

# Mean Cordial Labeling of Graphs

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Received August 12, 2012; revised September 4, 2012; accepted September 17, 2012

## ABSTRACT

Let  $f$  be a map from  $V(G)$  to  $\{0,1,2\}$ . For each edge  $uv$  assign the label  $\left\lceil \frac{f(u)+f(v)}{2} \right\rceil$ .  $f$  is called a mean cordial labeling if  $|v_f(i)-v_f(j)| \leq 1$  and  $|e_f(i)-e_f(j)| \leq 1$ ,  $i, j \in \{0,1,2\}$ , where  $v_f(x)$  and  $e_f(x)$  denote the number of vertices and edges respectively labelled with  $x$  ( $x = 0,1,2$ ). A graph with a mean cordial labeling is called a mean cordial graph. We investigate mean cordial labeling behavior of Paths, Cycles, Stars, Complete graphs, Combs and some more standard graphs.

**Keywords:** Path; Star; Complete Graph; Comb

## 1. Introduction

The graphs considered here are finite, undirected and simple. The vertex set and edge set of a graph  $G$  are denoted by  $V(G)$  and  $E(G)$  respectively. The cardinality of  $V(G)$  and  $E(G)$  are respectively called order and size of  $G$ . Labelled graphs are used in radar, circuit design, communication network, astronomy, cryptography etc. [1]. The concept of cordial labeling was introduced by Cahit in the year 1987 in [2]. Let  $f: V(G) \rightarrow \{0,1\}$  be a function. For each edge  $uv$  assign the label  $|f(u)-f(v)|$ .  $f$  is called a cordial labeling if  $|v_f(i)-v_f(j)| \leq 1$  and  $|e_f(i)-e_f(j)| \leq 1$ ,  $i, j \in \{0,1\}$  where  $v_f(x)$  and  $e_f(x)$  denote the number of vertices and edges respectively labelled with  $x$  ( $x = 0, 1$ ). A graph with admits a cordial labeling is called a cordial graph. Product cordial labeling was introduced by M. Sundaram, R. Ponraj and Somasundaram [3]. Here we introduce a new notion called mean cordial labeling. We investigate the mean cordial labeling behavior of some standard graphs. The symbol  $\lceil x \rceil$  stands for smallest integer greater than or equal to  $x$ . Terms not defined here are used in the sense of Harary [4].

## 2. Mean Cordial Labeling

**Definition 2.1.** Let  $f$  be a function from  $V(G)$  to  $\{0,1,2\}$ .

For each edge  $uv$  of  $G$ , assign the label  $\left\lceil \frac{f(u)+f(v)}{2} \right\rceil$ .

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$f$  is called a mean cordial labeling of  $G$  if

$$|v_f(i)-v_f(j)| \leq 1 \text{ and } |e_f(i)-e_f(j)| \leq 1,$$

$i, j \in \{0,1,2\}$  where  $v_f(x)$  and  $e_f(x)$  denote the number of vertices and edges labelled with  $x$  ( $x = 0,1,2$ ) respectively. A graph with a mean cordial labeling is called mean cordial graph.

**Remarks 2.2:** If we restrict the range set of  $f$  to  $\{0,1\}$ , the definition 2.1 coincides with that of product cordial labeling.

**Remarks 2.3:** If we try to extend the range set of  $f$  to  $\{0,1,\dots,k\}$  ( $k > 2$ ), the definition 2.1 shall not work out since  $v_f(0)$  becomes very small.

**Theorem 2.4:** Every graph is a sub graph of a connected mean cordial graph.

**Proof:** Let  $G$  be a given  $(p,q)$  graph. Take three copies of  $K_p$ . Let  $G_1, G_2, G_3$  respectively denote the first, second and third copies of  $K_p$ . Let  $u \in V(G_1), v \in V(G_2), w \in V(G_3)$ . Let  $G^*$  be the graph with

$$V(G^*) = V(G_1) \cup V(G_2) \cup V(G_3)$$

$$\text{and } E(G^*) = E(G_1) \cup E(G_2) \cup E(G_3) \cup \{uv, vw\}.$$

Clearly  $G^*$  is a super graph of  $G$ . Assign the label 0 to all the vertices of  $G_1$ , 1 to all the vertices of  $G_2$ , 2 to all the vertices of  $G_3$ . Then  $v_f(0) = v_f(1) = v_f(2) = p$  and

$$e_f(0) = \binom{p}{2}, \quad e_f(1) = \binom{p}{2} + 1, \quad e_f(2) = \binom{p}{2} + 1.$$

Therefore this labeling is a mean cordial labeling of  $G^*$ .

