

Block Extension of a Single-Step Hybrid Multistep Method for Directly Solving Fourth-Order Initial Value Problems

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Abstract

This article aims to derive, analyse, and implement an efficient one-step implicit hybrid method with block extension comprised of seven off-step points to directly solve Initial Value Problems (IVPs) of general four-order ordinary differential equations. For the resolution of the fourth-order IVPs, the exact was approximated by a polynomial termed basis function. The partial sum of the basis function and its fourth derivative were interpolated and collocated at some selected grid and off-grid points for the unknown parameters to be determined. The derived method, when tested, is found to be consistent, convergent, and zero-stable. The method's accuracy and usability were experimented with using specific sample problems, and the findings revealed that it surpassed some cited methods in terms of accuracy.

Keywords

Implicit, Absolute Stability, Grid Points, Off-Grid Points, Convergence, Interpolation, and Collocation

1. Introduction

This paper proposed a one-step hybrid method for directly solving fourth-order initial value problem of ordinary differential equations of the form:

$$y^{(4)} = f(x, y, y', y'', y'''), y(x_0) = a_0, y'(x_0) = a_1, y''(x_0) = a_2, y'''(x_0) = a_3 \quad (1)$$

where, f is a given continuous real value function. High-order linear and nonlinear IVPs have been used to represent engineering and other areas' problems. The static deflection of a uniform beam or a cantilever beam (with the left end

embedded and the right end free giving birth to fourth-order IVPs) is one of the applications of fourth-order problems (see [1]).

Traditionally, the reduction approach is adopted for numerically solving equation (1), as reported by [2] [3] [4] [5] and many others. Although this strategy has had much success, it does have certain disadvantages. For example, computer programs related to the method's implementation are frequently complex, particularly the subroutines that supply the starting values of the methods, resulting in longer computer time and computational work. However, [6] found that these approaches do not consider extra information connected with a particular ordinary differential equation, such as the solution's oscillatory nature.

A direct approach is introduced as an alternative method to overcome the setback inherent in the reduction approach. Several direct numerical methods exist in the literature, but a few are specially designed to solve fourth-order ordinary differential equations. For instance, [7] developed a Block Hybrid Collocation Method (BHCM) and applied it to solve fourth-order IVPs. Three off-grid points are utilized with the collocation. [8] developed a four-step implicit block method with three generalized off-step points and applied it to solve fourth-order IVPs directly. In a work by [9], an algorithmic collocation approach was presented for obtaining the approximation of fourth-order IVPs. [1] suggested Runge-Kutta type method for directly solving this kind of problem.

According to [10] [11], single-step methods are efficient in terms of accuracy due to the hybrid points incorporated into the method. The proposed method's efficiency is measured by the number of hybrid points included, either as solutions or function values. The success of [11] motivates this work where the approximation of (1) is sorted in the interval $[x_n, x_{n+1}]$ with seven (7) intermediate points.

2. Mathematical Formulation

Let's start by allowing the exact $y(x)$ of the fourth-order IVP of ordinary differential Equation (1) to be approximated by a partial sum of the polynomial $p(x)$ of the form:

$$y(x) \cong p(x) = \sum_{r=0}^{8k+5} a_r x^r. \quad (2)$$

Equation (2) is differentiated four times to obtain its fourth derivative given as:

$$y^{(4)}(x) \cong p^{(4)}(x) = \sum_{r=4}^{8k+5} r(r-1)(r-2)(r-3)a_r x^{r-4}. \quad (3)$$

Equating (3) and (1) yields the differential system:

$$f(x, y(x), y'(x), y''(x), y'''(x)) = \sum_{r=4}^{8k+5} r(r-1)(r-2)(r-3)a_r x^{r-4}. \quad (4)$$

Note that x is continuously differentiable, parameters a_r 's in (2), (3), and (4) are linear terms to be determined. By applying, $x = x_{n+r}, r = \frac{4}{8}, \frac{5}{8}, \frac{6}{8}, \frac{7}{8}$ to (2) and $x = x_{n+r}, r = 0\left(\frac{1}{8}\right)1$ to (4) yields the following system of algebraic equations:

$$y_{n+r} = \sum_{r=0}^{13} a_r x^r, \quad r = \frac{4}{8}, \frac{5}{8}, \frac{6}{8}, \frac{7}{8}, \quad (5)$$

$$f_{n+r} = \sum_{r=4}^{13} r(r-1)(r-2)(r-3)a_r x^{r-4}, \quad r = 0\left(\frac{1}{8}\right)1. \quad (6)$$

By allowing $x_{\frac{n+r}{8}} = x_n + \frac{r}{8}h$ and $x_n = 0$, (5) and (6) are written as matrix equation and solved using CAS in Mathematica to obtain the parameters $a_{\frac{r}{8}}$'s for $r = 0, 1, 2, \dots, 1$ which were substituted into (2) and yields the following continuous scheme after some simplifications:

$$y(t) = \sum_{r=4}^7 \alpha_{\frac{r}{8}}(t) y_{\frac{n+r}{8}}(t) + h^4 \sum_{r=0}^8 \beta_{\frac{r}{8}}(t) f_{\frac{n+r}{8}}(t) \quad (7)$$

where, $x = x_{n+it} = x_n + th$, $\alpha_{\frac{r}{8}}(t)$'s and $\beta_{\frac{r}{8}}(t)$'s are the coefficients that defined the scheme and are given as:

$$\alpha_{\frac{1}{2}}(t) = \frac{1}{3}(105 - 428t + 576t^2 - 256t^3), \quad \alpha_{\frac{5}{8}}(t) = 4(-21 + 94t - 136t^2 + 64t^3),$$

$$\alpha_{\frac{3}{4}}(t) = -2(-35 + 166t - 256t^2 + 128t^3), \quad \alpha_{\frac{7}{8}}(t) = \frac{4}{3}(-15 + 74t - 120t^2 + 64t^3),$$

$$\beta_0(t) = \frac{h^4}{490497638400}(-125875 - 7186478t + 232909496t^2 - 2931938944t^3 + 20437401600t^4 - 88873500672t^5 + 255465357312t^6 - 498906169344t^7 + 665831079935t^8 - 597939978240t^9 + 345467431872t^{10} - 115964116992t^{11} + 17179869184t^{12}),$$

$$\beta_{\frac{1}{8}}(t) = -\frac{h^4}{61312204800}(-2558325 + 58784710t - 528368824t^2 + 2030364160t^3 - 32699842560t^5 + 149796421632t^6 - 362293493760t^7 + 546450702336t^8 - 530579456000t^9 + 323330506752t^{10} - 112742891520t^{11} + 17179689184t^{12}),$$

