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Boundaries of Smooth Strictly Convex Sets in the Euclidean Plane R^2

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Abstract

We give a characterization of the boundaries of smooth strictly convex sets in the Euclidean plane R^2 based on the existence and uniqueness of inscribed triangles.

Keywords

Strict Convexity, Smoothness, Supporting Lines, Inscribed Triangles

1. Introduction

The reader unfamiliar with the theory of convex sets is referred to the books [1] [2] [3] [4] [5]. Let M be a set in the n-dimensional Euclidean space R^n. In the following we shall denote by $\operatorname{int} M$, clM, ϑM, $\operatorname{conv} M$ the interior, the closure, the boundary and respectively the convex hull of the set M. With $d(x,y)$ we denote the Euclidean distance of the points x and y and with $L(x,y)$ the line determined by the points x and y. The diameter $\operatorname{diam} M$ of a set M is $\operatorname{diam} M = \sup\{d(x,y): x,y \in M\}$. For a point $p \in R^2$ and a real number r we shall denote with $C(p,r)$ and $D(p,r)$ the circle and respectively the disc with center p and radius r. The distance $d(p,M)$ between a point p and a set M in R^2 is $d(p,M) = \inf\{d(p,x): x \in M\}$. With $]x,y[$ we denote the open line segment with endpoints x and y, that is $]x,y[= \operatorname{conv}\{x,y\} \setminus \{x,y\}$. For 3 nonlinear points x,y and z in R^2 we denote with $\max\angle(x,y,z)$ the maximum angle of the triangle $\Delta(x,y,z)$. A convex curve is a connected subset of the boundary of a convex set.

2. Preliminaries

In the chapter 8 of the book [4] of F.A. Valentine the author says the following: "It is interesting to see what kind of strong conclusions can be obtained from weak suppositions about any triplet of points of a plane set S." In [6] Menger gives such a characterization of the boundary of a convex plane set S based on intersection properties of S with the seven convex sets in which the space R^2 is subdivided by the lines $L(x_1,x_2), L(x_2,x_3)$ and $L(x_3,x_1)$ determined by an arbitrary triplet of noncollinear points $\{x_1,x_2,x_3\}$ from S. In [7] K. Juul proved the following:

Theorem 1. A plane set S fulfils

1) $\forall x,y,z \in S: S \cap \text{int}\{\text{conv}\{x,y,z\}\} = \varnothing$, if and only if S is either a subset of the boundary of a convex set, or an X-set, that is a set $\{x_1,x_2,x_3,x_4,x_5\}$ with $]x_1,x_2[\cap]x_3,x_4[=\{x_5\}$.

A survey of different characterizations of convex sets is given in the paper [8]. The results of K. Menger and that of K. Juul give characterizations of the boundaries of *convex* sets.

In the years 1978 [9] and 1979 [10] we have proved the following two theorems giving a characterization of the boundaries of smooth strictly convex sets:

Theorem 2. A plane compact set S is the boundary of a smooth strictly convex set if and only if the following two conditions hold:

1) $\forall x,y,z \in S: S \cap \text{int}\{\text{conv}\{x,y,z\}\} = \varnothing$,

2) For every triangle $\Delta(p_1,p_2,p_3)$ in R^2 there is one and only one triangle $\Delta(p_1',p_2',p_3')$ homothetic to the triangle $\Delta(p_1,p_2,p_3)$ inscribed in the set S, *i.e.* such that $p_1',p_2',p_3' \in S$.

Theorem 3. A plane compact set S is the boundary of a smooth strictly convex set if and only if the following two conditions hold:

1) For every $p \in S$ and every $\epsilon > 0$ there is a positive number $\delta(p,\epsilon)$ such that for every triplet of nonlinear points r,s,t in $S \cap \text{int}\{D(p,\delta(p,\epsilon))\}$ we have $\max \angle(r,s,t) > \pi - \epsilon$.

2) For every triangle $\Delta(p_1,p_2,p_3)$ in R^2 there is one and only one triangle $\Delta(p_1',p_2',p_3')$, homothetic to the triangle $\Delta(p_1,p_2,p_3)$ inscribed in the set S, *i.e.* such that $p_1',p_2',p_3' \in S$.

