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Arithmetic Operations of Generalized Trapezoidal Picture Fuzzy Numbers by Vertex Method

Mohammad Kamrul Hasan[1,2*], Abeda Sultana[1], Nirmal Kanti Mitra[2]

[1]Department of Mathematics, Jahangirnagar University, Savar, Bangladesh
[2]Department of Mathematics and Statistics, Bangladesh University of Business and Technology, Dhaka, Bangladesh
Email: *krul.habi@yahoo.com

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Abstract

In this article, we define the arithmetic operations of generalized trapezoidal picture fuzzy numbers by vertex method which is assembled on a combination of the (α, γ, β)-cut concept and standard interval analysis. Various related properties are explored. Finally, some computations of picture fuzzy functions over generalized picture fuzzy variables are illustrated by using our proposed technique.

Keywords

Picture Fuzzy Set, Generalized Trapezoidal Picture Fuzzy Number, (α, γ, β)-Cut, Arithmetic Operations, Vertex Method

1. Introduction

In the last few decades, the fuzzy set theory [1] and the intuitionistic fuzzy set theory [2] are two strong concepts that successfully handle the uncertain situations in many problems of our real life. In some problems of the uncertain situations, a few researchers also have realized some complications while considering the case of neutrality degree and that is the reason they have paid attention to picture fuzzy set theory [3] [4] and applied it to the field of artificial intelligence, pattern recognition, medical diagnosis and many other decision making problems. The concept of fuzzy numbers was introduced by Chang and Zadeh [5] with some arithmetic operations. The arithmetic operations of fuzzy numbers are the extensions of the operations of classical interval arithmetic operations which was introduced by R.E. Moore [6]. Then a number of researchers studied the concept of fuzzy numbers with their arithmetic operations (see [7]-[15]).

The difficulty arises when the direct interval arithmetic operations are used to compute the fuzzy number valued functions while occurring multivalued fuzzy variables, because classical interval arithmetic operations do not satisfy distributivity. Also the abnormal solution arises for discretization of fuzzy numbers by using the interval arithmetic operations.

To overcome, these difficulties the concept of vertex method was introduced by Dong and Shah [16] in 1987. This method was based on the combination of the α-cut concept and standard interval analysis. This method can prevent abnormality in the output membership function due to application of the discretization technique on the fuzzy variables' domain, and it can prevent the widening of the resulting function value set due to multiple occurrences of variables in the functional expression by conventional interval analysis methods. Also, the vertex method is discussed by Ahmad M.Z. *et al.* [17], Huey-Kuo Chen *et al.* [18] and Sharaf I. M. [19] in fuzzy number and D. Chakraborty *et al.* [20] in intuitionistic fuzzy numbers.

In this article, we define the arithmetic operations on generalized trapezoidal picture fuzzy numbers by vertex method. Some related properties of them are explored. At the end of this work, some computations of picture fuzzy functions over generalized picture fuzzy variables are illustrated by using our proposed technique.

The article is organized as follows: In section 2, some basic definitions and operations are given which are essential to rest of the paper. In section 3, the arithmetic operations of GTraPFNs by vertex method are illustrated. In section 4, some picture fuzzy valued functions are calculated by our proposed method.

2. Preliminaries

Definition 2.1 [1]. Let X be non-empty set. A fuzzy set A in X is given by

$$A=\left\{\left(x,\mu_A(x)\right):x\in X\right\}, \tag{2.1}$$

where $\mu_A:X\to[0,1]$.

Definition 2.2 [2]. An intuitionistic fuzzy set A in X is given by

$$A=\left\{\left(x,\mu_A(x),\nu_A(x)\right):x\in X\right\}, \tag{2.2}$$

where $\mu_A:X\to[0,1]$ and $\nu_A:X\to[0,1]$, with the condition $0\le\mu_A(x)+\nu_A(x)\le 1;\forall x\in X$.

The values $\mu_A(x)$ and $\nu_A(x)$ represent, respectively, the membership degree and non-membership degree of the element x to the set A.

For any intuitionistic fuzzy set A on the universal set X, for $x\in X$

$$h_A(x)=1-\left(\mu_A(x)+\nu_A(x)\right),$$

is called the hesitancy degree (or intuitionistic fuzzy index) of an element x in A. It is the degree of indeterminacy membership of the element x whether belonging to A or not.

Obviously, $0\le h_A(x)\le 1$ for any $x\in X$.

Particularly, $h_A(x)=1-\mu_A(x)-\nu_A(x)$ is always valid for any fuzzy set A on the universal set X. The set of all intuitionistic fuzzy sets in X will be denoted by $IFS(X)$.

Definition 2.3 [3] [4]. A picture fuzzy set A on a universe of discourse X is of the form

$$A=\{(x,\mu_A(x),\eta_A(x),\nu_A(x)):x\in X\}, \tag{2.3}$$

where $\mu_A(x)\in[0,1]$ is called the degree of positive membership of x in A, $\eta_A(x)\in[0,1]$ is called the degree of neutral membership of x in A and $\nu_A(x)\in[0,1]$ is called the degree of negative membership of x in A, and where $\mu_A(x),\eta_A(x)$ and $\nu_A(x)$ satisfy the following condition:

$$0\le\mu_A(x)+\eta_A(x)+\nu_A(x)\le1;\forall x\in X.$$

Here $1-(\mu_A(x)+\eta_A(x)+\nu_A(x));\forall x\in X$ is called the degree of refusal membership of x in A.

The set of all picture fuzzy sets in X will be denoted by $PFS(X)$.

Definition 2.4 [3] [4]. Let $A,B\in PFS(X)$, then the subset, equality, the union, intersection and complement are defined as follows:

1) $A\subseteq B$ iff $\forall x\in X,\mu_A(x)\le\mu_B(x),\eta_A(x)\le\eta_B(x)$ and $\nu_A(x)\ge\nu_B(x)$;

2) $A=B$ iff $\forall x\in X,\mu_A(x)=\mu_B(x),\eta_A(x)=\eta_B(x)$ and $\nu_A(x)=\nu_B(x)$;

3) $A\cup B=\{(x,\max(\mu_A(x),\mu_B(x)),\min(\eta_A(x),\eta_B(x)),\min(\nu_A(x),\nu_B(x))):x\in X\}$;

4) $A\cap B=\{(x,\min(\mu_A(x),\mu_B(x)),\min(\eta_A(x),\eta_B(x)),\max(\nu_A(x),\nu_B(x))):x\in X\}$;

5) $A^c=\{(x,\nu_A(x),\eta_A(x),\mu_A(x)):x\in X\}$.

Definition 2.5. Let $A=\{(x,\mu_A(x),\eta_A(x),\nu_A(x)):x\in X\}$ be a picture fuzzy set on X and $\alpha,\gamma,\beta\in[0,1]$, $\alpha+\gamma+\beta\le1$, then the upper (α,γ,β)-cut of A is given by

$$A^{(\alpha,\gamma,\beta)}=\{x\in X:\mu_A(x)\ge\alpha,\eta_A(x)\ge\gamma,\nu_A(x)\le\beta\} \tag{2.4}$$

That is, $\alpha_{\mu_A}=\{x:\mu_A(x)\ge\alpha\}$, $\gamma_{\eta_A}=\{x:\eta_A(x)\ge\gamma\}$ and $\beta_{\nu_A}=\{x:\nu_A(x)\le\beta\}$ are upper α, γ and β-cut of positive membership, neutral membership and negative membership of a picture fuzzy set A respectively.

Definition 2.6. Let $\omega_1,\omega_2,\omega_3\in[0,1]$ with $0\le\omega_1+\omega_2+\omega_3\le1$. A generalized picture fuzzy number (GPFN) $\tilde{\rho}$ is a special picture fuzzy set of real numbers $\mathbb{R}$ whose membership functions $\mu_{\tilde{\rho}}(x):\mathbb{R}\to[0,\omega_1]$, $\eta_{\tilde{\rho}}(x):\mathbb{R}\to[0,\omega_2]$ and $\nu_{\tilde{\rho}}(x):\mathbb{R}\to[\omega_3,1]$ satisfy the following conditions:

1) There exist at least three real numbers x_1, x_2 and x_3 such that $\mu_{\tilde{\rho}}(x_1)=\omega_1$, $\eta_{\tilde{\rho}}(x_2)=\omega_2$ and $\nu_{\tilde{\rho}}(x_3)=\omega_3$.

2) $\mu_{\tilde{\rho}}$ and $\eta_{\tilde{\rho}}$ are quasi concave and upper semi continuous on $\mathbb{R}$.

3) $\nu_{\tilde{\rho}}$ is quasi convex and lower semi continuous on $\mathbb{R}$.

4) The support of $\tilde{\rho}$ is compact.

Definition 2.7. A generalized trapezoidal picture fuzzy number (GTraPFN)

$\tilde{\rho}=\langle(a,b,c,d);\omega_1,\omega_2,\omega_3\rangle$ is a special picture fuzzy set on $\mathbb{R}$ whose positive, neutral and negative membership functions are defined as follows:

$$\mu_{\tilde{\rho}}(x)=\begin{cases}0; & x<a\\ \dfrac{\omega_1(x-a)}{(b-a)}; & a\le x<b\\ \omega_1; & b\le x\le c\\ \dfrac{\omega_1(d-x)}{(d-c)}; & c<x\le d\\ 0; & x>d\end{cases},$$

$$\eta_{\tilde{\rho}}(x)=\begin{cases}0; & x<a\\ \dfrac{\omega_2(x-a)}{(b-a)}; & a\le x<b\\ \omega_2; & b\le x\le c\\ \dfrac{\omega_2(d-x)}{(d-c)}; & c<x\le d\\ 0; & x>d\end{cases},\quad \nu_{\tilde{\rho}}(x)=\begin{cases}1; & x<a\\ \dfrac{b-x+\omega_3(x-a)}{(b-a)}; & a\le x<b\\ \omega_3; & b\le x\le c\\ \dfrac{x-c+\omega_3(d-x)}{(d-c)}; & c<x\le d\\ 1; & x>d\end{cases}$$

Following **Figure 1** is the graphical represent the GTraPFN:

Definition 2.8. A GTraPFN $A=\langle(a_1,a_2,a_3,a_4);\omega_1,\omega_2,\omega_3\rangle$ is said to be monotonic increasing if $a_1\le a_2\le a_3\le a_4$.

