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Conditional Independence Leads to Satisfaction of the Bell Inequality without Assuming Non-Locality or Non-Reality

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Abstract

The original Bell inequality was obtained in a statistical derivation assuming three mutually cross-correlated random variables (four in the later version). Given that observations destroy the particles, the physical realization of three variables from an experiment producing two particles per trial requires two separate trial runs. One assumed variable value (for particle 1) occurs at a fixed instrument setting in both trial runs while a second variable (for particle 2) occurs at alternative instrument settings in the two trial runs. Given that measurements on the two particles occurring in each trial are themselves correlated, measurements from independent realizations at mutually exclusive settings on particle 2 are conditionally independent, *i.e.*, conditionally dependent on particle 1, through probability. This situation is realized from variables defined by Bell using entangled particle pairs. Two correlations have the form that Bell computed from entanglement, but a third correlation from conditionally independent measurements has a different form. When the correlations are computed using quantum probabilities, the Bell inequality is satisfied without recourse to assumptions of non-locality, or non-reality.

Keywords

Bell Theorem, Bell Inequality, Entanglement, Cross-Correlations, Conditional Independence

1. Introduction: Problems in Physical Application of the Bell Inequality

The Bell theorem [1] and inequality, together with violation of the inequality under certain experimental procedures, have led to more than 50 years of speculation and controversy [2] [3]. However, inconsistencies between the mathematical structure of the Bell inequality and its experimental use require critical examination. Below, Bell's derivation of the three variable inequality is reviewed, followed by a proof that it is identically satisfied by cross-correlations of any three numerical data sets corresponding to the three cross-correlated, random variables used in Bell's original derivation [4].

How then have classic experiments such as [5] produced data claimed to violate the inequality in "test" experiments. The answer is, by using three independent correlations from six data sets rather than the three cross-correlations of three data sets used in the Bell derivation of the inequality. Given the statistical derivation of the inequality, and that experiments directly produce data for correlations produced by particle pairs, it seems to be casually assumed that the correct correlations are obtained from three (or four in the four variable case) statistically independent particle pairs, or six data sets. This assumption is inconsistent with the inequality derivation as shown in the following sections. The correlations from three independently produced particle pairs (six variables) bear only partial resemblance to the cross-correlations among three correctly used quantum variables because, in the latter case, the final correlation reuses data from the previous two correlations. Since the observations of two of the variables are conditionally dependent on a third in Bell's usage, the third correlation is different from the first two Bell correlations.

The original Bell inequality [1] results from the mutual cross-correlation of three variables. The resulting expression, as will be shown below, appears to be symmetric in the correlations of the variables. But the physical situation to which the result is applied is intrinsically asymmetric with the third correlation having a different form from the first two. A more general derivation of the inequality, as applies to correlations of actual finite physical data sets, rather than predicted correlations, shows that the inequality must be identically satisfied, and that the correlations in general have different functional forms. These facts will be illustrated below, using correlations satisfying the inequality in the quantum mechanical case. A similar but clearly more complex procedure, not considered here, is required to apply the four variable inequality to experimental photon pairs with logical consistency.

It is useful to consider the processes invoked in the Bell theorem in the context of other known random processes. There are processes in which any number of variables may be measured using different pairs of coordinates, and for which the resulting correlations all have the same functional dependence on coordinate differences. These processes are defined as second-order-stationary [6]. There are optical processes where the correlations gradually change their functional form, but are approximately second order stationary over finite regions [7]. At the other extreme, in the observation of photons, only one measurement per particle may be made in a statistical realization, since measurement destroys the particle. In this case, two measurements can be made by retrodiction –prior paths are deduced from a final detector location [8]. Each successive path is then conditionally dependent in terms of probability, on the prior path. This is described in quantum mechanics in the context of non-commutation and is central to the Bell situation [9]. It is important to recognize, however, that the term non-commutation does not in itself imply quantum idiosyncratic randomness. The Pauli matrices used to describe the non-commutative behavior of spin measurements were originally constructed to describe macroscopic, non-commutative 3-D rotations [10].