3. Main Results

The main result of this paper is Theorem 4 giving another characterization of the boundaries of smooth strictly convex sets in the Euclidean plane R^2 which uses also condition (2) of the Theorem 2 and Theorem 3.

Theorem 4. A compact set S in the Euclidean plane R^2 is the boundary of a smooth strictly convex set if and only if there are verified the following three conditions:

1) For every triangle $\Delta(p_1,p_2,p_3)$ in R^2 there is one and only one triangle $\Delta(p_1',p_2',p_3')$ homothetic to the triangle $\Delta(p_1,p_2,p_3)$ inscribed in the set S, *i.e.* such that $p_1',p_2',p_3' \in S$.

2) For any two distinct points $p \in S$ and $q \in S$ there are at least two points t_1 and t_2 such that $t_1 \in S \cap H_1$ and $t_2 \in S \cap H_2$, where H_1 and H_2 are the two open halfplanes generated in R^2 by the line $L(p,q)$.

3) The set S does not contain three collinear points.

For the proof of Theorem 4 we need the following theorem from the paper [11] and three lemmas:

Theorem 5. Let $\Delta(a,b,c)$ be a triangle in the Euclidean plane R^2. Suppose that S is a strictly convex closed arc of class C^1. Then there exists a single triangle $\Delta(a_1,b_1,c_1)$ homothetic to the triangle $\Delta(a,b,c)$ inscribed in the set S, in the sense that $a_1,b_1,c_1 \in S$.

Lemma 1. The convex hull convS of a compact set S in the Euclidean plane R^2 verifying the condition (2) from Theorem 4 is a strictly convex set.

Proof. Let us suppose the contrary. Then there are two distinct points $p,q \in \vartheta\{\text{conv}S\}$ such that the line segment $\text{conv}\{p,q\} \subset \text{conv}S$. The convex hull of a compact set is also a compact set (see [5] Theorem 2.2.6). The line $L(p,q)$ is thereby a supporting line for the compact set convS. Denote with H_1 and H_2 the two open halfplanes generated by the line $L(p,q)$ such that $\text{conv}S \subset cl\{H_1\}$ and $H_2 \cap \text{conv}S = \varnothing$. By Carathodory's Theorem (see [5] or [12] Theorem 2.2.4) the point $p \in \text{conv}S$ can be expressed as a convex combination of 3 or fewer points of S.

If the point p can be expressed only as a convex combination of three (and not of fewer) points x_1,x_2,x_3 of S then it follows that we must have $p \in \text{int}\{\text{conv}\{x_1,x_2,x_3\}\} \subset \text{int}\{\text{conv}S\}$ in contradiction to the fact that $p \in \vartheta\{\text{conv}S\}$.

If the point p can be expressed only as a convex combination of 2 (and not of fewer) points of S, there are $x_1 \in S$ and $x_2 \in S$ such that $p \in \text{conv}\{x_1,x_2\} \subset \text{conv}S \subset cl\{H_1\}$. Then the points x_1 and x_2 must be on the supporting line $L(p,q)$. As $H_2 \cap \text{conv}S = \varnothing,$ this is in contradiction with property (2) of the set S.

Thereby we must have $p \in S$. By an analog reasoning for the point q we can conclude that we have also: $q \in S$. Thus we have proved the existence of at least 2 different points of S on the supporting line $L(p,q)$ of convS in contradiction to the property (2) of the set S.

Lemma 2. The boundary $\vartheta\{\text{conv}S\}$ of the convex hull of a compact set S in the Euclidean plane R^2 verifying the condition (2) from Theorem 4 is a subset of the set *S*, *i.e.* $\vartheta\{\text{conv}S\} \subset S$.

Proof. Let $p \in \vartheta\{\text{conv}S\}$ be an arbitrary point from the boundary of the convex hull of the compact set S. Each boundary point of the compact convex set convS in R^2 is situated on at least one supporting line of the set convS (see for instance [3] pp. 6). We distinguish now the following two cases:

1) There is only one supporting line L_1 of the set convS going through the point p, *i.e.* the boundary $\vartheta\{\text{conv}S\}$ is smooth in the point p. By Lemma 1 it follows that the convex hull convS is a strictly convex set and thereby we have $\text{conv}S \cap L_1 = p$.