Definition 2.9. Let $A=\langle(a,b,c,d);\omega_1,\omega_2,\omega_3\rangle$ be a GTraPFN. Then the α-cut of A is a crisp subset of X which is defined as follows:

$$A^{\alpha}=\{x:\mu_A(x)\ge\alpha\}=[A_1(\alpha),A_2(\alpha)]=\left[a+\frac{\alpha}{\omega_1}(b-a),d-\frac{\alpha}{\omega_1}(d-c)\right];\ \alpha\in[0,\omega_1]$$

Definition 2.10. Let $A=\langle(a,b,c,d);\omega_1,\omega_2,\omega_3\rangle$ be a GTraPFN. Then the γ-cut of A is a crisp subset of X which is defined as follows:

$$A^{\gamma}=\{x:\mu_A(x)\ge\gamma\}=[A_1(\gamma),A_2(\gamma)]=\left[a+\frac{\gamma}{\omega_2}(b-a),d-\frac{\gamma}{\omega_2}(d-c)\right];\ \gamma\in[0,\omega_2]$$

Definition 2.11. Let $A=\langle(a,b,c,d);\omega_1,\omega_2,\omega_3\rangle$ be a GTraPFN. Then the β-cut of A is a crisp subset of X which is defined as follows:

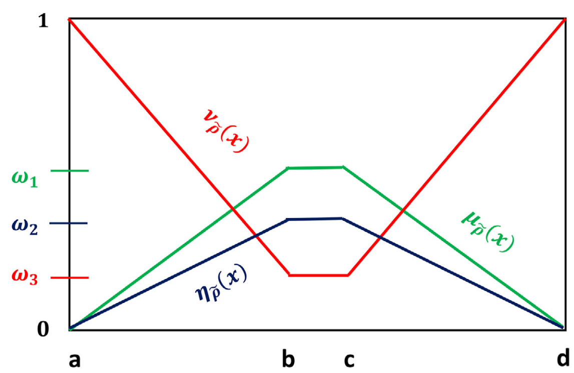


Figure 1. Generalized TraPFN.

$$A^{\beta}=\{x:\mu_A(x)\le\beta\}=[A_1(\beta),A_2(\beta)]=\left[\frac{(b-\omega_3 a)-\beta(b-a)}{1-\omega_3},\frac{\beta(d-c)-(\omega_3 d-c)}{1-\omega_3}\right];\ \beta\in[\omega_3,1]$$

Definition 2.12. Let $A=\langle(a,b,c,d);\omega_1,\omega_2,\omega_3\rangle$ be a GTraPFN. Then the (α,γ,β)-cut of A is given by

$$A^{(\alpha,\gamma,\beta)}=\{[A_1(\alpha),A_2(\alpha)],[A_1(\gamma),A_2(\gamma)],[A_1(\beta),A_2(\beta)]\};$$
$$0<\alpha+\gamma+\beta\le1,\ \alpha\in[0,\omega_1],\ \gamma\in[0,\omega_2],\ \beta\in[\omega_3,1].$$

Definition 2.13. Let A and B be two GTraPFNs and their corresponding (α,γ,β)-cuts are

$$A^{(\alpha,\gamma,\beta)}=\{[A_1(\alpha),A_2(\alpha)],[A_1(\gamma),A_2(\gamma)],[A_1(\beta),A_2(\beta)]\} \tag{2.5}$$

$$B^{(\alpha,\gamma,\beta)}=\{[B_1(\alpha),B_2(\alpha)],[B_1(\gamma),B_2(\gamma)],[B_1(\beta),B_2(\beta)]\} \tag{2.6}$$

for any $\alpha,\gamma,\beta\in[0,1]$ and $0\le\alpha+\gamma+\beta\le1$, then the four basic arithmetic operations such as addition, subtraction, multiplication and division are defined as follows:

Addition:

$$(A+B)^{(\alpha,\gamma,\beta)}$$
$$=\{[A_1(\alpha)+B_1(\alpha),A_2(\alpha)+B_2(\alpha)],[A_1(\gamma)+B_1(\gamma),A_2(\gamma)+B_2(\gamma)],[A_1(\beta)+B_1(\beta),A_2(\beta)+B_2(\beta)]\}$$

Subtraction:

$$A^{(\alpha,\gamma,\beta)}-B^{(\alpha,\gamma,\beta)}$$
$$=\{[A_1(\alpha)-B_2(\alpha),A_2(\alpha)-B_1(\alpha)],[A_1(\gamma)-B_2(\gamma),A_2(\gamma)-B_1(\gamma)],[A_1(\beta)-B_2(\beta),A_2(\beta)-B_1(\beta)]\}$$

Multiplication:

$$A^{(\alpha,\gamma,\beta)}\times B^{(\alpha,\gamma,\beta)}=\{[\min M^{\alpha},\max M^{\alpha}],[\min M^{\gamma},\max M^{\gamma}],[\min M^{\beta},\max M^{\beta}]\}$$

$$M^{\alpha}=\{A_1(\alpha).B_1(\alpha),A_1(\alpha).B_2(\alpha),A_2(\alpha).B_1(\alpha),A_2(\alpha).B_2(\alpha)\}$$

$$M^{\gamma}=\{A_1(\gamma).B_1(\gamma),A_1(\gamma).B_2(\gamma),A_2(\gamma).B_1(\gamma),A_2(\gamma).B_2(\gamma)\}$$

$$M^{\beta}=\{A_1(\beta).B_1(\beta),A_1(\beta).B_2(\beta),A_2(\beta).B_1(\beta),A_2(\beta).B_2(\beta)\}$$

Division:

$$A^{(\alpha,\gamma,\beta)}\div B^{(\alpha,\gamma,\beta)}=\{[\min D^{\alpha},\max D^{\alpha}],[\min D^{\gamma},\max D^{\gamma}],[\min D^{\beta},\max D^{\beta}]\}$$

$$D^{\alpha}=\left\{\frac{A_1(\alpha)}{B_1(\alpha)},\frac{A_1(\alpha)}{B_2(\alpha)},\frac{A_2(\alpha)}{B_1(\alpha)},\frac{A_2(\alpha)}{B_2(\alpha)}\right\}$$

$$D^{\gamma}=\left\{\frac{A_1(\gamma)}{B_1(\gamma)},\frac{A_1(\gamma)}{B_2(\gamma)},\frac{A_2(\gamma)}{B_1(\gamma)},\frac{A_2(\gamma)}{B_2(\gamma)}\right\}$$

$$D^{\beta}=\left\{\frac{A_1(\beta)}{B_1(\beta)},\frac{A_1(\beta)}{B_2(\beta)},\frac{A_2(\beta)}{B_1(\beta)},\frac{A_2(\beta)}{B_2(\beta)}\right\}$$

where $B_i(\alpha)\ne0,B_i(\gamma)\ne0,B_i(\beta)\ne0;i=1,2$.

Definition 2.14. Vertex Method [14].

When $y = f(x_1, x_2, \cdots, x_n)$ is continuous in the *n*-dimensional rectangular region and also no extreme point exists in this region (including the boundaries) then the value of interval function can be obtained by

$$Y = f(X_1, X_2, \cdots, X_n) = \left[\min_j \left(f(c_j)\right), \max_j \left(f(c_j)\right)\right], j = 1, 2, \cdots, N$$

where c_j is the ordinate of the *j*-th vertex and $X_1, X_2, \cdots, X_n$ are interval of real numbers.

3. Arithmetic Operations of GTraPFN by Vertex Method

Definition 3.1: Let $A = \langle (a_1, a_2, a_3, a_4); \omega_1, \omega_2, \omega_3 \rangle$ and $B = \langle (b_1, b_2, b_3, b_4); \omega_1, \omega_2, \omega_3 \rangle$ be two GTraPFNs. Let $C^{(\alpha,\gamma,\beta)} = A^{(\alpha,\gamma,\beta)} * B^{(\alpha,\gamma,\beta)}$, where, $* = +, -, \times, \div$.