$$\beta_{\frac{1}{4}}(t) = \frac{h^4}{122624409600}(39090975 - 476020370t + 1697474216t^2 - 86879232t^3 - 114449448960t^5 + 67688740992t^6 - 1905207017472t^7 + 3174272335872t^8 - 3303433830400t^9 + 2117150441472t^{10} - 766651662336t^{11} + 120256084288t^{12}),$$

$$\beta_{\frac{3}{8}}(t) = -\frac{h^4}{61312204800} \left(-68220075 + 543730330t - 1591532296t^2 + 2867507456t^3 - 76299632640t^5 + 485168775168t^6 - 1489244258304t^7 + 2670798569472t^8 - 2948176281600t^9 + 1979845705728t^{10} - 744103084032t^{11} + 120259084288t^{12} \right),$$

$$\beta_{\frac{1}{2}}(t) = \frac{h^4}{49049763840} \left(127767255 - 764794898t + 1418313224t^2 + 146591104t^3 - 57224724480t^5 + 376593186816t^6 - 1209997983744t^7 + 2282345201664t^8 - 26427470643t^9 + 1851399340032t^{10} - 721554505728t^{11} + 120259084288t^{12} \right),$$

$$\beta_{\frac{3}{4}}(t) = \frac{h^4}{122624409600} \left(103817175 - 509826050t + 7510153512t^2 - 323121920t^3 - 38149816320t^5 + 259539861504t^6 - 873552936960t^7 + 1745860165632t^8 - 2159227699200t^9 + 1621081718784t^{10} - 676457349120t^{11} + 120256084288t^{12} \right),$$

$$\beta_{\frac{7}{8}}(t) = -\frac{h^4}{61312204800} \left(768075 - 3551450t - 3513592t^2 + 897994424t^3 - 4671406080t^5 + 32076988416t^6 - 109414711296t^7 + 222566547456t^8 - 281437798400t^9 + 217030066176t^{10} - 93415538688t^{11} + 17179689184t^{12} \right),$$

$$\beta_1(t) = \frac{h^4}{490497638400} \left(594825 - 2755790t - 3575368t^2 + 78427008t^3 - 4087480320t^5 + 28262006784t^6 - 97372864512t^7 + 200766652416t^8 - 258369126400t^9 + 203742511104t^{10} - 90194313216t^{11} + 17179869184t^{12} \right).$$

Evaluating (7) at $t = 0, \frac{1}{8}, \frac{1}{4}, \frac{3}{8}, 1$ to obtain the discrete one-step formula,

$$y_n - 35y_{n+\frac{1}{2}} + 84y_{n+\frac{5}{8}} - 70y_{n+\frac{3}{4}} + 20y_{n+\frac{7}{8}} = \frac{-h^4}{424673280} \left(109f_n - 17720f_{n+\frac{1}{8}} - 135380f_{n+\frac{1}{4}} - 472520f_{n+\frac{3}{8}} - 1106210f_{n+\frac{1}{2}} - 1542344f_{n+\frac{5}{8}} - 359540f_{n+\frac{3}{4}} + 5320f_{n+\frac{7}{8}} - 515f_{n+1} \right) \tag{8a}$$

$$y_{n+\frac{1}{8}} - 20y_{n+\frac{1}{2}} + 45y_{n+\frac{5}{8}} - 36y_{n+\frac{3}{4}} + 10y_{n+\frac{7}{8}} = \frac{-h^4}{990904320} \left(125f_n - 1144f_{n+\frac{1}{8}} - 34660f_{n+\frac{1}{4}} - 332440f_{n+\frac{3}{8}} - 1087810f_{n+\frac{1}{2}} - 1758760f_{n+\frac{5}{8}} - 4197f_{n+\frac{3}{4}} + 6200f_{n+\frac{7}{8}} - 595f_{n+1} \right) \tag{8b}$$

$$\begin{aligned}
 & y_{n+\frac{1}{4}} - 10y_{n+\frac{1}{2}} + 20y_{n+\frac{5}{8}} - 15y_{n+\frac{3}{4}} + 4y_{n+\frac{7}{8}} \\
 &= \frac{-h^4}{2972712960} \left(205f_n - 1880f_{n+\frac{1}{8}} - 7228f_{n+\frac{1}{4}} - 130280f_{n+\frac{3}{8}} \right. \\
 & \quad \left. - 973010f_{n+\frac{1}{2}} - 2032040f_{n+\frac{5}{8}} - 506180f_{n+\frac{3}{4}} + 7912f_{n+\frac{7}{8}} - 755f_{n+1} \right) \tag{8c}
 \end{aligned}$$

$$\begin{aligned}
 & y_{n+\frac{3}{8}} - 4y_{n+\frac{1}{2}} + 6y_{n+\frac{5}{8}} - 4y_{n+\frac{3}{4}} + y_{n+\frac{7}{8}} \\
 &= \frac{-h^4}{2972712960} \left(41f_n - 328f_{n+\frac{1}{8}} + 908f_{n+\frac{1}{4}} + 152f_{n+\frac{3}{8}} - 125722f_{n+\frac{1}{2}} \right. \\
 & \quad \left. - 475288f_{n+\frac{5}{8}} - 127444f_{n+\frac{3}{4}} + 2120f_{n+\frac{7}{8}} - 199f_{n+1} \right) \tag{8d}
 \end{aligned}$$

$$\begin{aligned}
 & y_{n+1} - 4y_{n+\frac{7}{8}} + 6y_{n+\frac{3}{4}} - 4y_{n+\frac{5}{8}} + y_{n+\frac{1}{2}} \\
 &= \frac{h^4}{2972712960} \left(199f_n - 1832f_{n+\frac{1}{8}} + 7492f_{n+\frac{1}{4}} - 17624f_{n+\frac{3}{8}} + 24922f_{n+\frac{1}{2}} \right. \\
 & \quad \left. + 100648f_{n+\frac{5}{8}} + 492004f_{n+\frac{3}{4}} + 120280f_{n+\frac{7}{8}} - 329f_{n+1} \right) \tag{8e}
 \end{aligned}$$