**Theorem 2.5:** Any Path  $P_n$  is a mean cordial.

**Proof:** Let  $P_n$  be the Path  $u_1u_2 \cdots u_n$ .

**Case (1):**  $n \equiv 0 \pmod{3}$

Let  $n = 3t$ . Define  $f(u_i) = 2, 1 \leq i \leq t$ ,

$f(u_{t+i}) = 1, 1 \leq i \leq t, f(u_{2t+i}) = 0, 1 \leq i \leq t$ . Then

$v_f(0) = v_f(1) = v_f(2) = t$  and

$e_f(0) = t-1, e_f(1) = e_f(2) = t$ . Hence  $f$  is a mean cordial labeling.

**Case (2):**  $n \equiv 1 \pmod{3}$

Let  $n = 3t+1$ . Assign labels to the vertices

$u_i (1 \leq i \leq n-1)$  as in case (1). Then assign the label 0 to the vertex  $u_n$ . Here.  $v_f(0) = v_f(1) = v_f(2) = t$  and  $e_f(0) = t, e_f(1) = e_f(2) = t$ . Hence  $f$  is a mean cordial labeling.

**Case (3):**  $n \equiv 2 \pmod{3}$

Let  $n = 3t+2$ . Assign labels to the vertices

$u_i (1 \leq i \leq n-1)$  as in case (2). Then assign the label 1 to the vertex  $u_n$ . Here  $v_f(0) = v_f(1) = t+1, v_f(2) = t, e_f(1) = t+1$ . and  $e_f(0) = e_f(2) = t$ . Hence  $f$  is a mean cordial labeling.

**Illustration 2.6:** Mean cordial labeling of  $P_6$  is shown in **Figure 1**.

**Theorem 2.7:** The Star  $K_{1,n}$  is a mean cordial iff  $n \leq 2$ .

**Proof:** Let  $V(K_{1,n}) = \{u, u_i : 1 \leq i \leq n\}$  and  $E(K_{1,n}) = \{uu_i : 1 \leq i \leq n\}$ .

For  $n \leq 2$ , the result follows from Theorem 2.5. Assume  $n > 2$ . If possible let there be a mean cordial labeling  $f$ .

**Case (1):**  $f(u) = 0$ .

Then  $f(u) + f(v) \leq 2$  for all edge  $uv$ . This forces  $e_f(2) = 0$ . a contradiction.

**Case (2):**  $f(u) = 2$ .

In this case  $e_f(0) = 0$ , again a contradiction.

**Case (3):**  $f(u) = 1$ .

Here also  $e_f(0) = 0$ , a contradiction.

Hence  $K_{1,n}$  is not a mean cordial for all  $n > 2$ .

**Theorem 2.8:** The cycle  $C_n$  is mean cordial iff  $n \equiv 1, 2 \pmod{3}$ .

**Proof:** Let  $C_n$  be the cycle  $u_1u_2 \cdots u_nu_1$

**Case (1):**  $n \equiv 0 \pmod{3}$

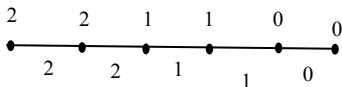
Let  $n = 3t$ . Then  $v_f(0) = v_f(1) = v_f(2) = t$ . In this case  $e_f(0) \leq t-1$ , a contradiction

**Case (2):**  $n \equiv 1 \pmod{3}$

Let  $n = 3t+1$ . Define  $f(u_i) = 0, 1 \leq i \leq t+1$

$f(u_{t+i}) = 1, 1 \leq i \leq t, f(u_{2t+i}) = 2, 1 \leq i \leq t$

Then  $v_f(0) = t+1, v_f(1) = v_f(2) = t$  and



**Figure 1.** Mean cordial labeling of  $P_6$ .

Then  $v_f(0) = t+1, v_f(1) = v_f(2) = t$  and  $e_f(1) = t+1, e_f(0) = e_f(2) = t$ . Hence  $f$  is a mean cordial labeling.

**Case (3):**  $n \equiv 2 \pmod{3}$

Let  $n = 3t+2$ . Define  $f(u_i) = 0, 1 \leq i \leq t+1$ ,

$f(u_{t+i}) = 1, 1 \leq i \leq t, f(u_{2t+i}) = 2, 1 \leq i \leq t+1$

Then  $v_f(1) = t, v_f(0) = v_f(2) = t+1$  and  $e_f(1) = t+1, e_f(0) = e_f(2) = t$ . Hence  $f$  is a mean cordial labeling.