Now, the ordinate of the vertices for the positive membership function are

$$c_1 = \left(a_1 + \frac{\alpha}{\omega_1}(a_2 - a_1), b_1 + \frac{\alpha}{\omega_1}(b_2 - b_1) \right),$$

$$c_2 = \left(a_1 + \frac{\alpha}{\omega_1}(a_2 - a_1), b_4 - \frac{\alpha}{\omega_1}(b_4 - b_3) \right),$$

$$c_3 = \left(a_4 - \frac{\alpha}{\omega_1}(a_4 - a_3), b_1 + \frac{\alpha}{\omega_1}(b_2 - b_1) \right)$$

and

$$c_4 = \left(a_4 - \frac{\alpha}{\omega_1}(a_4 - a_3), b_4 - \frac{\alpha}{\omega_1}(b_4 - b_3) \right)$$

Therefore,

$$f(c_1) = \left(a_1 + \frac{\alpha}{\omega_1}(a_2 - a_1) \right) * \left(b_1 + \frac{\alpha}{\omega_1}(b_2 - b_1) \right)$$
$$= a_1 * b_1 + \frac{\alpha}{\omega_1}\left\{ (a_2 - a_1) * (b_2 - b_1) \right\}$$

$$f(c_2) = \left(a_1 + \frac{\alpha}{\omega_1}(a_2 - a_1) \right) * \left(b_4 - \frac{\alpha}{\omega_1}(b_4 - b_3) \right)$$
$$= a_1 * b_4 + \frac{\alpha}{\omega_1}\left\{ (a_2 - a_1) * \left(-(b_4 - b_3)\right) \right\}$$

$$f(c_3) = \left(a_4 - \frac{\alpha}{\omega_1}(a_4 - a_3) \right) * \left(b_1 + \frac{\alpha}{\omega_1}(b_2 - b_1) \right)$$
$$= a_4 * b_1 + \frac{\alpha}{\omega_1}\left\{ \left(-(a_4 - a_3)\right) * (b_2 - b_1) \right\}$$

$$f(c_4) = \left(a_4 - \frac{\alpha}{\omega_1}(a_4 - a_3) \right) * \left(b_4 - \frac{\alpha}{\omega_1}(b_4 - b_3) \right)$$
$$= a_4 * b_4 + \frac{\alpha}{\omega_1}\left\{ \left(-(a_4 - a_3)\right) * \left(-(b_4 - b_3)\right) \right\}$$

where $\alpha \in [0,\omega_1]$, $\gamma \in [0,\omega_2]$, $\beta \in [\omega_3,1]$.

Now, since $a_1 \le a_2 \le a_3 \le a_4$ and $b_1 \le b_2 \le b_3 \le b_4$, so $f(c_1) < f(c_2) < f(c_3) < f(c_4)$. Hence,

$$C^{\alpha} = \left[\min\left\{f(c_1), f(c_2), f(c_3), f(c_4)\right\}, \max\left\{f(c_1), f(c_2), f(c_3), f(c_4)\right\}\right] = \left[f(c_1), f(c_4)\right].$$

By applying similar process for neutral membership, we can find C^{γ}.

Again, the ordinate of the negative membership function

$$c_1 = \left(\frac{(a_2-\omega_3 a_1)-\beta(a_2-a_1)}{1-\omega_3}, \frac{(b_2-\omega_3 b_1)-\beta(b_2-b_1)}{1-\omega_3}\right),$$

$$c_2 = \left(\frac{(a_2-\omega_3 a_1)-\beta(a_2-a_1)}{1-\omega_3}, \frac{\beta(b_4-b_3)-(\omega_3 b_4-b_3)}{1-\omega_3}\right),$$

$$c_3 = \left(\frac{\beta(a_4-a_3)-(\omega_3 a_4-a_3)}{1-\omega_3}, \frac{(b_2-\omega_3 b_1)-\beta(b_2-b_1)}{1-\omega_3}\right)$$

and

$$c_4 = \left(\frac{\beta(a_4-a_3)-(\omega_3 a_4-a_3)}{1-\omega_3}, \frac{\beta(b_4-b_3)-(\omega_3 b_4-b_3)}{1-\omega_3}\right)$$

Therefore,

$$f(c_1) = \frac{(a_2-\omega_3 a_1)-\beta(a_2-a_1)}{1-\omega_{3a}} * \frac{(b_2-\omega_3 b_1)-\beta(b_2-b_1)}{1-\omega_{3b}} = \frac{(a_2-\omega_3 a_1)-\beta(a_2-a_1)*(b_2-\omega_3 b_1)-\beta(b_2-b_1)}{1-\omega_3}$$

$$f(c_2) = \frac{(a_2-\omega_3 a_1)-\beta(a_2-a_1)}{1-\omega_3} * \frac{\beta(b_4-b_3)-(\omega_3 b_4-b_3)}{1-\omega_3} = \frac{(a_2-\omega_3 a_1)-\beta(a_2-a_1)*\beta(b_4-b_3)-(\omega_3 b_4-b_3)}{1-\omega_3}$$

$$f(c_3) = \frac{\beta(a_4-a_3)-(\omega_3 a_4-a_3)}{1-\omega_3} * \frac{(b_2-\omega_3 b_1)-\beta(b_2-b_1)}{1-\omega_3} = \frac{\beta(a_4-a_3)-(\omega_3 a_4-a_3)*(b_2-\omega_3 b_1)-\beta(b_2-b_1)}{1-\omega_3}$$

$$f(c_4) = \frac{\beta(a_4-a_3)-(\omega_3 a_4-a_3)}{1-\omega_3} * \frac{\beta(b_4-b_3)-(\omega_3 b_4-b_3)}{1-\omega_3} = \frac{\beta(a_4-a_3)-(\omega_3 a_4-a_3)*\beta(b_4-b_3)-(\omega_3 b_4-b_3)}{1-\omega_3}$$

Now, since $a_1 \le a_2 \le a_3 \le a_4$ and $b_1 \le b_2 \le b_3 \le b_4$, so $f(c_1) < f(c_2) < f(c_3) < f(c_4)$. Hence,

$$C^{\beta} = \left[\min\left\{f(c_1), f(c_2), f(c_3), f(c_4)\right\}, \max\left\{f(c_1), f(c_2), f(c_3), f(c_4)\right\}\right] = \left[f(c_1), f(c_4)\right]$$

The above method is known as vertex method for arithmetic operations of GTraPFN.

Proposition 3.2: Addition of two generalized trapezoidal picture fuzzy numbers is also a generalized trapezoidal picture fuzzy number.

Proof: Let $A=\langle(a_1,a_2,a_3,a_4);\omega_1,\omega_2,\omega_3\rangle$ and $B=\langle(b_1,b_2,b_3,b_4);\omega_1,\omega_2,\omega_3\rangle$ be two GTraPFNs and $C^{(\alpha,\gamma,\beta)}=A^{(\alpha,\gamma,\beta)}+B^{(\alpha,\gamma,\beta)}$.

Now, the ordinate of the vertices for the positive membership function are

$$c_1=\left(a_1+\frac{\alpha}{\omega_1}(a_2-a_1),b_1+\frac{\alpha}{\omega_1}(b_2-b_1)\right),$$

$$c_2=\left(a_1+\frac{\alpha}{\omega_1}(a_2-a_1),b_4-\frac{\alpha}{\omega_1}(b_4-b_3)\right)$$

$$c_3=\left(a_4-\frac{\alpha}{\omega_1}(a_4-a_3),b_1+\frac{\alpha}{\omega_1}(b_2-b_1)\right)$$

and

$$c_4=\left(a_4-\frac{\alpha}{\omega_1}(a_4-a_3),b_4-\frac{\alpha}{\omega_1}(b_4-b_3)\right)$$

Therefore,

$$f(c_1)=a_1+\frac{\alpha}{\omega_1}(a_2-a_1)+b_1+\frac{\alpha}{\omega_1}(b_2-b_1)=a_1+b_1+\frac{\alpha}{\omega_1}(b_2+a_2-a_1-b_1)$$

$$f(c_2)=a_1+\frac{\alpha}{\omega_1}(a_2-a_1)+b_4-\frac{\alpha}{\omega_1}(b_4-b_3)=a_1+b_4+\frac{\alpha}{\omega_1}(a_2-b_4-a_1+b_3)$$

$$f(c_3)=a_4-\frac{\alpha}{\omega_1}(a_4-a_3)+b_1+\frac{\alpha}{\omega_1}(b_2-b_1)=a_4+b_1+\frac{\alpha}{\omega_1}(b_2-a_4+a_3-b_1)$$

$$f(c_4)=a_4-\frac{\alpha}{\omega_1}(a_4-a_3)+b_4-\frac{\alpha}{\omega_1}(b_4-b_3)=a_4+b_4+\frac{\alpha}{\omega_1}(a_3+b_3-a_4-b_4)$$

Now, since $a_1\le a_2\le a_3\le a_4$ and $b_1\le b_2\le b_3\le b_4$, so $f(c_1)<f(c_2)<f(c_3)<f(c_4)$. Hence,

$$C^{\alpha}=\left[\min\{f(c_1),f(c_2),f(c_3),f(c_4)\},\max\{f(c_1),f(c_2),f(c_3),f(c_4)\}\right]=\left[f(c_1),f(c_4)\right]$$

$$=\left[a_1+b_1+\frac{\alpha}{\omega_1}(b_2+a_2-a_1-b_1),a_4+b_4+\frac{\alpha}{\omega_1}(a_3+b_3-a_4-b_4)\right]$$

$$=\left[a_1+b_1+\frac{\alpha}{\omega_1}(b_2+a_2-a_1-b_1),a_4+b_4+\frac{\alpha}{\omega_1}(a_3+b_3-a_4-b_4)\right]$$

$$=\left[a_1+b_1+\frac{\alpha}{\omega_1}\{(a_2-a_1)+(b_2-b_1)\},a_4+b_4-\frac{\alpha}{\omega_1}\{(a_4-a_3)+(b_4-b_3)\}\right]$$

Let $a_1+b_1+\frac{\alpha}{\omega_1}\{(a_2-a_1)+(b_2-b_1)\}\le z\le a_4+b_4-\frac{\alpha}{\omega_1}\{(a_4-a_3)+(b_4-b_3)\}$

Now,

$$a_1+b_1+\frac{\alpha}{\omega_1}\{(a_2-a_1)+(b_2-b_1)\}\le z$$

$$\Rightarrow \omega_1\frac{z-(a_1+b_1)}{\{(a_2+b_2)-(a_1+b_1)\}}\ge\alpha$$

Let $\mu_C^L(z)=\omega_1\dfrac{z-(a_1+b_1)}{\{(a_2+b_2)-(a_1+b_1)\}}$

Now, $\dfrac{\mathrm{d}}{\mathrm{d}z}\mu_C^L(z)=\dfrac{\mathrm{d}}{\mathrm{d}z}\omega_1\dfrac{z-(a_1+b_1)}{\{(a_2+b_2)-(a_1+b_1)\}}=\dfrac{\omega_1}{\{(a_2+b_2)-(a_1+b_1)\}}>0$; if $(a_2+b_2)>(a_1+b_1)$.