Derivation of the Block Method

In the spirit of [12], the normalized form of the general block method is given by

$$AY_i = Ey_n + h^{\mu-\rho} df(y_n) + h^{\mu-\rho} BF(y_n) \tag{9}$$

By combining the formulas in (8) and the additional first, second, and third derivatives formulas obtained from (7) and writing in block form, using the definition of the implicit block method in (9) to get the block formula described as follows:

$$h^{\alpha} \sum_{r=0}^q \phi_{m,r} y_{n+r}^{\rho} = h^{\rho} \sum_{r=0}^q \nabla_{m,r} y_n^{\rho} + h^{p-\rho} \left(\sum_{r=0}^q \Delta_{m,r} f_n + \sum_{r=0}^q \delta_{m,r} f_{n+r} \right) \tag{10}$$

where σ is the power of the derivative of the continuous method and p is the order of the problem to solve. Equation (10) is solved for $r = 0, \frac{1}{8}, \frac{2}{8}, \dots, 1$ to obtain the following proposed Single-step Hybrid Multistep Method (SHMM):

$$\begin{aligned}
 y_{n+\frac{1}{8}} &= y_n + \frac{1}{8}hy'_n + \frac{1}{128}h^2y''_n + \frac{1}{3072}h^3y'''_n + \frac{h^4}{3923981107200} \left(24396497f_n \right. \\
 & \quad \left. + 36501816f_{n+\frac{1}{8}} - 52883276f_{n+\frac{1}{4}} + 67126376f_{n+\frac{3}{8}} - 61500210f_{n+\frac{1}{2}} \right. \\
 & \quad \left. + 38838088f_{n+\frac{5}{8}} - 16041916f_{n+\frac{3}{4}} + 3904536f_{n+\frac{7}{8}} - 425111f_{n+1} \right), \\
 y_{n+\frac{1}{4}} &= y_n + \frac{1}{4}hy'_n + \frac{1}{32}h^2y''_n + \frac{1}{384}h^3y'''_n + \frac{h^4}{1532805100} \left(1035731f_n \right. \\
 & \quad \left. + 2719504f_{n+\frac{1}{8}} - 3139836f_{n+\frac{1}{4}} + 3933392f_{n+\frac{3}{8}} - 3577790f_{n+\frac{1}{2}} \right. \\
 & \quad \left. + 2249904f_{n+\frac{5}{8}} - 926764f_{n+\frac{3}{4}} + 225136f_{n+\frac{7}{8}} - 24477f_{n+1} \right),
 \end{aligned}$$

$$\begin{aligned}
 y_{n+\frac{3}{8}} &= y_n + \frac{3}{8}hy'_n + \frac{9}{128}h^2y''_n + \frac{9}{1024}h^3y'''_n + \frac{h^4}{16148070400} \left(4104531f_n \right. \\
 &\quad + 13764168f_{n+\frac{1}{8}} - 12476916f_{n+\frac{1}{4}} + 16614360f_{n+\frac{3}{8}} - 15168870f_{n+\frac{1}{2}} \\
 &\quad \left. + 9553464f_{n+\frac{5}{8}} - 3938148f_{n+\frac{3}{4}} + 957096f_{n+\frac{7}{8}} - 104085f_{n+1} \right), \\
 y_{n+\frac{1}{2}} &= y_n + \frac{1}{2}hy'_n + \frac{1}{8}h^2y''_n + \frac{1}{48}h^3y'''_n + \frac{h^4}{59875200} \left(38084f_n \right. \\
 &\quad + 145056f_{n+\frac{1}{8}} - 104780f_{n+\frac{1}{4}} + 160352f_{n+\frac{3}{8}} - 144375f_{n+\frac{1}{2}} \\
 &\quad \left. + 90976f_{n+\frac{5}{8}} - 37516f_{n+\frac{3}{4}} + 9120f_{n+\frac{7}{8}} - 992f_{n+1} \right), \\
 y_{n+\frac{5}{8}} &= y_n + \frac{5}{8}hy'_n + \frac{25}{128}h^2y''_n + \frac{125}{3072}h^3y'''_n + \frac{h^4}{156959244288} \left(201421625f_n \right. \\
 &\quad + 828115000f_{n+\frac{1}{8}} - 490807500f_{n+\frac{1}{4}} + 895985000f_{n+\frac{3}{8}} - 766681250f_{n+\frac{1}{2}} \\
 &\quad \left. + 487389000f_{n+\frac{5}{8}} - 201077500f_{n+\frac{3}{4}} + 48895000f_{n+\frac{7}{8}} - 5319375f_{n+1} \right), \\
 y_{n+\frac{3}{4}} &= y_n + \frac{3}{4}hy'_n + \frac{9}{32}h^2y''_n + \frac{9}{128}h^3y'''_n + \frac{h^4}{63078400} \left(142929f_n \right. \\
 &\quad + 618192f_{n+\frac{1}{8}} - 308772f_{n+\frac{1}{4}} + 672912f_{n+\frac{3}{8}} - 532170f_{n+\frac{1}{2}} \\
 &\quad \left. + 351216f_{n+\frac{5}{8}} - 143892f_{n+\frac{3}{4}} + 34992f_{n+\frac{7}{8}} - 3807f_{n+1} \right), \\
 y_{n+\frac{7}{8}} &= y_n + \frac{7}{8}hy'_n + \frac{49}{128}h^2y''_n + \frac{343}{3072}h^3y'''_n + \frac{h^4}{560568729600} \left(2048300303f_n \right. \\
 &\quad + 9184285992f_{n+\frac{1}{8}} - 3949712228f_{n+\frac{1}{4}} + 10148873336f_{n+\frac{3}{8}} \\
 &\quad - 7344827070f_{n+\frac{1}{2}} + 5205732952f_{n+\frac{5}{8}} - 2050386772f_{n+\frac{3}{4}} \\
 &\quad \left. + 504037128f_{n+\frac{7}{8}} - 54841241f_{n+1} \right), \\
 y_{n+1} &= y_n + hy'_n + \frac{1}{2}h^2y''_n + \frac{1}{6}h^3y'''_n + \frac{h^4}{3742200} \left(20648f_n + 95104f_{n+\frac{1}{8}} \right. \\
 &\quad - 35808f_{n+\frac{1}{4}} + 108880f_{n+\frac{3}{8}} - 70760f_{n+\frac{1}{2}} + 54912f_{n+\frac{5}{8}} \\
 &\quad \left. - 19744f_{n+\frac{3}{4}} + 6248f_{n+\frac{7}{8}} - 555f_{n+1} \right), \\
 y'_{n+\frac{1}{8}} &= y'_n + \frac{1}{8}hy''_n + \frac{1}{128}h^2y'''_n + \frac{h^3}{20437401600} \left(3619903f_n + 6779886f_{n+\frac{1}{8}} \right. \\
 &\quad - 9359135f_{n+\frac{1}{4}} + 11774146f_{n+\frac{3}{8}} - 10745445f_{n+\frac{1}{2}} + 6771082f_{n+\frac{5}{8}} \\
 &\quad \left. - 2792861f_{n+\frac{3}{4}} + 679110f_{n+\frac{7}{8}} - 73886f_{n+1} \right),
 \end{aligned}
 \tag{11}$$