**Theorem 2.9:** The Complete graph  $K_n$  is mean cordial iff  $n \leq 2$ .

**Proof:** Clearly  $K_1$  and  $K_2$  are mean cordial by Theorem 2.5. Assume  $n > 2$ . If possible let there be a mean cordial labeling  $f$ .

**Case (1):**  $n \equiv 0 \pmod{3}$

Let  $n = 3t, t \geq 1$ . Then  $v_f(0) = v_f(1) = v_f(2) = t$

$e_f(0) = \binom{t}{2}, e_f(1) = \binom{t}{2} + t^2 + t^2$ , and

$e_f(2) = \binom{t}{2} + t^2$ . Then  $e_f(1) - e_f(0) = 2t^2 > 1$ , a con-

tradiction.

**Case (2):**  $n \equiv 1 \pmod{3}$

Let  $n = 3t+1$

**Subcase (i):**  $v_f(0) = t+1, v_f(1) = v_f(2) = t$

$e_f(0) = \binom{t+1}{2}, e_f(1) = \binom{t}{2} + t(t+1) + t(t+1)$  and

$e_f(2) = \binom{t}{2} + t^2, e_f(1) - e_f(2) = t^2 + 2t > 1$ , a contra-

dition.

**Subcase (ii):**  $v_f(1) = t+1, v_f(0) = v_f(2) = t$

$e_f(0) = \binom{t}{2}, e_f(1) = \binom{t+1}{2} + t(t+1) + t^2$  and

$e_f(2) = \binom{t}{2} + t(t+1). e_f(2) - e_f(0) = t^2 + t > 1$ , a con-

tradiction.

**Subcase (iii):**  $v_f(2) = t+1, v_f(0) = v_f(1) = t$

$e_f(0) = \binom{t}{2}, e_f(1) = \binom{t}{2} + t(t+1) + t^2$  and

$e_f(2) = \binom{t+1}{2} + t(t+1)$ . Then

$e_f(1) - e_f(0) = 2t^2 + t > 1$ , a contradiction.

**Case (3):**  $n \equiv 2 \pmod{3}$

Let  $n = 3t+2$

**Subcase (i):**  $v_f(0) = t, v_f(1) = v_f(2) = t+1$

$e_f(0) = \binom{t}{2}, e_f(1) = \binom{t+1}{2} + t(t+1) + t(t+1)$  and

$e_f(2) = \binom{t+1}{2} + (t+1)^2$ . Then

$e_f(1) - e_f(0) = 2t^2 + 3t > 1$ , a contradiction

**Subcase (ii):**  $v_f(1) = t \quad v_f(0) = v_f(2) = t + 1$

$e_f(0) = \binom{t+1}{2}$ ,  $e_f(1) = \binom{t}{2} + t(t+1) + (t+1)^2$  and

$e_f(2) = \binom{t+1}{2} + t(t+1)$ . Then

$e_f(2) - e_f(0) = t^2 + t > 1$ , a contradiction.

**Subcase (iii):**  $v_f(2) = t \quad v_f(0) = v_f(1) = t + 1$

$e_f(0) = \binom{t+1}{2} \quad e_f(1) = \binom{t+1}{2} + (t+1)^2 + t(t+1)$  and

$e_f(2) = \binom{t}{2} + t(t+1)$

Then  $e_f(1) - e_f(0) = (t+1)^2 + t^2 + t > 1$ , a contradiction.

**Theorem 2.10:** The Wheel  $W_n$  is not a mean cordial graph for all  $n \geq 3$ .

**Proof:** Let  $W_n = C_n + K_1$  where  $C_n$  is the cycle  $u_1u_2 \dots u_nu_1$  and  $V\{K_1\} = u$ . If possible let there be a mean cordial labeling  $f$ .

**Case (1):**  $n \equiv 0 \pmod{3}$

Let  $n = 3t$  Then the size of the wheel is  $6t$ .

**Subcase (i):**  $f(u) = 0$ ,

Here  $e_f(0) \leq t + t - 1 \leq 2t - 1$ , a contradiction.

**Subcase (ii):**  $f(u) = 1$  or  $2$ .

In this case  $e_f(0) \leq t$ , again a contradiction.

**Case (2):**  $n \equiv 1 \pmod{3}$

Let  $n = 3t + 1$ . Then  $e_f(0) \leq 2t - 1$  or  $e_f(0) \leq t$  according as  $f(u) = 0$ , or  $f(u) \neq 0$ , this is a contradiction.

**Case (3):**  $n \equiv 2 \pmod{3}$

Similarly to case (1), we get a contradiction.