Therefore, $\mu_C^L(z)$ is an increasing function.

Also, $\mu_C^L(a_1+b_1)=0$, $\mu_C^L(a_2+b_2)=\omega_1$ and $\mu_C^L\left(\dfrac{a_1+b_1+a_2+b_2}{2}\right)>\dfrac{\omega_1}{2}$.

Again,

$$a_4+b_4-\frac{\alpha}{\omega_1}\{(a_4-a_3)+(b_4-b_3)\}\ge z$$

$$\Rightarrow \omega_1\frac{(a_4+b_4)-z}{\{(a_4+b_4)-(a_3+b_3)\}}\ge\alpha$$

Let $\mu_C^R(z)=\omega_1\dfrac{(a_4+b_4)-z}{\{(a_4+b_4)-(a_3+b_3)\}}$

Now, $\dfrac{\mathrm{d}}{\mathrm{d}z}\mu_C^R(z)=\dfrac{\mathrm{d}}{\mathrm{d}z}\omega_1\dfrac{(a_4+b_4)-z}{\{(a_4+b_4)-(a_3+b_3)\}}=-\dfrac{\omega_1}{\{(a_4+b_4)-(a_3+b_3)\}}<0$; if $(a_4+b_4)>(a_3+b_3)$

Therefore, $\mu_C^R(z)$ is an decreasing function.

Also, $\mu_C^R(a_4+b_4)=0$, $\mu_C^R(a_3+b_3)=\omega_1$ and $\mu_C^R\left(\dfrac{a_3+b_3+a_4+b_4}{2}\right)<\dfrac{\omega_1}{2}$.

So the positive membership function of $C=A+B$ is

$$\mu_C(z)=\begin{cases}\omega_1\dfrac{z-(a_1+b_1)}{\{(a_2+b_2)-(a_1+b_1)\}}; & a_1+b_1\le z\le a_2+b_2\\ \omega_1; & a_2+b_2\le z\le a_3+b_3\\ \omega_1\dfrac{(a_4+b_4)-z}{\{(a_4+b_4)-(a_3+b_3)\}}; & a_3+b_3\le z\le a_4+b_4\\ 0; & \text{otherwise}\end{cases}$$

Hence, the addition rule is proved for the positive membership function.

Similarly, we can prove the addition rule for the neutral membership function.

$$\eta_C(z)=\begin{cases}\omega_2\dfrac{z-(a_1+b_1)}{\{(a_2+b_2)-(a_1+b_1)\}}; & a_1+b_1\le z\le a_2+b_2\\ \omega_2; & a_2+b_2\le z\le a_3+b_3\\ \omega_2\dfrac{(a_4+b_4)-z}{\{(a_4+b_4)-(a_3+b_3)\}}; & a_3+b_3\le z\le a_4+b_4\\ 0; & \text{otherwise}\end{cases}$$

Now, the ordinate of the negative membership function

$$c_1=\left(\frac{(a_2-\omega_3a_1)-\beta(a_2-a_1)}{1-\omega_3},\frac{(b_2-\omega_3b_1)-\beta(b_2-b_1)}{1-\omega_3}\right),$$

$$c_2=\left(\frac{(a_2-\omega_3a_1)-\beta(a_2-a_1)}{1-\omega_3},\frac{\beta(b_4-b_3)-(\omega_3b_4-b_3)}{1-\omega_3}\right),$$

$$c_3=\left(\frac{\beta(a_4-a_3)-(\omega_3a_4-a_3)}{1-\omega_3},\frac{(b_2-\omega_3b_1)-\beta(b_2-b_1)}{1-\omega_3}\right)$$

and

$$c_4=\left(\frac{\beta(a_4-a_3)-(\omega_3a_4-a_3)}{1-\omega_3},\frac{\beta(b_4-b_3)-(\omega_3b_4-b_3)}{1-\omega_3}\right)$$

Therefore,

$$f(c_1)=\frac{(a_2-\omega_3a_1)-\beta(a_2-a_1)}{1-\omega_3}+\frac{(b_2-\omega_3b_1)-\beta(b_2-b_1)}{1-\omega_3}=\frac{(a_2-\omega_3a_1)-\beta(a_2-a_1)+(b_2-\omega_3b_1)-\beta(b_2-b_1)}{1-\omega_3}$$

$$f(c_2)=\frac{(a_2-\omega_3a_1)-\beta(a_2-a_1)}{1-\omega_3}+\frac{\beta(b_4-b_3)-(\omega_3b_4-b_3)}{1-\omega_3}=\frac{(a_2-\omega_3a_1)-\beta(a_2-a_1)+\beta(b_4-b_3)-(\omega_3b_4-b_3)}{1-\omega_3}$$

$$f(c_3)=\frac{\beta(a_4-a_3)-(\omega_3a_4-a_3)}{1-\omega_3}+\frac{(b_2-\omega_3b_1)-\beta(b_2-b_1)}{1-\omega_3}=\frac{\beta(a_4-a_3)-(\omega_3a_4-a_3)+(b_2-\omega_3b_1)-\beta(b_2-b_1)}{1-\omega_3}$$

$$f(c_4)=\frac{\beta(a_4-a_3)-(\omega_3a_4-a_3)}{1-\omega_3}+\frac{\beta(b_4-b_3)-(\omega_3b_4-b_3)}{1-\omega_3}=\frac{\beta(a_4-a_3)-(\omega_3a_4-a_3)+\beta(b_4-b_3)-(\omega_3b_4-b_3)}{1-\omega_3}$$

Now, since $a_1\le a_2\le a_3\le a_4$ and $b_1\le b_2\le b_3\le b_4$, so $f(c_1)<f(c_2)<f(c_3)<f(c_4)$. Hence,

$$C^\beta=\left[\min\{f(c_1),f(c_2),f(c_3),f(c_4)\},\max\{f(c_1),f(c_2),f(c_3),f(c_4)\}\right]=\left[f(c_1),f(c_4)\right]$$

$$=\left[\frac{(a_2-\omega_3a_1)-\beta(a_2-a_1)+(b_2-\omega_3b_1)-\beta(b_2-b_1)}{1-\omega_3},\frac{\beta(a_4-a_3)-(\omega_3a_4-a_3)+\beta(b_4-b_3)-(\omega_3b_4-b_3)}{1-\omega_3}\right]$$

$$=\left[\frac{a_2+b_2-\omega_3(a_1+b_1)-\beta\{(a_2-a_1)+(b_2-b_1)\}}{1-\omega_3},\frac{a_3+b_3-\omega_3(a_4+b_4)+\beta\{(a_4-a_3)+(b_4-b_3)\}}{1-\omega_3}\right]$$

Let

$$\frac{a_2+b_2-\omega_3(a_1+b_1)-\beta\{(a_2-a_1)+(b_2-b_1)\}}{1-\omega_3}\le z\le\frac{a_3+b_3-\omega_3(a_4+b_4)+\beta\{(a_4-a_3)+(b_4-b_3)\}}{1-\omega_3}$$

Now,

$$\frac{a_2+b_2-\omega_3(a_1+b_1)-\beta\{(a_2-a_1)+(b_2-b_1)\}}{1-\omega_3}\le z$$

$$\Rightarrow\frac{-(1-\omega_3)z-\omega_3(a_1+b_1)+(a_2+b_2)}{\{(a_2+b_2)-(a_1+b_1)\}}\le\beta$$

Let $\nu_C^L(z) = \dfrac{-(1-\omega_3)z - \omega_3(a_1+b_1) + (a_2+b_2)}{\{(a_2+b_2)-(a_1+b_1)\}}$

Now,

$\dfrac{\mathrm{d}}{\mathrm{d}z}\nu_C^L(z) = \dfrac{\mathrm{d}}{\mathrm{d}z}\dfrac{-(1-\omega_3)z - \omega_3(a_1+b_1) + (a_2+b_2)}{\{(a_2+b_2)-(a_1+b_1)\}} = \dfrac{-(1-\omega_3)}{\{(a_2+b_2)-(a_1+b_1)\}} < 0$; if $(a_2+b_2) > (a_1+b_1)$

Therefore, $\nu_C^L(z)$ is an decreasing function.

Also, $\nu_C^L(a_2+b_2) = \omega_3$, $\nu_C^L(a_1+b_1) = 1$ and $\nu_C^L\left(\dfrac{a_2+b_2+a_1+b_1}{2}\right) < \dfrac{(1+\omega_3)}{2}$.

Again,

$$\frac{a_3+b_3-\omega_3(a_4+b_4)+\beta\{(a_4-a_3)+(b_4-b_3)\}}{1-\omega_3} \ge z$$

$$\Rightarrow \frac{(1-\omega_3)z+\omega_3(a_4+b_4)-(a_3+b_3)}{\{(a_4+b_4)-(a_3+b_3)\}} \le \beta$$

Let $\nu_C^R(z) = \dfrac{(1-\omega_3)z+\omega_3(a_4+b_4)-(a_3+b_3)}{\{(a_4+b_4)-(a_3+b_3)\}}$

Now,

$\dfrac{\mathrm{d}}{\mathrm{d}z}\nu_C^R(z) = \dfrac{\mathrm{d}}{\mathrm{d}z}\dfrac{(1-\omega_3)z+\omega_3(a_4+b_4)-(a_3+b_3)}{\{(a_4+b_4)-(a_3+b_3)\}} = \dfrac{(1-\omega_3)}{\{(a_4+b_4)-(a_3+b_3)\}} > 0$; if $(a_4+b_4) > (a_3+b_3)$

Therefore, $\nu_C^R(z)$ is an increasing function.