$$\begin{aligned}
y'_{n+\frac{1}{4}} &= y'_n + \frac{1}{4}hy''_n + \frac{1}{32}h^2y''' + \frac{h^3}{319334400} \left(286967f_n + 911204f_{n+\frac{1}{8}} \right. \\
&\quad - 926646f_{n+\frac{1}{4}} + 1173140f_{n+\frac{3}{8}} - 1067950f_{n+\frac{1}{2}} + 67162f_{n+\frac{5}{8}} \\
&\quad \left. - 276634f_{n+\frac{3}{4}} + 67196f_{n+\frac{7}{8}} - 7305f_{n+1} \right), \\
y'_{n+\frac{3}{8}} &= y'_n + \frac{3}{8}hy''_n + \frac{9}{128}h^2y''' + \frac{h^3}{252313600} \left(550152f_n + 2135754f_{n+\frac{1}{8}} \right. \\
&\quad - 1563651f_{n+\frac{1}{4}} + 2298870f_{n+\frac{3}{8}} - 2099655f_{n+\frac{1}{2}} + 1323918f_{n+\frac{5}{8}} \\
&\quad \left. - 546129f_{n+\frac{3}{4}} + 132786f_{n+\frac{7}{8}} - 14445f_{n+1} \right), \\
y'_{n+\frac{1}{2}} &= y'_n + \frac{1}{2}hy''_n + \frac{1}{8}h^2y''' + \frac{h^3}{19958400} \left(80293f_n + 342816f_{n+\frac{1}{8}} \right. \\
&\quad - 188120f_{n+\frac{1}{4}} + 358816f_{n+\frac{3}{8}} - 310800f_{n+\frac{1}{2}} + 196192f_{n+\frac{5}{8}} \\
&\quad \left. - 80936f_{n+\frac{3}{4}} + 19680f_{n+\frac{7}{8}} - 2141f_{n+1} \right), \\
y'_{n+\frac{5}{8}} &= y'_n + \frac{5}{8}hy''_n + \frac{25}{128}h^2y''' + \frac{h^3}{817496064} \left(5253125f_n + 23702750f_{n+\frac{1}{8}} \right. \\
&\quad - 10296375f_{n+\frac{1}{4}} + 25537250f_{n+\frac{3}{8}} - 19680625f_{n+\frac{1}{2}} \\
&\quad \left. + 129202250f_{n+\frac{5}{8}} - 5327125f_{n+\frac{3}{4}} + 1295750f_{n+\frac{7}{8}} - 141000f_{n+1} \right), \\
y'_{n+\frac{3}{4}} &= y'_n + \frac{3}{4}hy''_n + \frac{9}{32}h^2y''' + \frac{h^3}{3942400} \left(37017f_n + 173124f_{n+\frac{1}{8}} \right. \\
&\quad - 61830f_{n+\frac{1}{4}} + 192564f_{n+\frac{3}{8}} - 129330f_{n+\frac{1}{2}} + 95148f_{n+\frac{5}{8}} \\
&\quad \left. - 37674f_{n+\frac{3}{4}} + 9180f_{n+\frac{7}{8}} - 999f_{n+1} \right), \\
y'_{n+\frac{7}{8}} &= y'_n + \frac{7}{8}hy''_n + \frac{49}{128}h^2y''' + \frac{h^3}{2919628800} \left(37701874f_n + 180838518f_{n+\frac{1}{8}} \right. \\
&\quad - 54639557f_{n+\frac{1}{4}} + 206894170f_{n+\frac{3}{8}} - 122270925f_{n+\frac{1}{2}} + 104842066f_{n+\frac{5}{8}} \\
&\quad \left. - 35782103f_{n+\frac{3}{4}} + 9423582f_{n+\frac{7}{8}} - 1020425f_{n+1} \right), \\
y'_{n+1} &= y'_n + hy''_n + \frac{1}{2}h^2y''' + \frac{h^3}{1247400} \left(21203f_n + 103616f_{n+\frac{1}{8}} \right. \\
&\quad - 27024f_{n+\frac{1}{4}} + 121280f_{n+\frac{3}{8}} - 63820f_{n+\frac{1}{2}} + 63552f_{n+\frac{5}{8}} \\
&\quad \left. - 16816f_{n+\frac{3}{4}} + 6464f_{n+\frac{7}{8}} - 555f_{n+1} \right), \tag{12}
\end{aligned}$$

$$\begin{aligned}
 y''_{n+\frac{1}{8}} &= y''_n + \frac{1}{8}hy''' + \frac{h^2}{464486400} \left(1624505f_n + 4124232f_{n+\frac{1}{8}} \right. \\
 &\quad - 5225624f_{n+\frac{1}{4}} + 6488192f_{n+\frac{3}{8}} - 5888310f_{n+\frac{1}{2}} \\
 &\quad \left. + 3698920f_{n+\frac{5}{8}} - 1522672f_{n+\frac{3}{4}} + 369744f_{n+\frac{7}{8}} - 40187f_{n+1} \right), \\
 y''_{n+\frac{1}{4}} &= y''_n + \frac{1}{4}hy''' + \frac{h^2}{7257600} \left(58193f_n + 235072f_{n+\frac{1}{8}} - 183708f_{n+\frac{1}{4}} \right. \\
 &\quad + 247328f_{n+\frac{3}{8}} - 227030f_{n+\frac{1}{2}} + 143232f_{n+\frac{5}{8}} - 59092f_{n+\frac{3}{4}} \\
 &\quad \left. + 14368f_{n+\frac{7}{8}} - 1563f_{n+1} \right), \\
 y''_{n+\frac{3}{8}} &= y''_n + \frac{3}{8}hy''' + \frac{h^2}{5734400} \left(71661f_n + 328608f_{n+\frac{1}{8}} - 150624f_{n+\frac{1}{4}} \right. \\
 &\quad + 315000f_{n+\frac{3}{8}} - 281430f_{n+\frac{1}{2}} + 177264f_{n+\frac{5}{8}} - 73128f_{n+\frac{3}{4}} \\
 &\quad \left. + 17784f_{n+\frac{7}{8}} - 1935f_{n+1} \right), \\
 y''_{n+\frac{1}{2}} &= y''_n + \frac{1}{2}hy''' + \frac{h^2}{453600} \left(7703f_n + 37248f_{n+\frac{1}{8}} - 11600f_{n+\frac{1}{4}} \right. \\
 &\quad + 40064f_{n+\frac{3}{8}} - 29610f_{n+\frac{1}{2}} + 19072f_{n+\frac{5}{8}} - 7888f_{n+\frac{3}{4}} \\
 &\quad \left. + 1920f_{n+\frac{7}{8}} - 209f_{n+1} \right), \\
 y''_{n+\frac{5}{8}} &= y''_n + \frac{5}{8}hy''' + \frac{h^2}{18579456} \left(398825f_n + 1987000f_{n+\frac{1}{8}} - 465000f_{n+\frac{1}{4}} \right. \\
 &\quad + 2294000f_{n+\frac{3}{8}} - 128350f_{n+\frac{1}{2}} + 1020600f_{n+\frac{5}{8}} - 412000f_{n+\frac{3}{4}} \\
 &\quad \left. + 100000f_{n+\frac{7}{8}} - 10875f_{n+1} \right), \\
 y''_{n+\frac{3}{4}} &= y''_n + \frac{3}{4}hy''' + \frac{h^2}{89600} \left(2325f_n + 11808f_{n+\frac{1}{8}} - 2196f_{n+\frac{1}{4}} + 14208f_{n+\frac{3}{8}} \right. \\
 &\quad \left. - 6390f_{n+\frac{1}{2}} + 7200f_{n+\frac{5}{8}} - 2268f_{n+\frac{3}{4}} + 576f_{n+\frac{7}{8}} - 63f_{n+1} \right), \\
 y''_{n+\frac{7}{8}} &= y''_n + \frac{7}{8}hy''' + \frac{h^2}{66355200} \left(2019731f_n + 10388784f_{n+\frac{1}{8}} - 1575056f_{n+\frac{1}{4}} \right. \\
 &\quad + 12811736f_{n+\frac{3}{8}} - 4826010f_{n+\frac{1}{2}} + 7068544f_{n+\frac{5}{8}} - 1018024f_{n+\frac{3}{4}} \\
 &\quad \left. + 589176f_{n+\frac{7}{8}} - 57281f_{n+1} \right),
 \end{aligned}$$

