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Vortex Solitons for a Class of Schrödinger Equation with Square Root Nonlinear Term

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Abstract

In this paper, we prove an existence theory for ring-profiled optical vortex solitons via constrained minimization, which are considered in the context of an electromagnetic light wave propagating in a nonlinear media and governed by a nonlinear Schrödinger type equation with square root nonlinear term.

Keywords

Optical Vortex Solitons, Square Root Nonliear, Constrained Minimization

1. Introduction

In optics research, a fundamental prototype situation is that the light waves are described by a complex-valued wave function governed by nonlinear Schrödinger equations [1]-[8]. These rigorous mathematical treatments of such nonlinear problems provide more possibilities for the existence and properties of optical vortices. Our interest is motivated by the work of Lin, Belić, Petrović, Hajaiej and Chen [9], the mathematical analysis of Lin and Ren [10].

In dimensionless form, consider the following nonlinear Schrödinger equation,

$$i\partial_z E+\frac{1}{2}\nabla_\perp^2 E+f(I)E=0, \tag{1}$$

where E is the evolution of the slowly varing electric field envelope propagating in the longitudinal z-direction; $\nabla_\perp^2$ is the Laplace operator over the transverse plane of coordinates (x,y) which is perpendicular to the z-axis. The function f depends on the total field intensity, I, *i.e.* $I=|E|^2$, and we will concentrate henceforth on the model of the self-focusing square-root nonlinearity.

$$f\left(|E|^2\right)=1-\frac{1}{\sqrt{1+|E|^2}}, \tag{2}$$

which describes narrow-gap semiconductors [11] [12].

We focus on spatial optical solitons. Spatially localized solutions of (1), which do not change their intensity profile during propagation, can be described under the spatial soliton ansatz

$$E(r,\theta,z)=u(r)\mathrm{e}^{i(n\theta+\alpha z)} \tag{3}$$

where r and θ are real polar coordinates over $\mathbb{R}^2$, and $r=\sqrt{x^2+y^2}$, $\theta=\arctan(y/x)$, $u(r)$ is the radial profile function which gives rise to the intensity of light waves, $n\in\mathbb{Z}$ is the winding number, and $\alpha\in\mathbb{R}$ is the wave propagation constant. This ansatz describe a vortex wave centered around the *z*-axis. Inserting (3) into (1) and in a square root nonlinear media, we arrive at the following equation

$$(ru_r)_r-\frac{n^2}{r}u+2ru-\frac{2ru}{\sqrt{1+u^2}}-2\alpha ru=0 \tag{4}$$

Due to the presence of the vortex core, in other words, the regularity of u at $r=0$, we impose the condition $u(0)=0$. Besides, such ring-like beams remain localized that allows us to mathematically impose the "boundary" condition $u(R)=0$ for $R>0$ sufficiently large, where R represents the distance from the vortex core.

Therefore, in view of (4), we can get the *n*-vortex equation with boundary conditions.

$$\begin{cases}(ru_r)_r-\dfrac{n^2}{r}u+2ru-\dfrac{2ru}{\sqrt{1+u^2}}-2\alpha ru=0\\ u(0)=0,\ u(R)=0\end{cases} \tag{5}$$

In this paper, we treat (5) as a nonlinear eigenvalue problem and prove the existence of positive solution pairs (u,α) by a constrained minimization approach, with a prescribed energy flux constrained.

2. Preliminary Setting and Main Theorems

In this section, we give some basic notations and lemmas which will be used in next section. In order to approach the Equation (5), we write down the action functional $I_\alpha:H\to\mathbb{R}$ defined as

$$I_\alpha(u)=\frac{1}{2}\int_0^R\left\{ru_r^2+\frac{n^2}{r}u^2-2(1-\alpha)ru^2+4r\sqrt{1+u^2}\right\}\mathrm{d}r \tag{6}$$

where $|n|\ge 1$, H is the completion of

$$X=\left\{u\in C^1[0,R]\,|\,u(0)=0=u(R)\right\} \tag{7}$$

equipped with the inner product

$$(u,v)=\int_0^R\left\{ru_rv_r+\frac{1}{r}uv\right\}\mathrm{d}r,\ u,v\in H \tag{8}$$

As a Hilbert space, H may be considered as an embedded subspace of

$W_0^{1,2}(B_R)$ which is composed of radially symmetric functions such that any element $u \in H$ enjoys the desired property $u(0)=0$, where $B_R := \{(x,y) \in \mathbb{R}^2 : x^2+y^2 \le R^2\}$.

