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Weak Insertion of a Continuous Function between Two Comparable α-Continuous (C-Continuous) Functions*

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Abstract

A sufficient condition in terms of lower cut sets is given for the insertion of a continuous function between two comparable real-valued functions.

Keywords

Weak Insertion, Strong Binary Relation, *C*-Open Set, Semi-Preopen Set, *α*-Open Set, Lower Cut Set

Subject Areas: **Topology**

1. Introduction

The concept of a C-open set in a topological space was introduced by E. Hatir, T. Noiri and S. Yksel in 1996 [1]. The authors define a set S to be a C-open set if $S = U \cap A$, where U is open and A is semi-preclosed. A set S is a C-closed set if its complement is C-open set or equivalently if $S = U \cup A$, where U is closed and A is semi-preopen. The authors show that a subset of a topological space is open if and only if it is an α-open set and a C-open set. This enable them to provide the following decomposition of continuity: a function is continuous if and only if it is α-continuous and C-continuous.

Recall that a subset A of a topological space (X,τ) is called α-open if A is the difference of an open and a nowhere dense subset of X. A set A is called α-closed if its complement is α-open or equivalently if A is union of a closed and a nowhere dense set. Sets which are dense in some regular closed subspace are called semi-preopen or β-open. A set is semi-preclosed or β-closed if its complement is semi-preopen or β-open.

The concept of a set A was β-open if and only if $A \subseteq Cl(Int(Cl(A)))$ was introduced by J. Dontchev in 1998 [2].

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Recall that a real-valued function *f* defined on a topological space *X* was called *A*-continuous if the preimage of every open subset of $\mathbb{R}$ belongs to *A*, where *A* was a collection of subset of *X* and this the concept was introduced by M. Przemski in 1993 [3]. Most of the definitions of function used throughout this paper are consequences of the definition of *A*-continuity. However, for unknown concepts, the reader might refer to papers introduced by J. Dontchev in 1995 [4], M. Ganster and I. Reilly in 1990 [5].

Hence, a real-valued function *f* defined on a topological space *X* is called *C*-continuous (resp. *α*-continuous) if the preimage of every open subset of $\mathbb{R}$ is *C*-open (resp. *α*-open) subset of *X*.

Results of Katĕtov in 1951 [6] and in 1953 [7] concerning binary relations and the concept of an indefinite lower cut set for a real-valued function, which was due to Brooks in 1971 [8], were used in order to give necessary and sufficient conditions for the strong insertion of a continuous function between two comparable real-valued functions.

If *g* and *f* are real-valued functions defined on a space *X*, we write $g \le f$ in case $g(x) \le f(x)$ for all *x* in *X*.

The following definitions were modifications of conditions considered in paper introduced by E. Lane in 1976 [9].

A property *P* defined relative to a real-valued function on a topological space is a *c-property* provided that any constant function has property *P* and provided that the sum of a function with property *P* and any continuous function also has property *P*. If P_1 and P_2 are *c*-property, the following terminology is used: A space *X* has the *weak c-insertion property* for (P_1, P_2) if and only if for any functions *g* and *f* on *X* such that $g \le f, g$ has property P_1 and *f* has property P_2, then there exists a continuous function *h* such that $g \le h \le f$.

In this paper, it is given a sufficient condition for the weak *c*-insertion property. Also several insertion theorems are obtained as corollaries of this result.

2. The Main Result

Before giving a sufficient condition for insertability of a continuous function, the necessary definitions and terminology are stated.

Let (X, τ) be a topological space, the family of all *α*-open, *α*-closed, *C*-open and *C*-closed will be denoted by $\alpha O(X, \tau)$, $\alpha C(X, \tau)$, $CO(X, \tau)$ and $CC(X, \tau)$, respectively.

Definition 2.1. Let *A* be a subset of a topological space (X, τ). Respectively, we define the *α-closure*, *α-interior*, *C-closure* and *C-interior* of a set *A*, denoted by $\alpha Cl(A), \alpha Int(A), CCl(A)$ and $CInt(A)$ as follows:

$$\alpha Cl(A) = \cap\{F : F \supseteq A, F \in \alpha C(X, \tau)\}$$

$$\alpha Int(A) = \cup\{O : O \subseteq A, O \in \alpha O(X, \tau)\}$$

$$CCl(A) = \cap\{F : F \supseteq A, F \in CC(X, \tau)\}$$

and

$$CInt(A) = \cup\{O : O \subseteq A, O \in CO(X, \tau)\}.$$

Respectively, we have $\alpha Cl(A), CCl(A)$ are *α*-closed, semi-preclosed and $\alpha Int(A), CInt(A)$ are *α*-open, semi-preopen.