$$\begin{aligned}
y''_{n+1} &= y''_n + hy''' + \frac{h^2}{28350} \left(989f_n + 5152f_{n+\frac{1}{8}} - 696f_{n+\frac{1}{4}} + 6560f_{n+\frac{3}{8}} \right. \\
&\quad \left. - 2270f_{n+\frac{1}{2}} + 3936f_{n+\frac{5}{8}} - 232f_{n+\frac{3}{4}} + 736f_{n+\frac{7}{8}} \right), \tag{13} \\
y'''_{n+\frac{1}{8}} &= y''' + \frac{h}{29030400} \left(1070017f_n + 4467094f_{n+\frac{1}{8}} - 4604594f_{n+\frac{1}{4}} \right. \\
&\quad + 5595358f_{n+\frac{3}{8}} - 5033120f_{n+\frac{1}{2}} + 31463388f_{n+\frac{5}{8}} - 1291214f_{n+\frac{3}{4}} \\
&\quad \left. + 312874f_{n+\frac{7}{8}} - 33953f_{n+1} \right), \\
y'''_{n+\frac{1}{4}} &= y''' + \frac{h}{907200} \left(32377f_n + 182584f_{n+\frac{1}{8}} - 42494f_{n+\frac{1}{4}} + 120088f_{n+\frac{3}{8}} \right. \\
&\quad \left. - 116120f_{n+\frac{1}{2}} + 74728f_{n+\frac{5}{8}} - 31154f_{n+\frac{3}{4}} + 7624f_{n+\frac{7}{8}} - 833f_{n+1} \right), \\
y'''_{n+\frac{3}{8}} &= y''' + \frac{h}{358400} \left(12881f_n + 70902f_{n+\frac{1}{8}} + 3438f_{n+\frac{1}{4}} + 79934f_{n+\frac{3}{8}} \right. \\
&\quad \left. - 56160f_{n+\frac{1}{2}} + 34434f_{n+\frac{5}{8}} - 14062f_{n+\frac{3}{4}} + 3402f_{n+\frac{7}{8}} - 369f_{n+1} \right), \\
y'''_{n+\frac{1}{2}} &= y''' + \frac{h}{113400} \left(4063f_n + 22576f_{n+\frac{1}{8}} + 244f_{n+\frac{1}{4}} + 32752f_{n+\frac{3}{8}} \right. \\
&\quad \left. - 9080f_{n+\frac{1}{2}} + 9232f_{n+\frac{5}{8}} - 3956f_{n+\frac{3}{4}} + 976f_{n+\frac{7}{8}} - 107f_{n+1} \right), \\
y'''_{n+\frac{5}{8}} &= y''' + \frac{h}{1161216} \left(41705f_n + 230150f_{n+\frac{1}{8}} + 7550f_{n+\frac{1}{4}} + 318350f_{n+\frac{3}{8}} \right. \\
&\quad \left. - 4000f_{n+\frac{1}{2}} + 170930f_{n+\frac{5}{8}} - 49150f_{n+\frac{3}{4}} + 11450f_{n+\frac{7}{8}} - 1225f_{n+1} \right), \\
y'''_{n+\frac{3}{4}} &= y''' + \frac{h}{11200} \left(401f_n + 2232f_{n+\frac{1}{8}} + 18f_{n+\frac{1}{4}} + 3224f_{n+\frac{3}{8}} \right. \\
&\quad \left. - 360f_{n+\frac{1}{2}} + 2664f_{n+\frac{5}{8}} + 158f_{n+\frac{3}{4}} + 72f_{n+\frac{7}{8}} - 9f_{n+1} \right), \\
y'''_{n+\frac{7}{8}} &= y''' + \frac{h}{4147200} \left(149527f_n + 816634f_{n+\frac{1}{8}} + 48706f_{n+\frac{1}{4}} + 1085938f_{n+\frac{3}{8}} \right. \\
&\quad \left. + 54880f_{n+\frac{1}{2}} + 736078f_{n+\frac{5}{8}} + 522046f_{n+\frac{3}{4}} + 223174f_{n+\frac{7}{8}} - 8183f_{n+1} \right), \\
y'''_{n+1} &= y''' + \frac{h}{28350} \left(989f_n + 5888f_{n+\frac{1}{8}} - 928f_{n+\frac{1}{4}} + 10496f_{n+\frac{3}{8}} \right. \\
&\quad \left. - 4540f_{n+\frac{1}{2}} + 10496f_{n+\frac{5}{8}} - 928f_{n+\frac{3}{4}} + 5888f_{n+\frac{7}{8}} + 989f_{n+1} \right). \tag{14}
\end{aligned}$$

3. Analysis of the Properties of the Method

This section presented the analysis of the basic properties of the proposed single-step hybrid multistep method.