**Theorem 2.11:**  $S(K_{1,n})$  is mean cordial, where  $S(G)$  denotes subdivision of  $G$ .

**Proof:**

Let  $V(S(K_{1,n})) = \{u, u_i, v_i : 1 \leq i \leq n\}$  and

$E(S(K_{1,n})) = \{uu_i, u_i v_i : 1 \leq i \leq n\}$

**Case (i):**  $n \equiv 0 \pmod{3}$

Let  $n = 3t$ .

Define  $f(u) = 0, \quad f(u_i) = 0, 1 \leq i \leq t,$

$f(u_{t+i}) = 1, 1 \leq i \leq 2t, \quad f(v_i) = 0, 1 \leq i \leq t,$

$f(v_{t+i}) = 2, 1 \leq i \leq 2t,$  Then  $v_f(0) = 2t + 1,$

$v_f(1) = v_f(2) = 2t$  and  $e_f(0) = e_f(1) = 2t, \quad e_f(2) = 2t.$  Hence  $f$  is a mean cordial labeling.

**Case (2):**  $n \equiv 1 \pmod{3}$

Let  $n = 3t + 1$  Assign labels to the vertices  $u, u_i$  and  $v_i (1 \leq i \leq n - 1)$  as in case (1). Then assign the label 1 and 2 to the vertices  $u_n$  and  $v_n$  respectively. Here  $v_f(0) = v_f(1) = v_f(2) = 2t + 1$  and  $e_f(0) = 2t,$

$e_f(1) = e_f(2) = 2t + 1,$  Hence  $f$  is a mean cordial labeling.

**Case (3):**  $n \equiv 2 \pmod{3}$

Let  $n = 3t + 2$ . Assign labels to the vertices  $u, u_i$  and  $v_i (1 \leq i \leq n - 1)$  as in case (2). Then assign the label 0 and 2 to the vertices  $u_n, v_n$  respectively. Here  $v_f(0) = v_f(2) = 2t + 2, \quad v_f(1) = 2t + 1$  and  $e_f(1) = 2t + 2, \quad e_f(0) = e_f(2) = 2t + 1,$  Hence  $f$  is a mean cordial labeling.

**Illustration 2.12:** Mean cordial labeling of  $S(K_{1,6})$  is shown in **Figure 2**.

**Theorem 2.13:** The comb  $P_n \odot K_1$  is mean cordial where  $G_1 \odot G_2$  denotes the corona of  $G_1$  and  $G_2$ .

**Proof:** Let  $P_n$  be the Path  $u_1u_2 \dots u_n$ . Let

$V(P_n \odot K_1) = V(P_n) \cup \{v_i : 1 \leq i \leq n\},$

$E(P_n \odot K_1) = E(P_n) \cup \{u_i v_i : 1 \leq i \leq n\}$

**Case (1):**  $n \equiv 0 \pmod{3}$

Let  $n = 3t$ .

$f(u_i) = 2, 1 \leq i \leq t, \quad f(u_{t+i}) = 1, 1 \leq i \leq t$

$f(u_{2t+i}) = 0, 1 \leq i \leq t, \quad f(v_i) = 2, 1 \leq i \leq t$

$f(v_{t+i}) = 1, 1 \leq i \leq t, \quad f(v_{2t+i}) = 0, 1 \leq i \leq t$

Then  $v_f(0) = v_f(1) = v_f(2) = 2t$  and

$e_f(1) = e_f(2) = 2t, \quad e_f(0) = 2t - 1,$

Hence  $f$  is a mean cordial labeling.

**Case (2):**  $n \equiv 1 \pmod{3}$

Let  $n = 3t + 1$ . Assign labels to the vertices  $u_i$  and  $v_i (1 \leq i \leq n - 1)$  as in case (1). Then assign the label 0 and 1 to the vertices  $u_n$  and  $v_n$ . Here

$v_f(0) = v_f(1) = 2t + 1, \quad v_f(2) = 2t,$  and

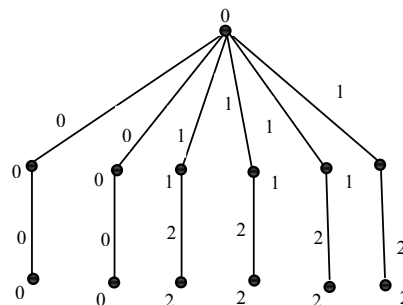
$e_f(0) = e_f(2) = 2t,$  and  $e_f(1) = 2t + 1$

Hence  $f$  is a mean cordial labeling.