Lemma 2.1. From the inequalities

$$\left(\sqrt{1+u^2}-1\right)^2 \le \frac{1}{4}u^4 \quad \text{for } u \in \mathbb{R} \tag{9}$$

and

$$\int_0^R ru^2 \mathrm{d}r \le R^2 \int_0^R \frac{u^2}{r} \mathrm{d}r \tag{10}$$

we get that there exists a constant $C>0$, such that $I_\alpha(u) \le C\|u\|_H^2$.

For convenience, we define the "energy" functional as

$$\varepsilon(u) = \frac{1}{2}\int_0^R \left\{ ru_r^2 + \frac{u^2}{r} + 4r\sqrt{1+u^2} \right\} \mathrm{d}r \tag{11}$$

Now, we state our main theorem in this paper.

Theorem 2.2. For any parameters $|n| \ge 1$, consider the n-vortex Equation (5) with boundary conditions, describing ring-profile vortex solitons in a square-root nonlinear media, with the prescribed energy flux $\Phi(u)=\Phi_0>0$, and $R>0$.

1) There exists a solution pair (u,α) with $u(r)>0$, $r \in (0,R)$ and $\alpha \in \mathbb{R}$.

2) For $r \in [0,R]$, the energy flux $\Phi(u)=\Phi_0 \le \frac{1}{4}$, and there exists no non-trivial solution, if $n^2+2r^2\alpha>0$.

3. Existence of Vortices via Constrained Minimization

In this section, we consider the wave propagation constant α as a Lagrange multiplier, we prove the existence of solution of the Equation (5) with constrained minimization approach.

We rewrite the n-vortex Equation (5) as

$$\begin{cases} (ru_r)_r - \dfrac{n^2}{r}u + 2ru - \dfrac{2ru}{\sqrt{1+u^2}} = 2\alpha ru \\ u(0)=0,\ u(R)=0 \end{cases} \tag{12}$$

Define the function I and the soliton energy flux as

$$I(u) = \frac{1}{2}\int_0^R \left\{ ru_r^2 + \frac{n^2}{r}u^2 - 2ru^2 + 4r\sqrt{1+u^2} \right\} \mathrm{d}r$$

$$\Phi(u) = \int_0^{2\pi} \mathrm{d}\theta \int_0^R ru^2 \mathrm{d}r = 2\pi \int_0^R ru^2 \mathrm{d}r$$

Thus, to get a solution of (12), it suffices to show that a solution to the following exists:

$$\min\{I(u) \mid u \in \Lambda, \Phi(u)=\Phi_0\}, \Phi_0>0 \tag{13}$$

where the nonempty admissible class Λ is defined by

$$\Lambda = \{u(r) \text{ is absolutely continuous over } [0,R], u(0)=u(R)=0, \varepsilon(u)<\infty\} \quad (14)$$

with $\varepsilon(u)$ being defined by (11).

The proof of Theorem 2.2. 1) Using the energy flux Φ_0, we have

$$I(u) \geq \frac{1}{2}\left(\int_0^R r u_r^2 \mathrm{d}r + n^2 \int_0^R \frac{u^2}{r} \mathrm{d}r\right) - \frac{\Phi_0}{2\pi} \quad (15)$$