3.1. Order of the Method

Since $y(x)$ is continuously differentiable, following [2], the linear difference operator associated with formulas in (11) is defined by:

$$\begin{aligned} & \Upsilon_{\frac{r}{8}} \left\{ y(x) : h \right\} \\ &= y \left(x_n + \frac{r}{8} h \right) - \left\{ \sum_{b=0}^3 \alpha_b \left(\frac{r}{8} \right) y^{(b)}(x) h^b - h^4 \sum_{b=0}^3 \beta_b \left(\frac{r}{8} \right) y^{(4)} \left(x_n + \frac{r}{8} h \right), r = 1, 2, 3 \right\}. \end{aligned} \tag{16}$$

Taking $y(x)$ as the valid solution of (1), the Taylor series expansion about the point x after using (16) gives a formula for the local truncation error written as:

$$\begin{aligned} \Upsilon_{\frac{r}{8}} \left\{ y(x) : h \right\} &= c_0 y(x) + c_1 h y'(x) + c_2 h^2 y''(x) + \dots + c_{p+3} h^{p+3} y^{(p+3)}(x) \\ &+ c_{p+4} h^{p+4} y^{(p+4)}(x). \end{aligned} \tag{17}$$

The term c_{p+4} is called the error constant and implies that the local truncation error is given by:

$$\Upsilon_{\frac{r}{8}} \left\{ y(x) : h \right\} = c_{p+4} h^{p+4} y^{(p+4)}(x_n) + Oh^{p+5} \tag{18}$$

Since $c_0 = c_1 = \dots = c_{p+3} = 0, c_{p+4} \neq 0$, refer to [13]; then formulas in (11) have uniform order $p = 9$ with error constant given by

$$\begin{aligned} c_{p+4} &= \left(\frac{100009549}{143780317296063204556800}, \frac{100009549}{143780317296063204556800}, \right. \\ &\frac{2851897}{280820932218873446400}, \frac{8093367}{197229516181156659200}, \\ &\frac{14479}{137119595809996800}, \frac{1243375625}{5751212691842528182272}, \\ &\left. \frac{148383}{385213898791321600}, \frac{1833526051}{2934292189715575603200}, \frac{8123}{8569974738124800} \right) \end{aligned}$$

This procedure can be repeated for the formulas in (12)-(14) to obtain their respective error constants.

3.2. Zero Stability of the Block Method

According to [3] and [14], a block method is zero stable if the zeros of the characteristic polynomial satisfy $|\eta_i| \leq 1$ and the root $|\eta_i| = 1$ has multiplicity not exceeding the order of the differential equation. Moreover, this kind of stability tells the behaviour of the numerical method as $h^p \rightarrow 0$. Setting h to zero in the formulas (11)-(14), its reduced to the form that satisfies $\Pi(\eta_i) = |\eta_i(\phi_{m,r}) - \nabla_{m,r}|$, whose the characteristics equation is $\eta_i^{28}(\eta_i - 1)^4 = 0$. It is evident that the SHMM is zero stable since the multiplicity of roots $|\eta_i| = 1$ does not exceed the order of

the differential equation under study.

3.3. Convergence

According to [15], the necessary and sufficient condition for a numerical method to be convergent is to be consistent and Zero stable. The satisfactory condition for the proposed method SHMM to be consistent is that it must have order $p \geq 1$ (see [16]). Section 3.2 established zero stability of the proposed SHMM, and it is correct to conclude that SHMM is convergent.

3.4. Region of Absolute Stability of the Method SHMM

This section studies the region of absolute stability of the proposed Single-step Hybrid Multistep Method (SHMM). Substituting the test problems:

$$y' = -vy, \quad y'' = -v^2y, \quad y''' = -v^3y, \quad y^{(4)} = -v^4y$$

into formulas in (11) and then combined as a block:

$$\widehat{U}_0 Y_{n+r} = \widehat{U}_1 Y_n + h\widehat{U}_2 Y'_n + h^2\widehat{U}_3 Y''_n + h^3\widehat{U}_4 Y'''_n + h^4 (\mathfrak{B}_0 F_n + \mathfrak{B}_1 F_{n+1}), \quad (19)$$

where vectors,

$$\begin{aligned} Y_{n+r} &= \left(y_{n+\frac{1}{8}}, y_{n+\frac{2}{8}}, y_{n+\frac{3}{8}}, y_{n+\frac{4}{8}}, y_{n+\frac{5}{8}}, y_{n+\frac{6}{8}}, y_{n+\frac{7}{8}}, y_{n+1} \right)^T, \\ Y_n &= \left(y_{n-\frac{1}{8}}, y_{n-\frac{2}{8}}, y_{n-\frac{3}{8}}, y_{n-\frac{4}{8}}, y_{n-\frac{5}{8}}, y_{n-\frac{6}{8}}, y_{n-\frac{7}{8}}, y_n \right)^T, \\ Y'_n &= \left(y'_{n-\frac{1}{8}}, y'_{n-\frac{2}{8}}, y'_{n-\frac{3}{8}}, y'_{n-\frac{4}{8}}, y'_{n-\frac{5}{8}}, y'_{n-\frac{6}{8}}, y'_{n-\frac{7}{8}}, y'_n \right)^T, \\ Y''_n &= \left(y''_{n-\frac{1}{8}}, y''_{n-\frac{2}{8}}, y''_{n-\frac{3}{8}}, y''_{n-\frac{4}{8}}, y''_{n-\frac{5}{8}}, y''_{n-\frac{6}{8}}, y''_{n-\frac{7}{8}}, y''_n \right)^T, \\ Y'''_n &= \left(y'''_{n-\frac{1}{8}}, y'''_{n-\frac{2}{8}}, y'''_{n-\frac{3}{8}}, y'''_{n-\frac{4}{8}}, y'''_{n-\frac{5}{8}}, y'''_{n-\frac{6}{8}}, y'''_{n-\frac{7}{8}}, y'''_n \right)^T, \\ F_n &= \left(f_{n-\frac{1}{8}}, f_{n-\frac{2}{8}}, f_{n-\frac{3}{8}}, f_{n-\frac{4}{8}}, f_{n-\frac{5}{8}}, f_{n-\frac{6}{8}}, f_{n-\frac{7}{8}}, f_n \right)^T, \\ F_{n+1} &= \left(f_{n+\frac{1}{8}}, f_{n+\frac{2}{8}}, f_{n+\frac{3}{8}}, f_{n+\frac{4}{8}}, f_{n+\frac{5}{8}}, f_{n+\frac{6}{8}}, f_{n+\frac{7}{8}}, f_{n+1} \right)^T, \end{aligned}$$

$\widehat{U}_0, \widehat{U}_1, \widehat{U}_2, \widehat{U}_3, \widehat{U}_4, \mathfrak{B}_0,$ and \mathfrak{B}_1 are matrices of dimension eight (8) whose entries are the coefficients of formulas in (11). Regarding (19) the amplification matrix obtained is:

$$M(\zeta) = \Gamma^{-1} \mathbf{Z} \quad (20)$$

where $\Gamma = (\widehat{U}_0 - \mathfrak{B}_1)$ and $\mathbf{Z} = (\widehat{U}_1 + \widehat{U}_2 + \widehat{U}_3 + \widehat{U}_4 + \mathfrak{B}_0)$. Further analysis of the amplification matrix $M(\zeta)$ give the eigenvalues

$\{\gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5, \gamma_6, \gamma_7, \gamma_8\} = \{0, 0, 0, 0, 0, 0, 0, \gamma_8\}$. The dominant eigenvalue γ_8

is a function of ζ where $\zeta = v^4 h^4$. **Figure 1** shows the absolute stability region where the proposed method exhibited the behaviours of the true solution.