Also, $\nu_C^R(a_3+b_3) = \omega_3$, $\nu_C^R(a_4+b_4) = 1$ and $\nu_C^R\left(\dfrac{a_3+b_3+a_4+b_4}{2}\right) > \dfrac{(1+\omega_3)}{2}$.

So the negative membership function of $C = A + B$ is

$$\nu_C(z) = \begin{cases} \dfrac{-(1-\omega_3)z-\omega_3(a_1+b_1)+(a_2+b_2)}{\{(a_2+b_2)-(a_1+b_1)\}}; & a_1+b_1 \le z \le a_2+b_2 \\ \omega_3; & a_2+b_2 \le z \le a_3+b_3 \\ \dfrac{(1-\omega_3)z+\omega_3(a_4+b_4)-(a_3+b_3)}{\{(a_4+b_4)-(a_3+b_3)\}}; & a_3+b_3 \le z \le a_4+b_4 \\ 1; & \text{otherwise} \end{cases}$$

Proposition 3.3: Subtraction of two generalized trapezoidal picture fuzzy numbers is also a generalized trapezoidal picture fuzzy number.

Proof: Let $A = \langle (a_1,a_2,a_3,a_4);\omega_1,\omega_2,\omega_3 \rangle$ and $B = \langle (b_1,b_2,b_3,b_4);\omega_1,\omega_2,\omega_3 \rangle$ be two GTraPFNs and $C^{(\alpha,\gamma,\beta)} = A^{(\alpha,\gamma,\beta)} - B^{(\alpha,\gamma,\beta)}$.

Now, the ordinate of the vertices for the membership function are

$$c_1 = \left(a_1 + \frac{\alpha}{\omega_1}(a_2-a_1), b_1 + \frac{\alpha}{\omega_1}(b_2-b_1)\right),$$

$$c_2 = \left(a_1 + \frac{\alpha}{\omega_1}(a_2-a_1), b_4 - \frac{\alpha}{\omega_1}(b_4-b_3)\right),$$

$$c_3=\left(a_4-\frac{\alpha}{\omega_1}(a_4-a_3),b_1+\frac{\alpha}{\omega_1}(b_2-b_1)\right)$$

and

$$c_4=\left(a_4-\frac{\alpha}{\omega_1}(a_4-a_3),b_4-\frac{\alpha}{\omega_1}(b_4-b_3)\right)$$

Therefore,

$$f(c_1)=a_1+\frac{\alpha}{\omega_1}(a_2-a_1)-b_1-\frac{\alpha}{\omega_1}(b_2-b_1)=a_1-b_1+\frac{\alpha}{\omega_1}(a_2-b_2-a_1+b_1)$$

$$f(c_2)=a_1+\frac{\alpha}{\omega_1}(a_2-a_1)-b_4+\frac{\alpha}{\omega_1}(b_4-b_3)=a_1-b_4+\frac{\alpha}{\omega_1}(a_2+b_4-a_1-b_3)$$

$$f(c_3)=a_4-\frac{\alpha}{\omega_1}(a_4-a_3)-b_1-\frac{\alpha}{\omega_1}(b_2-b_1)=a_4-b_1+\frac{\alpha}{\omega_1}(b_1-a_4+a_3-b_2)$$

$$f(c_4)=a_4-\frac{\alpha}{\omega_1}(a_4-a_3)-b_4+\frac{\alpha}{\omega_1}(b_4-b_3)=a_4-b_4+\frac{\alpha}{\omega_1}(a_3+b_4-a_4-b_3)$$

Now, since $a_1\le a_2\le a_3\le a_4$ and $b_1\le b_2\le b_3\le b_4$, so $f(c_2)<f(c_1)<f(c_4)<f(c_3)$. Hence,

$$C^\alpha=\left[\min\{f(c_1),f(c_2),f(c_3),f(c_4)\},\max\{f(c_1),f(c_2),f(c_3),f(c_4)\}\right]=\left[f(c_2),f(c_3)\right]$$

$$=\left[a_1-b_4+\frac{\alpha}{\omega_1}(a_2+b_4-a_1-b_3),a_4-b_1+\frac{\alpha}{\omega_1}(b_1-a_4+a_3-b_2)\right]$$

$$=\left[a_1-b_4+\frac{\alpha}{\omega_1}(a_2+b_4-a_1-b_3),a_4-b_1-\frac{\alpha}{\omega_1}(-b_1+a_4-a_3+b_2)\right]$$

$$=\left[a_1-b_4+\frac{\alpha}{\omega_1}\{(a_2-a_1)+(b_4-b_3)\},a_4-b_1-\frac{\alpha}{\omega_1}\{(a_4-a_3)+(b_2-b_1)\}\right]$$

$$=\left[a_1-b_4+\frac{\alpha}{\omega_1}\{(a_2-b_3)-(a_1-b_4)\},a_4-b_1-\frac{\alpha}{\omega_1}\{(a_4-b_1)-(a_3-b_2)\}\right]$$

Let $a_1-b_4+\frac{\alpha}{\omega_1}\{(a_2-b_3)-(a_1-b_4)\}\le z\le a_4-b_1-\frac{\alpha}{\omega_1}\{(a_4-b_1)-(a_3-b_2)\}$

Now,

$$a_1-b_4+\frac{\alpha}{\omega_1}\{(a_2-b_3)-(a_1-b_4)\}\le z$$

$$\Rightarrow \omega_1\frac{z-(a_1-b_4)}{\{(a_2-b_3)-(a_1-b_4)\}}\ge\alpha$$

Let $\mu_C^L(z)=\omega_1\frac{z-(a_1-b_4)}{\{(a_2-b_3)-(a_1-b_4)\}}$

Now, $\frac{\mathrm{d}}{\mathrm{d}z}\mu_C^L(z)=\frac{\mathrm{d}}{\mathrm{d}z}\omega_1\frac{z-(a_1-b_4)}{\{(a_2-b_3)-(a_1-b_4)\}}=\frac{\omega_1}{\{(a_2-b_3)-(a_1-b_4)\}}>0$; if $(a_2-b_3)>(a_1-b_4)$

Therefore, $\mu_C^L(z)$ is an increasing function.

Also, $\mu_C^L(a_1-b_4)=\omega_1$, $\mu_C^L(a_2-b_3)=\omega_1$ and $\mu_C^L\left(\frac{a_1-b_4+a_2-b_3}{2}\right)>\frac{\omega_1}{2}$.

Again,

$$a_4-b_1-\frac{\alpha}{\omega_1}\{(a_4-b_1)-(a_3-b_2)\}\ge z$$

$$\Rightarrow \omega_1\frac{(a_4-b_1)-z}{\{(a_4-b_1)-(a_3-b_2)\}}\ge\alpha$$

Let $\mu_C^R(z)=\omega_1\frac{(a_4-b_1)-z}{\{(a_4-b_1)-(a_3-b_2)\}}$

Now, $\frac{\mathrm{d}}{\mathrm{d}z}\mu_C^R(z)=\frac{\mathrm{d}}{\mathrm{d}z}\omega_1\frac{(a_4-b_1)-z}{\{(a_4-b_1)-(a_3-b_2)\}}=-\frac{\omega_1}{\{(a_4-b_1)-(a_3-b_2)\}}<0$; if $(a_4-b_1)>(a_3-b_2)$

Therefore, $\mu_C^R(z)$ is an decreasing function.

Also, $\mu_C^R(a_4-b_1)=0$, $\mu_C^R(a_3-b_2)=\omega_1$ and $\mu_C^R\left(\frac{a_4-b_1+a_3-b_2}{2}\right)<\frac{\omega_1}{2}$.

So the positive membership function of $C=A-B$ is

$$\mu_C(z)=\begin{cases}\omega_1\frac{z-(a_1-b_4)}{\{(a_2-b_3)-(a_1-b_4)\}}; & a_1-b_4\le z\le a_2-b_3\\ \omega_1; & a_2-b_3\le z\le a_3-b_2\\ \omega_1\frac{(a_4-b_1)-z}{\{(a_4-b_1)-(a_3-b_2)\}}; & a_3-b_2\le z\le a_4-b_1\\ 0; & \text{otherwise}\end{cases}$$

Hence, the subtraction rule is proved for the positive membership function.

Similarly, we can prove the subtraction rule for the neutral membership function.

$$\eta_C(z)=\begin{cases}\omega_2\frac{z-(a_1-b_4)}{\{(a_2-b_3)-(a_1-b_4)\}}; & a_1-b_4\le z\le a_2-b_3\\ \omega_2; & a_2-b_3\le z\le a_3-b_2\\ \omega_2\frac{(a_4-b_1)-z}{\{(a_4-b_1)-(a_3-b_2)\}}; & a_3-b_2\le z\le a_4-b_1\\ 0; & \text{otherwise}\end{cases}$$

Now, the ordinate of the negative membership function

$$c_1=\left(\frac{(a_2-\omega_3a_1)-\beta(a_2-a_1)}{1-\omega_3},\frac{(b_2-\omega_3b_1)-\beta(b_2-b_1)}{1-\omega_3}\right),$$

$$c_2=\left(\frac{(a_2-\omega_3a_1)-\beta(a_2-a_1)}{1-\omega_3},\frac{\beta(b_4-b_3)-(\omega_3b_4-b_3)}{1-\omega_3}\right),$$

$$c_3=\left(\frac{\beta(a_4-a_3)-(\omega_3a_4-a_3)}{1-\omega_3},\frac{(b_2-\omega_3b_1)-\beta(b_2-b_1)}{1-\omega_3}\right)$$

and

$$c_4 = \left(\frac{\beta(a_4 - a_3) - (\omega_3 a_4 - a_3)}{1-\omega_3}, \frac{\beta(b_4 - b_3) - (\omega_3 b_4 - b_3)}{1-\omega_3} \right)$$