The following first two definitions are modifications of conditions considered in [6] [7].

Definition 2.2. If *ρ* is a binary relation in a set *S* then $\bar{\rho}$ is defined as follows: $x \, \bar{\rho} \, y$ if and only if $y \rho v$ implies $x \rho v$ and $u \rho x$ implies $u \rho y$ for any *u* and *v* in *S*.

Definition 2.3. A binary relation *ρ* in the power set $P(X)$ of a topological space *X* is called a *strong binary relation* in $P(X)$ in case *ρ* satisfies each of the following conditions:

1) If $A_i \, \rho \, B_j$ for any $i \in \{1, \cdots, m\}$ and for any $j \in \{1, \cdots, n\}$, then there exists a set *C* in $P(X)$ such that $A_i \, \rho \, C$ and $C \, \rho \, B_j$ for any $i \in \{1, \cdots, m\}$ and any $j \in \{1, \cdots, n\}$.

2) If $A \subseteq B$, then $A \, \bar{\rho} \, B$.

3) If $A \, \rho \, B$, then $Cl(A) \subseteq B$ and $A \subseteq Int(B)$.

The concept of a lower indefinite cut set for a real-valued function was defined [8] as follows:

Definition 2.4. If *f* is a real-valued function defined on a space *X* and if

$\{x \in X : f(x) < \ell\} \subseteq A(f,\ell) \subseteq \{x \in X : f(x) \le \ell\}$ for a real number ℓ, then $A(f,\ell)$ is called a *lower indefinite cut set* in the domain of *f* at the level ℓ.

We now give the following main result:

Theorem 2.1. Let *g* and *f* be real-valued functions on a topological space *X* with $g \le f$. If there exists a strong binary relation *ρ* on the power set of *X* and if there exist lower indefinite cut sets $A(f,t)$ and $A(g,t)$ in the domain of *f* and *g* at the level *t* for each rational number *t* such that if $t_1 < t_2$ then $A(f,t_1)\,\rho\, A(g,t_2)$, then there exists a continuous function *h* defined on *X* such that $g \le h \le f$.

Proof. Let *g* and *f* be real-valued functions defined on *X* such that $g \le f$. By hypothesis there exists a strong binary relation *ρ* on the power set of *X* and there exist lower indefinite cut sets $A(f,t)$ and $A(g,t)$ in the domain of *f* and *g* at the level *t* for each rational number *t* such that if $t_1 < t_2$ then $A(f,t_1)\,\rho\, A(g,t_2)$.

Define functions *F* and *G* mapping the rational numbers $\mathbb{Q}$ into the power set of *X* by $F(t) = A(f,t)$ and $G(t) = A(g,t)$. If t_1 and t_2 are any elements of $\mathbb{Q}$ with $t_1 < t_2$, then $F(t_1)\,\bar{\rho}\, F(t_2), G(t_1)\,\bar{\rho}\, G(t_2)$, and $F(t_1)\,\rho\, G(t_2)$. By Lemmas 1 and 2 of [7] it follows that there exists a function *H* mapping $\mathbb{Q}$ into the power set of *X* such that if t_1 and t_2 are any rational numbers with $t_1 < t_2$, then $F(t_1)\,\rho\, H(t_2), H(t_1)\,\rho\, H(t_2)$ and $H(t_1)\,\rho\, G(t_2)$.

For any *x* in *X*, let $h(x) = \inf\{t \in \mathbb{Q} : x \in H(t)\}$.

We first verify that $g \le h \le f$: If *x* is in $H(t)$ then *x* is in $G(t')$ for any $t' > t$; since *x* is in $G(t') = A(g,t')$ implies that $g(x) \le t'$, it follows that $g(x) \le t$. Hence $g \le h$. If *x* is not in $H(t)$, then *x* is not in $F(t')$ for any $t' < t$; since *x* is not in $F(t') = A(f,t')$ implies that $f(x) > t'$, it follows that $f(x) \ge t$. Hence $h \le f$.