4. Numerical Experiments

Four sample problems are considered as numerical examples to test the usability of the proposed SHMM. The accuracy of the method was evaluated by calculating the absolute error generated when applied to the sample problems.

4.1. Test Problem 1

The general fourth-order IVP of ordinary differential equation

$$y^{(4)} = y''' + y'' + y' + 2y, \quad y(0) = 1, \quad y'(0) = 0, \quad y''(0) = 0, \quad y'''(0) = 30$$

whose exact solution is reported as $y(x) = 2e^{2x} - 5e^{-x} + 3\cos x - 9\sin x$ is considered as the first test problem. The solutions to problem 1 were obtained within $[0, 1]$ over 20 iterations and are compared with the exact solution, as presented in **Figure 2(a)**. It is clear from **Table 1** and **Figure 2(b)** that SHMM shows good performance over the methods in [7] and [8] up to at least seven decimal digits.

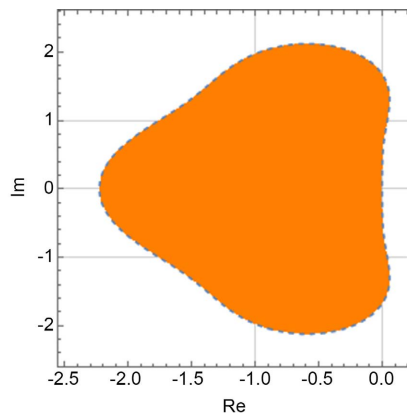


Figure 1. Stability plot of SHMM.

Table 1. Solution of problem 2 obtained using the proposed method.

x	y-computed	y-exact	Error SHMM	Error in [8]	Error in [7]
0.2	8.229478917040623	8.229478917040623	1.5152 E-20	3.5129E-13	2.319E-13
0.4	7.061099988393325	7.06109998839333	1.4159E-19	4.1833E-12	2.2603E-12
0.6	6.778516610116943	6.778516610116943	5.6530E-19	1.4302E-11	1.9651E-11
0.8	7.786624979322129	7.786624979322128	1.6039E-18	3.5924E-11	9.9145E-11
1.	10.665177458051863	10.665177458051861	3.7919E-18	7.2762E-11	3.3114E-10
1.2	16.251045310569197	16.251045310569197	8.0188E-18	1.3360E-10	9.0000E-10
1.4	25.763132220706712	25.76313222070671	1.5750E-17	2.2345E-10	2.1176E-09
1.6	40.99078198991456	40.99078198991457	2.9382E-17	3.5796E-10	4.5506E-09
1.8	64.57672836648089	64.57672836648088	5.2804E-17	5.4334E-10	9.1180E-09
2.	100.44085913139898	100.44085913139897	9.2291E-17	8.0796E-10	1.7409E-08

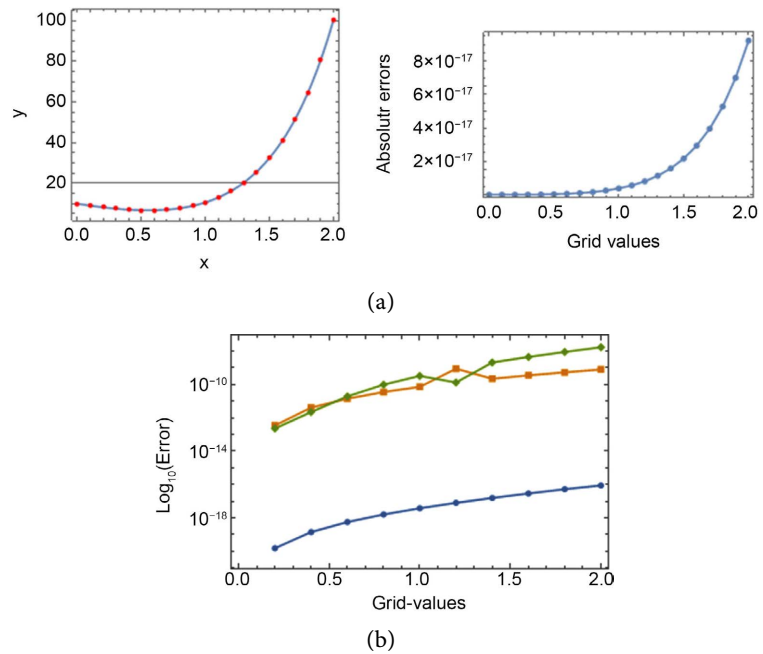


Figure 2. (a) Curves of solutions and behaviour of absolute errors on test problem 21 using the SHMM; (b) Efficiency curves of the maximum absolute errors against grid-values for problem 1.

4.2. Test Problem 2

The second example considered as a test problem is another fourth-order IVP of ordinary differential equation

$$y^{(4)} = y'', y(0) = 0, y'(0) = \frac{-1.1}{72 - 50\pi}, y''(0) = \frac{1}{144 - 100\pi}, y'''(0) = \frac{1.2}{144 - 100\pi},$$

whose exact solution is given as $y(x) = 1 - x - \cos x - y''(0) = \frac{1.2 \sin x}{144 - 100\pi}$. Problem 2 was iterated within $[0, 1]$ for 320 steps. The results are as presented in

Table 2 and Figure 3(a) and Figure 3(b). Table 2 and Figure 3(b) make it evident that SHMM outperforms the approach in [17] up to at least seven decimal digits.

4.3. Test Problem 3: Application: Ship Dynamic Problem (See [8])

We applied the proposed method to solve a physical problem that occurs in ship dynamics. In particular, this problem has been studied and solved numerically by [18], and [19], which describes how the sinusoidal wave of frequency ∇ passes along a ship or offshore structure to lead to a fourth-order differential equation relates to the action of the fluids with time x as below

$$y^{(4)} = -3y'' - y(2 + \phi \cos(\nabla x)), y(0) = 1, y'(0) = 0, y''(0) = 0, y'''(0) = 0,$$

whose exact solution is $y(x) = 2 \cos x - \cos(x\sqrt{2})$ for when $\phi = 0$. This problem was solved within $[0, 15]$ over 150 iterations. The results are reported

in **Table 3** and **Figure 4**. The results of the problem for $h = 0.25$ and 0.1 are compared with those of cited [7] [8] and [18], the suggested method compares well with the cited works (see **Table 4**).

