{e_8}(x)\mathrm{d}x = \int_0^{f_{e_8}} (6x^2+65)\mathrm{d}x = 2 f_{e_8}^3 + 65 f_{e_8} = 2 f_7^3 + 65 f_7,$$

$$\int_0^{f_{e_9}} t_{e_9}(x)\mathrm{d}x = \int_0^{f_{e_9}} (x^2+18)\mathrm{d}x = \frac{1}{3} f_{e_9}^3 + 18 f_{e_9} = \frac{1}{3} f_5^3 + 18 f_5,$$

$$\int_0^{f_{e_{10}}} t_{e_{10}}(x)\mathrm{d}x = \int_0^{f_{e_{10}}} (x^2+15)\mathrm{d}x = \frac{1}{3} f_{e_{10}}^3 + 15 f_{e_{10}} = \frac{1}{3}(f_2+f_4)^3 + 15(f_2+f_4),$$

$$\int_0^{f_{e_{11}}} t_{e_{11}}(x)\mathrm{d}x = \int_0^{f_{e_{11}}} (2x^2+10)\mathrm{d}x = \frac{2}{3} f_{e_{11}}^3 + 10 f_{e_{11}} = \frac{2}{3}(f_6+f_8)^3 + 10(f_6+f_8).$$

Thus the generalization of Beckmann's formula Q is:

$$\begin{aligned}\mathrm{Min}\, z(f) = {} & \frac{4}{3}(f_3+f_4)^3 + 17(f_3+f_4) + (f_7+f_8)^3 + 18(f_7+f_8) + 10 f_1^3 + 120 f_1 \\ & + \frac{2}{3} f_2^3 + 84 f_2 + \frac{1}{3} f_3^3 + 112 f_3 + (f_4+f_8)^2 + 18(f_4+f_8) + \frac{8}{3} f_6^3 + 62 f_6 \\ & + 2 f_7^3 + 65 f_7 + \frac{1}{3} f_5^3 + 18 f_5 + \frac{1}{3}(f_2+f_4)^3 + 15(f_2+f_4) + \frac{2}{3}(f_6+f_8)^3 + 10(f_6+f_8)\end{aligned}$$

$$s.t. \begin{cases} f_1+f_2+f_3+f_4 = 9 \\ f_5+f_6+f_7+f_8 = 8 \\ f_2+f_4 \le 11 \\ f_3+f_4 \le 6 \\ f_6+f_8 \le 7 \\ f_7+f_8 \le 7 \\ f_4+f_8 \le 5 \\ 10 \ge f_1 \ge 0, 8 \ge f_2 \ge 0, 6 \ge f_3 \ge 0, 5 \ge f_4 \ge 0, \\ 9 \ge f_5 \ge 0, 7 \ge f_6 \ge 0, 7 \ge f_7 \ge 0, 5 \ge f_8 \ge 0. \end{cases}$$

It is easy to verify that $f = (1.44, 3.64, 0, 3.92, 6.92, 0, 0, 1.08)^{\mathrm{T}}$ is the solution of the generalization of Beckmann's formula Q $(\mathrm{Min}\, z(f) = 1327.31)$. Clearly, f is an equilibrium flow for the TEPCCA.

Note that $g = (1.46, 3.74, 0, 3.80, 6.80, 0, 0, 1.20)^{\mathrm{T}}$ is also an equilibrium flow for the TEPCCA, but it is not a solution of the generalization of Beckmann's formula Q, *i.e.*, Theorem 3.1 is a sufficient condition only, not a necessary condition.

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