Therefore,

$$f(c_1) = \frac{(a_2 - \omega_3 a_1) - \beta(a_2 - a_1)}{1-\omega_3} - \frac{(b_2 - \omega_3 b_1) - \beta(b_2 - b_1)}{1-\omega_3}$$
$$= \frac{(a_2 - \omega_3 a_1) - \beta(a_2 - a_1) - (b_2 - \omega_3 b_1) + \beta(b_2 - b_1)}{1-\omega_3}$$

$$f(c_2) = \frac{(a_2 - \omega_3 a_1) - \beta(a_2 - a_1)}{1-\omega_3} - \frac{\beta(b_4 - b_3) - (\omega_3 b_4 - b_3)}{1-\omega_3}$$
$$= \frac{(a_2 - \omega_3 a_1) - \beta(a_2 - a_1) - \beta(b_4 - b_3) + (\omega_3 b_4 - b_3)}{1-\omega_3}$$

$$f(c_3) = \frac{\beta(a_4 - a_3) - (\omega_3 a_4 - a_3)}{1-\omega_3} - \frac{(b_2 - \omega_3 b_1) - \beta(b_2 - b_1)}{1-\omega_3}$$
$$= \frac{\beta(a_4 - a_3) - (\omega_3 a_4 - a_3) - (b_2 - \omega_3 b_1) + \beta(b_2 - b_1)}{1-\omega_3}$$

$$f(c_4) = \frac{\beta(a_4 - a_3) - (\omega_3 a_4 - a_3)}{1-\omega_3} - \frac{\beta(b_4 - b_3) - (\omega_3 b_4 - b_3)}{1-\omega_3}$$
$$= \frac{\beta(a_4 - a_3) - (\omega_3 a_4 - a_3) - \beta(b_4 - b_3) + (\omega_3 b_4 - b_3)}{1-\omega_3}$$

Now, since $a_1 \le a_2 \le a_3 \le a_4$ and $b_1 \le b_2 \le b_3 \le b_4$, so $f(c_2) < f(c_1) < f(c_4) < f(c_3)$. Hence,

$$C^\beta = \left[\min\{f(c_1), f(c_2), f(c_3), f(c_4)\}, \max\{f(c_1), f(c_2), f(c_3), f(c_4)\}\right] = \left[f(c_2), f(c_3)\right]$$
$$= \left[\frac{(a_2 - \omega_3 a_1) - \beta(a_2 - a_1) - \beta(b_4 - b_3) + (\omega_3 b_4 - b_3)}{1-\omega_3}, \frac{\beta(a_4 - a_3) - (\omega_3 a_4 - a_3) - (b_2 - \omega_3 b_1) + \beta(b_2 - b_1)}{1-\omega_3}\right]$$
$$= \left[\frac{a_2 - b_3 + \omega_3(b_4 - a_1) - \beta\{(a_2 - b_3) - (a_1 - b_4)\}}{1-\omega_3}, \frac{a_3 - b_2 + \omega_3(b_1 - a_4) + \beta\{(a_4 - b_1) - (a_3 - b_2)\}}{1-\omega_3}\right]$$

Let

$$\frac{a_2 - b_3 + \omega_3(b_4 - a_1) - \beta\{(a_2 - b_3) - (a_1 - b_4)\}}{1-\omega_3} \le z \le \frac{a_3 - b_2 + \omega_3(b_1 - a_4) + \beta\{(a_4 - b_1) - (a_3 - b_2)\}}{1-\omega_3}$$

Now,

$$\frac{a_2 - b_3 + \omega_3(b_4 - a_1) - \beta\{(a_2 - b_3) - (a_1 - b_4)\}}{1-\omega_3} \le z$$

$$\Rightarrow \frac{-(1-\omega_3)z + \omega_3(b_4 - a_1) + (a_2 - b_3)}{\{(a_2 - b_3) - (a_1 - b_4)\}} \le \beta$$

Let $\nu_C^L(z) = \dfrac{-(1-\omega_3)z + \omega_3(b_4 - a_1) + (a_2 - b_3)}{\{(a_2 - b_3) - (a_1 - b_4)\}}$

Now,

$$\frac{\mathrm{d}}{\mathrm{d}z}\nu_C^L(z)=\frac{\mathrm{d}}{\mathrm{d}z}\frac{-(1-\omega_3)z+\omega_3(b_4-a_1)+(a_2-b_3)}{\{(a_2-b_3)-(a_1-b_4)\}}=-\frac{(1-\omega_3)}{\{(a_2-b_3)-(a_1-b_4)\}}<0\text{ ; if }(a_2-b_3)>(a_1-b_4)$$

Therefore, $\nu_C^L(z)$ is a decreasing function.

Also, $\nu_C^L(a_2-b_3)=\omega_3$, $\nu_C^L(a_1-b_4)=1$ and $\nu_C^L\left(\frac{a_2-b_3+a_1-b_4}{2}\right)<\frac{(1+\omega_3)}{2}$.

Again,

$$\frac{a_3-b_2+\omega_3(b_1-a_4)+\beta\{(a_4-b_1)-(a_3-b_2)\}}{1-\omega_3}\geq z$$

$$\Rightarrow\frac{(1-\omega_3)z-\omega_3(b_1-a_4)-(a_3-b_2)}{\{(a_4-b_1)-(a_3-b_2)\}}\leq\beta$$

Let $\nu_C^R(z)=\frac{(1-\omega_3)z-\omega_3(b_1-a_4)-(a_3-b_2)}{\{(a_4-b_1)-(a_3-b_2)\}}$

Now,

$$\frac{\mathrm{d}}{\mathrm{d}z}\nu_C^R(z)=\frac{\mathrm{d}}{\mathrm{d}z}\frac{(1-\omega_3)z-\omega_3(b_1-a_4)-(a_3-b_2)}{\{(a_4-b_1)-(a_3-b_2)\}}=\frac{(1-\omega_3)}{\{(a_4-b_1)-(a_3-b_2)\}}>0\text{; if }(a_4-b_1)>(a_3-b_2)$$

Therefore, $\nu_C^R(z)$ is an increasing function.

Also, $\nu_C^R(a_3-b_2)=\omega_3$, $\nu_C^R(a_4-b_1)=1$ and $\nu_C^R\left(\frac{a_3-b_2+a_4-b_1}{2}\right)>\frac{(1+\omega_3)}{2}$.

So the negative membership function of $C=A-B$ is

$$\nu_C(z)=\begin{cases}\frac{-(1-\omega_3)z+\omega_3(b_4-a_1)+(a_2-b_3)}{\{(a_2-b_3)-(a_1-b_4)\}}; & a_1-b_4\leq z\leq a_2-b_3\\ \omega_3; & a_2-b_3\leq z\leq a_3-b_2\\ \frac{(1-\omega_3)z-\omega_3(b_1-a_4)-(a_3-b_2)}{\{(a_4-b_1)-(a_3-b_2)\}}; & a_3-b_2\leq z\leq a_4-b_1\\ 1; & \text{otherwise}\end{cases}$$

Hence, the subtraction rule is proved for negative membership.

Proposition 3.4: Scalar multiplication of a generalized trapezoidal picture fuzzy number is also a generalized trapezoidal picture fuzzy number.

Proof: Trivial.

4. Computation of Picture Fuzzy Functions

Example 1: Consider

$$F(X)=X+X^2 \tag{4.1}$$

Then we want to compute $F(A)$, where, the positive, neutral and negative membership functions of the generalized trapezoidal picture fuzzy number $A=(2,3,4,5);0.6,0.2,0.1$ are given as:

$$\mu_A(x)=\begin{cases}0; & x<2\\ 0.6(x-2); & 2\le x<3\\ 0.6; & 3\le x\le 4\\ 0.6(5-x); & 4<x\le 5\\ 0; & x>5\end{cases},\quad \eta_A(x)=\begin{cases}0; & x<2\\ 0.2(x-2); & 2\le x<3\\ 0.2; & 3\le x\le 4\\ 0.2(5-x); & 4<x\le 5\\ 0; & x>5\end{cases},$$

$$\nu_A(x)=\begin{cases}1; & x<2\\ 0.9(2-x)+1; & 2\le x<3\\ 0.1; & 3\le x\le 4\\ 0.9(x-4)+0.1; & 4<x\le 5\\ 1; & x>5\end{cases}.$$

Following Figure 2 graphically represents the GTraPFN A:

The corresponding (α,γ,β)-cut of the above GTraPFN A is as follows:

$$A^{(\alpha,\gamma,\beta)}=\left\{\left[2+\frac{\alpha}{0.6},5-\frac{\alpha}{0.6}\right],\left[2+\frac{\gamma}{0.2},5-\frac{\gamma}{0.2}\right],\left[2-\frac{\beta-1}{0.9},4+\frac{\beta-0.1}{0.9}\right]\right\}.$$

In order to find $(F(A))^{\alpha}$, we have,

$$c_1=2+\frac{\alpha}{0.6},\ c_2=5-\frac{\alpha}{0.6}$$

$$f(c_1)=2+\frac{\alpha}{0.6}+\left(2+\frac{\alpha}{0.6}\right)^2=6+\frac{5\alpha}{0.6}+\frac{\alpha^2}{0.36}$$

$$f(c_2)=5-\frac{\alpha}{0.6}+\left(5-\frac{\alpha}{0.6}\right)^2=30-\frac{11\alpha}{0.6}+\frac{\alpha^2}{0.36}$$