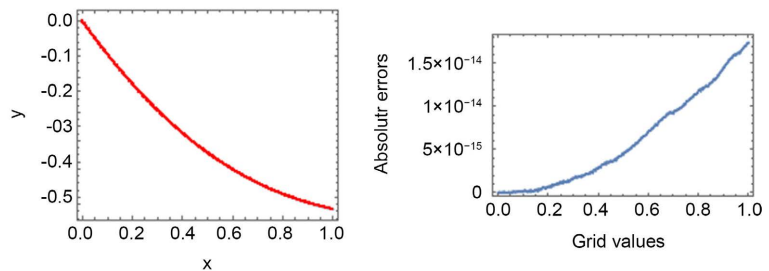
4.4. Test Problem 4

The following nonlinear fourth-order IVP of ordinary differential equation

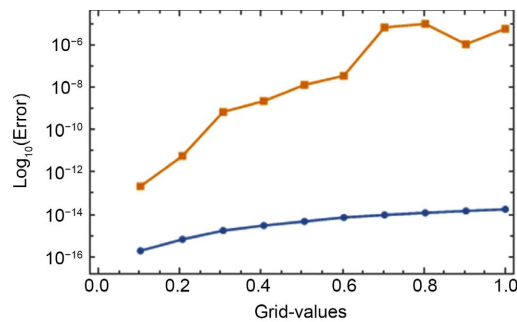
$$y^{(4)} = (y')^2 - yy'' - 4x^2 + e^x (1 + x^2 - 4x),$$

$$y(0) = 1, y'(0) = 1, y''(0) = 3, y'''(0) = 1, [0, 1]$$

with the exact solution $y(x) = x^2 + e^x$ is also considered as a test problem. The test problem was approximated using SHMM within $[0, 1]$ over 5 and 10 iterations, respectively. See **Table 5** for the results as compared with some cited works in the literatures.



(a)



(b)

Figure 3. (a) Curves of solutions and behaviour of absolute errors on test problem 2 using the SHMM; (b) Efficiency curves of the maximum absolute errors against grid-values for Example 2.

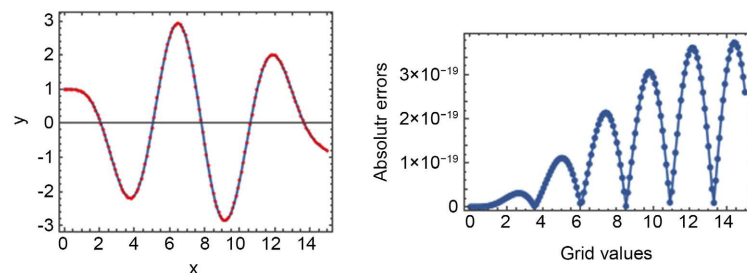


Figure 4. Curves of solutions and behaviours of absolute errors on test problem 3 using the SHMM.

Table 2. Solution of test problem 2 obtained by the proposed method.

x	y-computed	y-exact	Error in SHMM	Error in [16]
0.103125	-0.09708635645894467	-0.09708635645894448	1.9429E-16	2.116E-13
0.206250	-0.18361153147140596	-0.1836115314714053	6.6613E-16	5.6987E-12
0.306250	-0.2575947017186572	-0.2575947017186555	1.7208E-15	6.8031E-10
0.406250	-0.3220723501790842	-0.3220723501790812	2.9976E-15	2.2072E-9
0.506250	-0.3773994044471421	-0.37739940444713743	4.6629E-15	1.2741E-8
0.603125	-0.4226919296391457	-0.4226919296391385	7.2165E-15	3.4561E-8
0.703125	-0.46139027951012934	-0.4613902795101199	9.4369E-15	6.5534E-6
0.803125	-0.492512293357425	-0.4925122933574131	1.1935E-14	9.5865E-6
0.903125	-0.5167461772506352	-0.5167461772506207	1.4544E-14	1.0493E-6
1.	-0.5343680701999484	-0.5343680701999309	1.7431E-14	5.6962E-6

Table 3. Solution of the applied example obtained using SHMM.

x	y-computed	y-exact	Error
1.	0.92466091697090490	0.9246609169709051	2.3651E-21
2.	0.11906945503156266	0.11906945503156263	2.1163E-20
3.	-1.5273231359085384	-1.5273231359085389	2.6887E-20
4.	-2.1174708448420190	-2.1174708448420194	4.3095E-20
5.	-0.13802353538198978	-0.13802353538198964	1.1155E-19
6.	2.510535059206008	2.5105350592060085	8.3824E-21
7.	2.3972266325194904	2.3972266325194904	1.8763E-19
8.	-0.603795009129371	-0.6037950091293718	1.4646E-19
9.	-2.809239445368881	-2.8092394453688807	1.6758E-19
10.	-1.6731743960203111	-1.6731743960203105	2.9449E-19
11.	0.9973799806376236	0.9973799806376238	3.4453E-20
12.	1.9910488550789984	1.9910488550789966	3.5679E-19
13.	0.920973191409115	0.9209731914091138	1.3780E-19
14.	-0.30866899231111894	-0.30866899231111894	3.1111E-19
15.	-0.8070186485444422	-0.8070186485444416	2.6094E-19

Table 4. Comparison of the absolute error on test problem 3.

h	Method	Error at $x_N = 15$
0.25	SHMM	2.48E-15
	[8]	8.15E-08
	[7]	5.40E-07
	[19]	1.90E-04
0.1	SHMM	2.61E-19
	[8]	3.60E-11
	[7]	2.80E-10
	[18]	3.7E-05

Table 5. Comparison of the absolute error on test problem 4.

h	Method	Error at $x_N = 15$
0.2	SHMM	9.45E-23
	[20]	1.10E-16
	BHCM4 [20]	5.20E-12
	Adams [20]	5.01E-07
	[9]	5.84E-04
0.1	SHMM	9.71E-20
	[20]	1.77E-11
	BHCM4 [20]	1.95E-14
	Adams [20]	2.44E-06
	[9]	9.26E-05

5. Conclusion

This work proposes a single-step, linear multistep formula (LMF) with block extension for the direct solution of fourth-order ordinary differential equations. The ship dynamics problem and three other standard fourth-order ordinary differential equations are considered test problems to establish the methods' usefulness. The analysis and numerical experiments revealed that the proposed method is efficient with a better degree of accuracy for handling the direct solution of fourth-order ordinary differential equations.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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