Let $\{u_m\}$ be a minimizing sequence of (13). Then (15) gives the bound

$$\int_0^R r u_{m,r}^2 \mathrm{d}r + \int_0^R \frac{u_m^2}{r} \mathrm{d}r \leq C \quad (16)$$

where $C>0$ is a constant independent of *m*. We know the fact that the distributional derivative of *u* must satisfy $\left||u|_r\right| \leq |u_r|$, and the functionals I and Φ are even. Thus, we may assume that the sequence $\{u_m\}$ consists of non-negative valued functions. Therefore, it is clear that we may view these functions as radially symmetric over the disk B_R and vanishing on its boundary. Moreover, with (16) and (10), it can be seen that $\{u_m\}$ belongs in $W_0^{1,2}(B_R)$ under the reduced norm,

$$\|u\|^2 = \int_0^R r u^2 \mathrm{d}r + \int_0^R r u_r^2 \mathrm{d}r \quad (17)$$

Therefore, $\{u_m\}$ is bounded in $W_0^{1,2}(B_R)$. Without loss of generality, we get the weak convergence of $\{u_m\}$ to an element $u \in W_0^{1,2}(B_R)$. Using the compact embedding $W_0^{1,2}(B_R) \to L^p(B_R)$ for $p \geq 1$, $u_m \to u$ strongly in $L^p(B_R)$ as $m \to \infty$. Hence, *u* is radially symmetric as well with $u(R)=0$.

In view of (16) and Fatou's lemma, Let (X, Σ, μ) be a measure space and $\{f_n : X \to [0,\infty)\}$ a sequence of nonnegative measurable functions. Then the function $\liminf_{n\to\infty} f_n$ is measurable and

$$\int_X \liminf_{n\to\infty} f_n \mathrm{d}_\mu \leq \liminf_{n\to\infty} \int_X f_n \mathrm{d}_\mu \quad (18)$$

we have

$$\int_0^R r u_r^2 \mathrm{d}r \leq \liminf_{m\to\infty} \int_0^R r u_{m,r}^2 \mathrm{d}r \quad (19)$$

$$\int_0^R \frac{u^2}{r} \mathrm{d}r \leq \liminf_{m\to\infty} \int_0^R \frac{u_m^2}{r} \mathrm{d}r \quad (20)$$

$$\int_0^R 2r\sqrt{1+u^2} \mathrm{d}r \leq \liminf_{m\to\infty} \int_0^R 2r\sqrt{1+u_m^2} \mathrm{d}r \quad (21)$$

Therefore, from (10) and (19)-(21), we get

$$I(u) \leq \liminf_{m\to\infty} I(u_m)$$

Following as in [13]. Let $\{u_m\}$ be a sequence in $W^{1,2}(\epsilon, R)$ where $\varepsilon \in (0,R)$. It is clear that for any $\epsilon \in (0,R)$, $\{u_m\}$ is bounded in $W^{1,2}(\epsilon,R)$. We may get that $u_m \to u$ uniformly over $[\epsilon, R]$ as $m \to \infty$ applying the compact embedding $W^{1,2}(\epsilon,R) \to C[\epsilon,R]$. Thus, we have for any pair $r_1, r_2 \in (0,R), r_1 < r_2$, with (16),

$$\left|u_m^2(r_2)-u_m^2(r_1)\right| \le 2\left(\int_{r_1}^{r_2} ru_{m,r}^2(r)\,\mathrm{d}r\right)^{1/2}\left(\int_{r_1}^{r_2}\frac{u_m^2(r)}{r}\mathrm{d}r\right)^{1/2}$$
$$\le 2C^{1/2}\left(\int_{r_1}^{r_2}\frac{u_m^2(r)}{r}\mathrm{d}r\right)^{1/2} \tag{22}$$

Taking $m\to\infty$, we get

$$\left|u^2(r_2)-u^2(r_1)\right| \le 2C^{1/2}\left(\int_{r_1}^{r_2}\frac{u^2(r)}{r}\mathrm{d}r\right)^{1/2} \tag{23}$$

Since $\frac{u^2}{r}\in L(0,R)$, the right-hand side of (23) tends to zero as $r_1,r_2\to 0$. Hence,

$$\zeta_0=\lim_{r\to 0}u^2(r)=0$$

As a consequence, the boundary condition $u(0)=0$ is achieved. With (13), u is a solution to (13), and there is a real number α such that (u,α) satisfies (12).