Now,

$$(F(A))^{\alpha}$$

$$=\left[\min\left\{6+\frac{5\alpha}{0.6}+\frac{\alpha^2}{0.36},30-\frac{11\alpha}{0.6}+\frac{\alpha^2}{0.36}\right\},\max\left\{6+\frac{5\alpha}{0.6}+\frac{\alpha^2}{0.36},30-\frac{11\alpha}{0.6}+\frac{\alpha^2}{0.36}\right\}\right]$$

$$=\left[6+\frac{5\alpha}{0.6}+\frac{\alpha^2}{0.36},30-\frac{11\alpha}{0.6}+\frac{\alpha^2}{0.36}\right].$$

Again, in order to find $(F(A))^{\gamma}$, we have,

$$c_1=2+\frac{\gamma}{0.2},\ c_2=5-\frac{\gamma}{0.2}$$

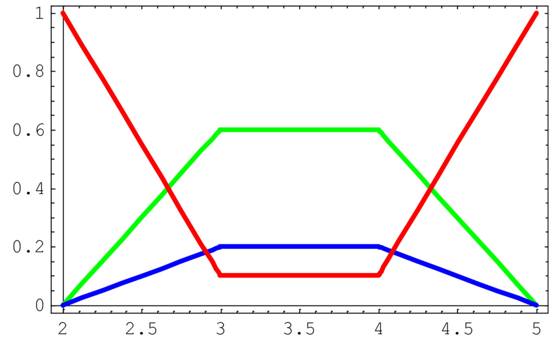
$$f(c_1)=2+\frac{\gamma}{0.2}+\left(2+\frac{\gamma}{0.2}\right)^2=6+\frac{5\gamma}{0.2}+\frac{\gamma^2}{0.04}$$

$$f(c_2)=5-\frac{\gamma}{0.2}+\left(5-\frac{\gamma}{0.2}\right)^2=30-\frac{11\gamma}{0.2}+\frac{\gamma^2}{0.04}$$

$$(F(A))^{\gamma}$$

$$=\left[\min\left\{6+\frac{5\gamma}{0.2}+\frac{\gamma^2}{0.04},30-\frac{11\gamma}{0.2}+\frac{\gamma^2}{0.04}\right\},\max\left\{6+\frac{5\gamma}{0.2}+\frac{\gamma^2}{0.04},30-\frac{11\gamma}{0.2}+\frac{\gamma^2}{0.04}\right\}\right]$$

$$=\left[6+\frac{5\gamma}{0.2}+\frac{\gamma^2}{0.04},30-\frac{11\gamma}{0.2}+\frac{\gamma^2}{0.04}\right].$$



The horizontal axes indicates the real numbers $\mathbb{R}$ and the vertical axes indicates the membership degrees of the positive, neutral and negative membership functions from 0 to 1.

Figure 2. GTraPFN A.

Again, in order to find $(F(A))^{\beta}$, we have,

$$c_1 = 2 - \frac{\beta - 1}{0.9},\ c_2 = 4 + \frac{\beta - 0.1}{0.9}$$

$$f(c_1) = 2 - \frac{\beta - 1}{0.9} + \left(2 - \frac{\beta - 1}{0.9}\right)^2 = \frac{10.36}{0.81} - \frac{6.5\beta}{0.81} + \frac{\beta^2}{0.81}$$

$$f(c_2) = 4 + \frac{\beta - 0.1}{0.9} + \left(4 + \frac{\beta - 0.1}{0.9}\right)^2 = \frac{15.4}{0.81} + \frac{7.9\beta}{0.81} + \frac{\beta^2}{0.81}$$

$$\begin{aligned}&(F(A))^{\beta}\\ &= \left[\min\left\{\frac{10.36}{0.81} - \frac{6.5\beta}{0.81} + \frac{\beta^2}{0.81}, \frac{15.4}{0.81} + \frac{7.9\beta}{0.81} + \frac{\beta^2}{0.81}\right\}, \max\left\{\frac{10.36}{0.81} - \frac{6.5\beta}{0.81} + \frac{\beta^2}{0.81}, \frac{15.4}{0.81} + \frac{7.9\beta}{0.81} + \frac{\beta^2}{0.81}\right\}\right]\\ &= \left[\frac{10.36}{0.81} - \frac{6.5\beta}{0.81} + \frac{\beta^2}{0.81}, \frac{15.4}{0.81} + \frac{7.9\beta}{0.81} + \frac{\beta^2}{0.81}\right].\end{aligned}$$

Thus,

$$\begin{aligned}&F(A)^{(\alpha,\gamma,\beta)}\\ &= \left\{\begin{matrix}\left[6 + \frac{5\alpha}{0.6} + \frac{\alpha^2}{0.36}, 30 - \frac{11\alpha}{0.6} + \frac{\alpha^2}{0.36}\right], \left[6 + \frac{5\gamma}{0.2} + \frac{\gamma^2}{0.04}, 30 - \frac{11\gamma}{0.2} + \frac{\gamma^2}{0.04}\right],\\ \left[\frac{10.36}{0.81} - \frac{6.5\beta}{0.81} + \frac{\beta^2}{0.81}, \frac{15.4}{0.81} + \frac{7.9\beta}{0.81} + \frac{\beta^2}{0.81}\right]\end{matrix}\right\}\end{aligned}$$

The corresponding positive, neutral and negative membership functions are as follows:

$$\mu_{F(A)}(x) = \begin{cases} 0; & x < 6 \\ \dfrac{-15 \pm 3\sqrt{1+4x}}{10}; & 6 \le x < 12 \\ 0.6; & 12 \le x \le 20 \\ \dfrac{33 \pm 3\sqrt{1+4x}}{10}; & 20 < x \le 30 \\ 0; & x > 30 \end{cases},\quad \eta_{F(A)}(x) = \begin{cases} 0; & x < 6 \\ \dfrac{-5 \pm \sqrt{1+4x}}{10}; & 6 \le x < 12 \\ 0.2; & 12 \le x \le 20 \\ \dfrac{11 \pm \sqrt{1+4x}}{10}; & 20 < x \le 30 \\ 0; & x > 30 \end{cases}$$

$$\nu_{F(A)}(x)=\begin{cases}1; & x<6\\ \dfrac{6.5\pm 0.9\sqrt{1+4x}}{2}; & 6\le x<12\\ 0.1; & 12\le x\le 20\\ \dfrac{-7.9\pm 0.9\sqrt{1+4x}}{2}; & 20<x\le 30\\ 1; & x>30\end{cases}$$

Figure 3 represents the value of $F(A)$ as follows:

Example 2: Consider

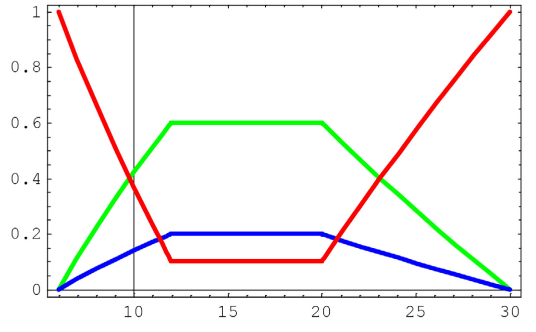
$$F(A,B)=A+B \tag{4.2}$$

Then we want to compute $F(A,B)$, where $A=\langle(2,3,4,5);0.6,0.2,0.1\rangle$ and $B=\langle(1,3,5,7);0.6,0.2,0.1\rangle$ are two GTraPFNs with the following membership functions

$$\mu_A(x)=\begin{cases}0; & x<2\\ 0.6(x-2); & 2\le x<3\\ 0.6; & 3\le x\le 4\\ 0.6(5-x); & 4<x\le 5\\ 0; & x>5\end{cases},$$

$$\eta_A(x)=\begin{cases}0; & x<2\\ 0.2(x-2); & 2\le x<3\\ 0.2; & 3\le x\le 4\\ 0.2(5-x); & 4<x\le 5\\ 0; & x>5\end{cases}$$

$$\nu_A(x)=\begin{cases}1; & x<2\\ 0.9(2-x)+1; & 2\le x<3\\ 0.1; & 3\le x\le 4\\ 0.9(x-4)+0.1; & 4<x\le 5\\ 1; & x>5\end{cases}$$



The horizontal axes indicates the real numbers $\mathbb{R}$ and the vertical axes indicates the membership degrees of the positive, neutral and negative membership functions from 0 to 1.

Figure 3. Value of *F*(*A*).

and

$$\mu_B(x)=\begin{cases}0; & x<1\\ \dfrac{0.6(x-1)}{2}; & 1\le x<3\\ 0.6; & 3\le x\le 5\\ \dfrac{0.6(7-x)}{2}; & 5<x\le 7\\ 0; & x>7\end{cases},\quad \eta_B(x)=\begin{cases}0; & x<1\\ \dfrac{0.2(x-1)}{2}; & 1\le x<3\\ 0.2; & 3\le x\le 5\\ \dfrac{0.2(7-x)}{2}; & 5<x\le 7\\ 0; & x>7\end{cases}$$

$$\nu_B(x)=\begin{cases}1; & x<1\\ 0.45(1-x)+1; & 1\le x<3\\ 0.1; & 3\le x\le 5\\ 0.45(x-5)+0.1; & 5<x\le 7\\ 1; & x>7\end{cases}.$$

Following **Figure 4(a)** and **Figure 4(b)** show the graphical representations of the GTraPFNs A and B respectively:

Now, we want to compute $F(A,B)$ by the vertex method.