Also, for any rational numbers t_1 and t_2 with $t_1 < t_2$, we have $h^{-1}(t_1,t_2) = Int(H(t_2)) \setminus Cl(H(t_1))$. Hence $h^{-1}(t_1,t_2)$ is an open subset of *X*, *i.e.*, *h* is a continuous function on *X*. ■

The above proof used the technique of proof of Theorem 1 of [6].

3. Applications

The abbreviations αc and Cc are used for *α*-continuous and *C*-continuous, respectively.

Corollary 3.1. If for each pair of disjoint *α*-closed (resp. *C*-closed) sets F_1, F_2 of X, there exist open sets G_1 and G_2 of *X* such that $F_1 \subseteq G_1$, $F_2 \subseteq G_2$ and $G_1 \cap G_2 = \varnothing$ then *X* has the weak *c*-insertion property for $(\alpha c, \alpha c)$ (resp. (Cc, Cc)).

Proof. Let *g* and *f* be real-valued functions defined on the *X*, such that *f* and *g* are αc (resp. Cc), and $g \le f$. If a binary relation *ρ* is defined by $A\,\rho\, B$ in case $\alpha Cl(A) \subseteq \alpha Int(B)$ (resp. $CCl(A) \subseteq CInt(B)$), then by hypothesis *ρ* is a strong binary relation in the power set of *X*. If t_1 and t_2 are any elements of $\mathbb{Q}$ with $t_1 < t_2$, then

$$A(f,t_1) \subseteq \{x \in X : f(x) \le t_1\} \subseteq \{x \in X : g(x) < t_2\} \subseteq A(g,t_2);$$

since $\{x \in X : f(x) \le t_1\}$ is an *α*-closed (resp. *C*-closed) set and since $\{x \in X : g(x) < t_2\}$ is an *α*-open (resp. *C*-open) set, it follows that $\alpha Cl(A(f,t_1)) \subseteq \alpha Int(A(g,t_2))$ (resp. $CCl(A(f,t_1)) \subseteq CInt(A(g,t_2))$). Hence $t_1 < t_2$ implies that $A(f,t_1)\,\rho\, A(g,t_2)$. The proof follows from Theorem 2.1. ■

Corollary 3.2. If for each pair of disjoint *α*-closed (resp. *C*-closed) sets F_1, F_2, there exist open sets G_1 and G_2 such that $F_1 \subseteq G_1$, $F_2 \subseteq G_2$ and $G_1 \cap G_2 = \varnothing$ then every *α*-continuous (resp. *C*-continuous) function is continuous.

Proof. Let *f* be a real-valued *α*-continuous (resp. *C*-continuous) function defined on the *X*. Set $g = f$, then by Corollary 3.1, there exists a continuous function *h* such that $g = h = f$. ■

Corollary 3.3. If for each pair of disjoint subsets F_1, F_2 of X, such that F_1 is *α*-closed and F_2 is *C*-closed, there exist open subsets G_1 and G_2 of *X* such that $F_1 \subseteq G_1$, $F_2 \subseteq G_2$ and $G_1 \cap G_2 = \varnothing$ then *X* have the weak *c*-insertion property for $(\alpha c, Cc)$ and $(Cc, \alpha c)$.

Proof. Let *g* and *f* be real-valued functions defined on the *X*, such that *g* is *ac* (resp. Cc) and *f* is Cc (resp. *ac*), with $g \le f$. If a binary relation *ρ* is defined by $A\,\rho\, B$ in case $CCl(A) \subseteq \alpha Int(B)$ (resp. $\alpha Cl(A) \subseteq CInt(B)$), then by hypothesis *ρ* is a strong binary relation in the power set of *X*. If t_1 and t_2 are any elements of $\mathbb{Q}$ with $t_1 < t_2$, then

$$A(f,t_1) \subseteq \{x \in X : f(x) \le t_1\} \subseteq \{x \in X : g(x) < t_2\} \subseteq A(g,t_2);$$

since $\{x \in X : f(x) \le t_1\}$ is a *C*-closed (resp. *α*-closed) set and since $\{x \in X : g(x) < t_2\}$ is an *α*-open (resp.

C-open) set, it follows that $CCl(A(f,t_1)) \subseteq \alpha Int(A(g,t_2))$ (resp. $\alpha Cl(A(f,t_1)) \subseteq CInt(A(g,t_2))$). Hence $t_1 < t_2$ implies that $A(f,t_1)\,\rho\,A(g,t_2)$. The proof follows from Theorem 2.1. ■

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