Moreover, we may suppose that there is a point $r_0\in(0,R)$ such that $u(r_0)=0$, then $u_r(r_0)=0$ since r_0 is a minimum point for the function $u(r)$. By the uniqueness theorem of the initial value problem of ordinary differential equations, we have $u(r)=0$ for all $r\in(0,R)$, thus contradicting the fact $\Phi(u)=\Phi_0>0$. Hence, $u(r)>0$ for all $r\in(0,R)$.

2) We establish

$$\liminf_{r\to 0}\left\{ru(r)\left|u_r(r)\right|\right\}=0 \tag{24}$$

Suppose otherwise that (24) is not valid, equivalently, $\liminf_{r\to 0}\left\{ru(r)\left|u_r(r)\right|\right\}\ne 0$, then there is a $\epsilon_0>0$ and $r_0\in(0,R]$ so that $ru(r)\left|u_r(r)\right|\ge\epsilon_0$ for all $r\in(0,r_0)$. However,

$$\infty=\int_0^{r_0}\frac{\epsilon_0}{r}\mathrm{d}r\le\int_0^{r_0}u\left|u_r\right|\mathrm{d}r\le\left(\int_0^{r_0}\frac{u^2}{r}\mathrm{d}r\right)^{1/2}\left(\int_0^{r_0}ru_r^2\mathrm{d}r\right)^{1/2} \tag{25}$$

which contradicts with $\varepsilon(u)<\infty$. So, (24) is valid. From (24), we can find a sequence $\{r_j\}$ such that $r_j\to 0$ as $j\to\infty$ and

$$\lim_{j\to\infty}\left\{r_ju(r_j)u_r(r_j)\right\}=0 \tag{26}$$

Multiplying (5) by u, integrating over $[r_j,R]$, letting $j\to\infty$. Appealing to (26), we obtain

$$-\int_0^R ru_r^2\mathrm{d}r=\int_0^R\left\{\frac{n^2}{r}u^2+2\alpha ru^2-2ru^2+\frac{2ru^2}{\sqrt{1+u^2}}\right\}\mathrm{d}r \tag{27}$$

Using $\left(\sqrt{1+u^2}-1\right)^2\le\frac{1}{4}u^4$, we have that

$$-\int_0^R ru_r^2\mathrm{d}r=\int_0^R\left\{\frac{n^2}{r}u^2+2\alpha ru^2-2ru^2+\frac{2ru^2}{\sqrt{1+u^2}}\right\}\mathrm{d}r$$
$$\ge\int_0^R\left\{\frac{n^2}{r}u^2+2\alpha ru^2-2ru^4\right\}\mathrm{d}r\ge\int_0^R\left(\frac{n^2}{r^2}+2\alpha\right)ru^2\mathrm{d}r-2\int_0^R ru^4\mathrm{d}r \tag{28}$$

We may treat u as a radially symmetric function defined over $\mathbb{R}^2$ with its support contained in the disk B_R. Hence, from the classical $G-N$ inequality over $\mathbb{R}^2$, we deduce

$$\int_0^R ru^4 \mathrm{d}r \le 4\pi \int_0^R ru^2 \mathrm{d}r \int_0^R ru_r^2 \mathrm{d}r$$

with $\Phi(u) = \Phi_0$, we have

$$\left(4\Phi_0 - 1\right)\int_0^R ru_r^2 \mathrm{d}r - \int_0^R \left(\frac{n^2}{r^2} + 2\alpha\right) ru^2 \mathrm{d}r \ge 0$$

Therefore, when $\Phi_0 \le \frac{1}{4}, \frac{n^2}{r^2} + 2\alpha > 0$ for $r \in (0, R]$, $u \equiv 0$. as claimed.

4. Conclusion

Through the prove of the theorem 2.2, we get that the existence of positive solution pairs (u, α) by a constrained minimization approach. In other words, we get the existence of ring-profiled optical vortex solitons propagating in a square-root nonlinear media. Moreover, we obtain that there is no nontrivial small-energy-flux solution satisfying $\Phi(u) = \Phi_0 \le 1/4$, if $n^2 + 2r^2\alpha > 0$ for $r \in [0, R]$.

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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