Let us now suppose the point $p \notin S$. From $\text{conv}S \cap L_1 = p$ and $p \notin S$ follows then $S \cap L_1 = \varnothing$. Denote with H_o the open halfplane generated by the line L_1, which contains the set S. As S is a compact set we have then $r = \min\{d(x,L_1) : x \in S\} > 0$. Consider then in the open halfplane H_o a line

L_1' parallel to the line L_1 at distance r to the line L_1. Denote with H_o' the open halfplane generated by the line L_1' and such that $H_o' \subset H_o$. It is evident that $p \notin cl\{H_o'\}$. From the definition of the constant, r folows $S \subset cl\{H_o'\}$ and $\vartheta\{\text{conv}S\} \subset cl\{H_o'\}$ in contradiction to $p \in \vartheta\{\text{conv}S\}$. Thereby our supposition $p \notin S$ is false, *i.e.* we must have $p \in S$.

2) There are two supporting lines L_1 and L_2 of the set conv$S$ going through the point p. Denote then with L_1' and L_2' the two halflines with endpoint p of the line L_1 and respectively L_2 such that $\text{conv}S \subset \text{conv}\{L_1' \cup L_2'\}$.

Let us suppose that $p \notin S$. From the compactness of S follows then the existence of a real number $r > 0$ such that for the disc $D(p,r)$ with the center p and the radius r we have: $D(p,r) \cap S = \varnothing$. Consider then the points $q_1 = C(p,r) \cap L_1'$ and $q_2 = C(p,r) \cap L_2'$, where $C(p,r)$ is the circle with center p and radius r. Let H_1 be the open halfplane generated by the line $L(q_1,q_2)$, which contains the point p and H_2 the other open halfplane generated by the line $L(q_1,q_2)$. We have then evidently $S \cap clH_1 = \varnothing$ and thereby $S \subset H_2$. From the inclusion $S \subset H_2$ it follows also that $\text{conv}S \subset H_2$. As $\vartheta\{\text{conv}S\} \subset S$ we have also: $\vartheta\{\text{conv}S\} \subset H_2$ in contradiction to our supposition $p \in \vartheta\{\text{conv}S\}$. Therefore the point p must belong to the set S.

So we have proved in both cases (1) and (2) that $p \in \vartheta\{\text{conv}S\}$ implies $p \in S$, *i.e.* $\vartheta\{\text{conv}S\} \subset S$.

A characterization of compact sets S in the Euclidean plane R^2 for which we have $S = \vartheta\{\text{conv}S\}$ is given in the following:

Lemma 3. A compact set S in the Euclidean plane R^2 has a strictly convex hull and coincides with the boundary of its convex hull $\vartheta\{\text{conv}S\}$ if and only if there are verified the conditions (2) and (3).

Proof. Let S be a compact set in the Euclidean plane R^2, which has a strictly convex hull conv$S$ and such that $S = \vartheta\{\text{conv}S\}$. Consider then two arbitrary points p_1 and p_2 of the set S and the two open halfplanes generated by the line $L(p_1,p_2)$ in R^2. Because S has a strictly convex hull it is then evident that we have verified condition (2) and (3).

To prove the only if part of the lemma let us consider a compact set S in the Euclidean plane R^2, which verifies conditions (2) and (3). By Lemma 1 the convex hull conv$S$ of S is a strictly convex set. By Lemma 2 we have then for the set S the inclusion $\vartheta\{\text{conv}S\} \subset S$. Let us now suppose that we would have $S \not\subset \vartheta\{\text{conv}S\}$, *i.e.* there is a point $p \in S$ such that $p \notin \vartheta\{\text{conv}S\}$. Then the point p must be an interior point of the convex hull conv$S$. Let L be an arbitrary line such that $p \in L$. Then it is obvious that the line L intersects $\vartheta\{\text{conv}S\}$ in two different points t_1 and t_2 such that $p \in \text{conv}\{t_1,t_2\}$. From $\vartheta\{\text{conv}S\} \subset S$ it follows that $t_1 \in S$ and $t_2 \in S$ in contradiction to the condition (3) of the set S. So we conclude that $S \subset \vartheta\{\text{conv}S\}$. This inclusion together with the inclusion $\vartheta\{\text{conv}S\} \subset S$ gives then $S \subset \vartheta\{\text{conv}S\}$.