Here, the corresponding (α,γ,β) -cut of the above GTraPFNs A and B are as follows:

$$A^{(\alpha,\gamma,\beta)}=\left\{\left[2+\frac{\alpha}{0.6},5-\frac{\alpha}{0.6}\right],\left[2+\frac{\gamma}{0.2},5-\frac{\gamma}{0.2}\right],\left[2-\frac{\beta-1}{0.9},4+\frac{\beta-0.1}{0.9}\right]\right\}$$

$$B^{(\alpha,\gamma,\beta)}=\left\{\left[1+\frac{\alpha}{0.3},7-\frac{\alpha}{0.3}\right],\left[1+\frac{\gamma}{0.1},7-\frac{\gamma}{0.1}\right],\left[1-\frac{\beta-1}{0.45},5+\frac{\beta-0.1}{0.45}\right]\right\}$$

In order to find $(F(A,B))^{\alpha}$, we have,

$$c_1=\left(2+\frac{\alpha}{0.6},1+\frac{\alpha}{0.3}\right),\ c_2=\left(2+\frac{\alpha}{0.6},7-\frac{\alpha}{0.3}\right),\ c_3=\left(5-\frac{\alpha}{0.6},1+\frac{\alpha}{0.3}\right)\ \text{and}$$

$$c_4=\left(5-\frac{\alpha}{0.6},7-\frac{\alpha}{0.3}\right).$$

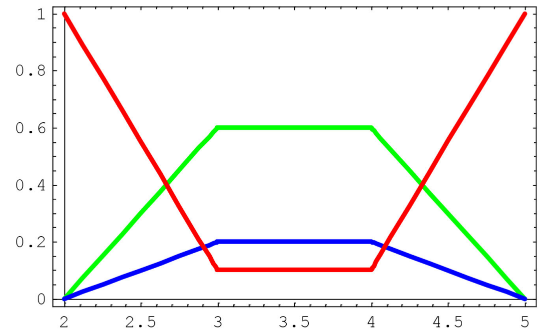
$$f(c_1)=2+\frac{\alpha}{0.6}+1+\frac{\alpha}{0.3}=3+5\alpha,\ f(c_2)=2+\frac{\alpha}{0.6}+7-\frac{\alpha}{0.3}=9-\frac{\alpha}{0.6},$$

$$f(c_3)=5-\frac{\alpha}{0.6}+1+\frac{\alpha}{0.3}=6+\frac{\alpha}{0.6}\ \text{and}\ f(c_4)=5-\frac{\alpha}{0.6}+7-\frac{\alpha}{0.3}=12-5\alpha.$$

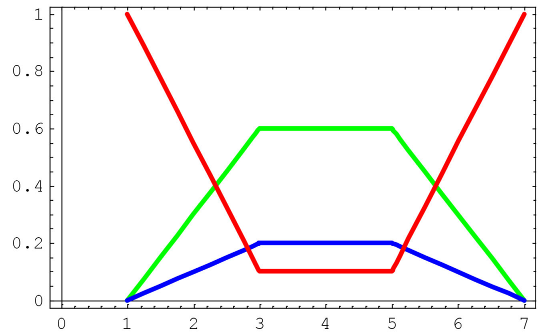
Now,

$$(F(A,B))^{\alpha}$$
$$=\left[\min\left\{3+5\alpha,9-\frac{\alpha}{0.6},6+\frac{\alpha}{0.6},12-5\alpha\right\},\max\left\{3+5\alpha,9-\frac{\alpha}{0.6},6+\frac{\alpha}{0.6},12-5\alpha\right\}\right]$$

$$(F(A,B))^{\alpha}$$
$$=\left[\min\left\{3+5\alpha,9-\frac{\alpha}{0.6},6+\frac{\alpha}{0.6},12-5\alpha\right\},\max\left\{3+5\alpha,9-\frac{\alpha}{0.6},6+\frac{\alpha}{0.6},12-5\alpha\right\}\right]$$
$$=[3+5\alpha,12-5\alpha].$$



(a)



(b)

The horizontal axes indicates the real numbers $\mathbb{R}$ and the vertical axes indicates the membership degrees of the positive, neutral and negative membership functions from 0 to 1.

Figure 4. (a) GTraPFN A; (b) GTraPFN B.

In order to find $(F(A,B))^{\gamma}$, we have,

$$c_1=\left(2+\frac{\gamma}{0.2},1+\frac{\gamma}{0.1}\right),\quad c_2=\left(2+\frac{\gamma}{0.2},7-\frac{\gamma}{0.1}\right),$$

$$c_3=\left(5-\frac{\gamma}{0.2},1+\frac{\gamma}{0.1}\right)\text{ and } c_4=\left(5-\frac{\gamma}{0.2},7-\frac{\gamma}{0.1}\right).$$

$$f(c_1)=2+\frac{\gamma}{0.2}+1+\frac{\gamma}{0.1}=3+15\gamma,\quad f(c_2)=2+\frac{\gamma}{0.2}+7-\frac{\gamma}{0.1}=9-5\gamma,$$

$$f(c_3)=5-\frac{\gamma}{0.2}+1+\frac{\gamma}{0.1}=6+5\gamma\text{ and } f(c_4)=5-\frac{\gamma}{0.2}+7-\frac{\gamma}{0.1}=12-15\gamma.$$

Now,

$$\begin{aligned}&(F(A,B))^{\gamma}\\&=\left[\min\{3+15\gamma,9-5\gamma,6+5\gamma,12-15\gamma\},\max\{3+15\gamma,9-5\gamma,6+5\gamma,12-15\gamma\}\right]\\&=[3+15\gamma,12-15\gamma].\end{aligned}$$

In order to find $(F(A,B))^{\gamma}$, we have,

$$c_1=\left(2-\frac{\beta-1}{0.9},1-\frac{\beta-1}{0.45}\right),\quad c_2=\left(2-\frac{\beta-1}{0.9},5+\frac{\beta-0.1}{0.45}\right),$$

$$c_3=\left(4+\frac{\beta-0.1}{0.9},1-\frac{\beta-1}{0.45}\right) \text{ and } c_4=\left(4+\frac{\beta-0.1}{0.9},5+\frac{\beta-0.1}{0.45}\right).$$

$$f(c_1)=2-\frac{\beta-1}{0.9}+1-\frac{\beta-1}{0.45}=\frac{19}{3}-\frac{10}{3}\beta,$$

$$f(c_2)=2-\frac{\beta-1}{0.9}+5+\frac{\beta-0.1}{0.45}=\frac{71}{9}+\frac{10}{9}\beta,$$

$$f(c_3)=4+\frac{\beta-0.1}{0.9}+1-\frac{\beta-1}{0.45}=\frac{64}{9}-\frac{10}{9}\beta$$

and

$$f(c_4)=4+\frac{\beta-0.1}{0.9}+5+\frac{\beta-0.1}{0.45}=\frac{26}{3}+\frac{10}{3}\beta.$$

Now,

$$\begin{aligned}&(F(A,B))^{\beta}\\&=\left[\min\left\{\frac{19}{3}-\frac{10}{3}\beta,\frac{71}{9}+\frac{10}{9}\beta,\frac{64}{9}-\frac{10}{9}\beta,\frac{26}{3}+\frac{10}{3}\beta\right\},\max\left\{\frac{19}{3}-\frac{10}{3}\beta,\frac{71}{9}+\frac{10}{9}\beta,\frac{64}{9}-\frac{10}{9}\beta,\frac{26}{3}+\frac{10}{3}\beta\right\}\right]\\&=\left[\frac{19}{3}-\frac{10}{3}\beta,\frac{26}{3}+\frac{10}{3}\beta\right].\end{aligned}$$

Thus,

$$F(A,B)^{(\alpha,\gamma,\beta)}=\left\{[3+5\alpha,12-5\alpha],[3+15\gamma,12-15\gamma],\left[\frac{19}{3}-\frac{10}{3}\beta,\frac{26}{3}+\frac{10}{3}\beta\right]\right\}$$

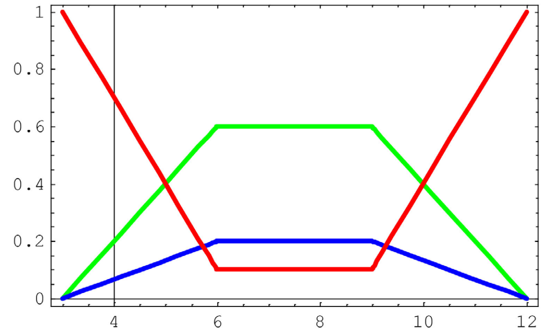
The corresponding positive, neutral and negative membership functions are as follows:

$$\mu_{F(A,B)}(x)=\begin{cases}0; & x<3\\ \frac{x-3}{5}; & 3\le x<6\\ 0.6; & 6\le x\le 9\\ \frac{12-x}{5}; & 9<x\le 12\\ 0; & x>12\end{cases},$$

$$\eta_{F(A,B)}(x)=\begin{cases}0; & x<3\\ \frac{x-3}{15}; & 3\le x<6\\ 0.2; & 6\le x\le 9\\ \frac{12-x}{15}; & 9<x\le 12\\ 0; & x>12\end{cases},\quad \nu_{F(A,B)}(x)=\begin{cases}1; & x<3\\ \frac{19-3x}{10}; & 3\le x<6\\ 0.1; & 6\le x\le 9\\ \frac{3x-26}{10}; & 9<x\le 12\\ 1; & x>12\end{cases}$$

Figure 5 represents the value of $F(A,B)$ as follows:

The method described in this paper significantly expands the technique for calculating the value of picture fuzzy functions and is computationally easy to implement. The MATHEMATICA program is used for graphical representations. The vertex method can be used directly without conducting an extreme analysis for some of the most commonly used monotonic functions.



The horizontal axes indicates the real numbers $\mathbb{R}$ and the vertical axes indicates the membership degrees of the positive, neutral and negative membership functions from 0 to 1.

Figure 5. Value of $F(A, B)$.

5. Conclusion

Picture fuzzy number plays a vital role in the field of uncertainty. In this paper, the arithmetic operations of generalized trapezoidal picture fuzzy numbers by vertex method are developed. Finally, some computations of picture fuzzy functions over generalized picture fuzzy variables are illustrated by using our proposed method.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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