**Case (3):**  $n \equiv 2 \pmod{3}$

Let  $n = 3t + 2$ . Assign labels to the vertices  $u_i$  &  $v_i (1 \leq i \leq n - 2)$  as in case (1). Then assign the label 0, 2 and 0, 1 to the vertices  $u_{n-1}, u_n$  and  $v_{n-1}, v_n$  respectively. Here  $v_f(0) = 2t + 2, \quad v_f(1) = v_f(2) = 2t + 1$  and  $e_f(0) = e_f(1) = e_f(2) = 2t + 1.$  Hence  $f$  is a mean cordial labeling.

**Theorem 2.14:**  $P_n \odot 2K_1$  is mean cordial.



**Figure 2.** Mean cordial labeling of  $S(K_{1,6})$ .

**Proof:** Let  $P_n$  be the Path  $u_1u_2 \cdots u_n$ . Let  $v_i$  and  $w_i$  be the pendant vertices which are adjacent to  $u_i, 1 \leq i \leq n$ .

**Case (1):**  $n$  is even.

$$\text{Define } f(u_i) = 0, 1 \leq i \leq \frac{n}{2}$$

$$f\left(u_{\frac{n}{2}+i}\right) = 1, 1 \leq i \leq \frac{n}{2}$$

$$f(v_i) = 0, 1 \leq i \leq \frac{n}{2}$$

$$f(w_i) = 1, 1 \leq i \leq \frac{n}{2}$$

$$f\left(v_{\frac{n}{2}+i}\right) = 2, 1 \leq i \leq \frac{n}{2}$$

$$f\left(w_{\frac{n}{2}+i}\right) = 2, 1 \leq i \leq \frac{n}{2}$$

Then  $v_f(0) = v_f(1) = v_f(2) = n$  and  $e_f(0) = n-1, e_f(1) = e_f(2) = n$ .

Hence  $f$  is a mean cordial labeling.

**Case (2):**  $n$  is odd.

$$\text{Define } f(u_i) = 0, 1 \leq i \leq \frac{n-1}{2}$$

$$f\left(u_{\frac{n-1}{2}+i}\right) = 1, 1 \leq i \leq \frac{n+1}{2}$$

$$f(v_i) = 0, 1 \leq i \leq \frac{n-3}{2}$$

$$f(w_i) = 1, 1 \leq i \leq \frac{n-3}{2}$$

$$f\left(v_{\frac{n-1}{2}}\right) = f\left(w_{\frac{n-1}{2}}\right) = 0$$

$$f\left(v_{\frac{n+1}{2}}\right) = 1, f\left(w_{\frac{n+1}{2}}\right) = 2$$

$$f\left(v_{\frac{n+1}{2}+i}\right) = f\left(w_{\frac{n+1}{2}+i}\right) = 2, 1 \leq i \leq \frac{n-1}{2}$$

Then  $v_f(0) = v_f(1) = v_f(2) = n$  and  $e_f(0) = n-1, e_f(1) = e_f(2) = n$ .

Hence  $f$  is a mean cordial labeling.

**Theorem 2.15:** The  $K_{2,n}$  is a mean cordial iff  $n \leq 2$ .

**Proof:**

Let  $V(K_{2,n}) = \{u, v, u_i : 1 \leq i \leq n\}$  and

$E(K_{2,n}) = \{uu_i, vu_i : 1 \leq i \leq n\}$ .  $K_{2,1}$  and  $K_{2,2}$  are mean cordial by Theorem 2.5 and 2.8 respectively. Assume  $n > 2$ . Suppose  $f$  is a mean cordial labeling of  $K_{2,n}$ . Clearly either  $f(u) = 0$  or  $f(v) = 0$ . Without loss generality we can assume  $f(u) = 0$  so that  $f(v) \neq 0$ .

**Case (1):**  $n \equiv 0 \pmod{3}$

Let  $n = 3t$ . Then  $e_f(0) = t$  or  $t-1$ , a contradiction since the size of  $K_{2,n}$  is  $6t$ .

**Case (2):**  $n \equiv 1 \pmod{3}$

Let  $n = 3t + 1$ . Here  $ce_f(0) = t$ , again a contradiction.

**Case (3):**  $n \equiv 2 \pmod{3}$

Let  $n = 3t + 2$ . Here  $e_f(0) = t$  or  $t+1$ , again a contradiction to the size of  $K_{2,n}$ .

### 3. Conclusion

In this paper we introduced the concept of mean cordial labeling and studied the mean cordial labeling behavior of few standard graph. The authors are of the opinion that the study of mean cordial labeling behavior of graph obtained from standard graphs using the graph operation shall be quite interesting and also will lead to newer results.

### 4. Acknowledgements

The authors are thankful to the referee for their valuable comments and suggestions.

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