A similar result as that of Lemma 3 without the compactness of the set S but with the additional assumption of the connectedness of the set S was obtained by K. Juul in [7]:

Theorem 6. A connected set S in R^2 is a convex curve if and only if it verifies condition (1) from Theorem 1.

Proof of Theorem 4.

For the proof of the if-part of the theorem let S be the boundary of a compact smooth strictly convex set in the Euclidean plane R^2. It is then easy to verify conditions (2) and (3) for the set S. Condition (1) follows immediately from Theorem 5.

For the proof of the "only if"—part of the theorem let S be a compact set in the Euclidean plane R^2, which verifies conditions (1), (2) and (3). By Lemma 3 it follows that the convex hull $\text{conv}S$ of the set S is strictly convex and that $S = \vartheta\{\text{conv}S\}$.

It remains to prove that $\text{conv}S$ is also a smooth set. Let us assume the contrary: there is a point $a_1 \in \vartheta\{\text{conv}S\}$, which is not a smooth point of the boundary of S, *i.e.* there exist two supporting lines L_1 and L_2 for the set $\text{conv}S$ at the point a_1. For $i \in \{1,2\}$ denote with H_i the closed half-plane generated by the supporting line L_i, which contains the set S. Denote with C the convex cone $C = H_1 \cap H_2$. We have then evidently the inclusions: $S \subset C$ and $\text{conv}S \subset C$. As $\text{conv}S$ is a strictly convex set we have also the inclusion $S \setminus a_1 \subset \text{int}\, C$. For $i \in \{1,2\}$ denote with L_i' the closed halfline of the line L_i with origin a_1 such that $L_i' \subset \vartheta C$. Consider then the isosceles triangle $\Delta(a_1, a_2, a_3)$ with $d(a_1, a_2) = d(a_1, a_3)$ and such that angle $\angle a_2 a_1 a_3$ has the same angle bisector as the boundary angle of the cone C formed by the halflines L_1' and L_2' with the vertex a_1 and such that the angle $\angle a_2 a_1 a_3$ is greater than the boundary angle of the cone C. By condition (1) there exists then three points $a_i' \in S, i = 1,2,3$ such that triangle $\Delta(a_1', a_2', a_3')$ is homothetic to the triangle $\Delta(a_1, a_2, a_3)$. Because the angle $\angle a_2 a_1 a_3$ is greater than the boundary angle of the cone C the point a_1' cannot coincide with the point a_1. From this fact and the inclusion $S \setminus a_1 \subset \text{int}\, C$ we can conclude that we have: $a_i' \in \text{int}\, C$ for $i = 1,2,3$. From the homothety of the triangles $\Delta(a_1', a_2', a_3')$ and $\Delta(a_1, a_2, a_3)$ it follows then that $a_1' \in \text{int}\{\text{conv}\{a_1, a_2', a_3'\}\} \subset \text{int}\{\text{conv}S\}$ in contradiction to $a_1' \in S = \vartheta\{\text{conv}S\}$. So we have proved that the convex hull $\text{conv}S$ is a smooth strictly convex set.

4. Conclusions

As we have seen condition (1) is used and is essential in the proofs of the Theorem 2, Theorem 3 and Theorem 4. We emit now the following:

Conjecture: A compact set S in the Euclidean plane R^2 is the boundary of a smooth strictly convex set if and only if there is verified the condition:

For every triangle $\Delta(p_1, p_2, p_3)$ in R^2 there is one and only one triangle $\Delta(p_1', p_2', p_3')$ homothetic to the triangle $\Delta(p_1, p_2, p_3)$ and inscribed in the set S *i.e.* such that $p_1', p_2', p_3' \in S$.

P. Mani-Levitska cites in his survey [8] the papers [7] and [9] and says reffering to these, that he has not encountered extensions of these results to higher dimensions. We also don't know generalizations of our results to higher dimensions.

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