2. Bell's Statistical Inequality Derivation Using Three Cross-Correlated Variables

Bell [1] hypothesized three measurements applied to a pair of entangled spins or photons, produced as shown in the experimental schematic of **Figure 1**. He then derived a statistical expression assuming three observations on two particles to impose a further condition that would shed light on the nature of the random processes involved. However, only one measurement may be performed per particle since a measurement destroys the particle, and that has led to a logical riddle. How is a mathematical construction using three variables to be logically applied to an experiment yielding two observations per trial? The answer to this lies at the heart of the Bell inequality derivation and mystery. More difficult yet, how is the four variable version of the inequality to be applied to such particle pairs?

First, the experimental situation must be presented based on the apparatus represented in **Figure 1**. Measurements on the A-side at angular setting a are represented in Bell's notation by the function $A(a,\lambda)$, and on the B-side at angular setting b by $B(b,\lambda)$. The variables λ are random variables with a probability density $\rho(\lambda)$ assumed to determine the results of measurements represented by functions A and B, postulated to be deterministic and with outcomes independent of each other's settings. The random results of measurements might then be interpreted causally as due to uncontrolled common initial conditions sampled randomly. The measurements have values equal to ±1 in Bell's definition with the additional stipulation that $B(a,\lambda) = -A(a,\lambda) = \pm 1$, so that the same setting on opposite sides in **Figure 1** yields outcomes of opposite sign as predicted from the condition of entanglement [1].

From the functions Bell defined, three mutual cross-correlations were considered and computed between one measurement on the A-side and two alternative measurements on the B-side of **Figure 1**. Using the Bell representation, the first of these cross-correlations is:

$$C(a,b) = \int A(a,\lambda) B(b,\lambda) \rho(\lambda) \mathrm{d}\lambda \tag{2.1}$$

(Bell used the variable P instead of C for the correlation, and this has occasionally led to misinterpreting the result as a probability rather than a correlation.) From this and a similar expression using an alternative hypothetical measurement on the B-side, Bell computed the absolute value of the difference of correlations for one observable on the left and two alternatives on the right of **Figure 1**:

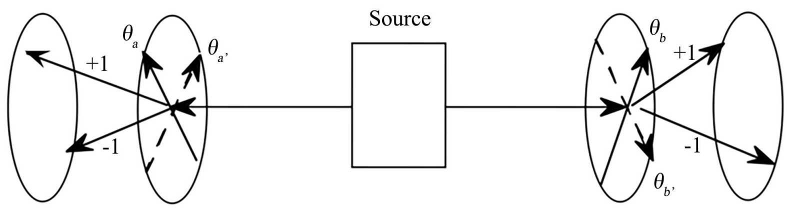

Figure 1. Schematic of Bell experiment in which a source sends two particles (photons most often used) to two detectors having angular settings θ_A and θ_B, (denoted as *a* and *b* in Bell's notation) and alternative settings $\theta_{A'}$ and $\theta_{B'}$. While one measurement operation on the *A*-side, e.g. at setting θ_A, commutes with one on the *B*-side at θ_B, any additional measurements at either $\theta_{A'}$ or $\theta_{B'}$ are non-commutative with prior measurements at θ_A and θ_B, respectively. This figure was drawn by the author and modified in notation for use in [4], as well as other papers.

$$\begin{aligned}
&\left|C(a,b)-C(a,b')\right| \\
&=\left|\int A(a,\lambda)\left(B(b,\lambda)-B(b',\lambda)\right)\rho(\lambda)\mathrm{d}\lambda\right| \\
&\le \int\left|A(a,\lambda)B(b,\lambda)\right|\left|1-B(b,\lambda)B(b',\lambda)\right|\rho(\lambda)\mathrm{d}\lambda \\
&\le \int\left|1-B(b,\lambda)B(b',\lambda)\right|\rho(\lambda)\mathrm{d}\lambda \\
&=1-C(b,b').
\end{aligned} \tag{2.2}$$

Two photons per realization emerge from the source in **Figure 1** so that data to compute either $C(a,b)$ or $C(a,b')$ is obtained from a given run, but not $C(b,b')$, since that requires two runs. Bell explicitly indicated (see Chap 8 of Bell's collected papers [1]) that the third random variable at b' is defined simply as the outcome that would have occurred at that setting had it been used in place of *b*, in the same experimental trial for a given set of variable values λ. However, one cannot undo a random observation at b that depends on random values of λ to obtain another at b'. *Thus, it must be concluded that the Bell inequality, if interpreted as inextricably dependent on the notation used to derive it and Bell's prescription for its interpretation, does not represent any experimental observation that may be obtained using a single photon pair.*

To construct an experimental realization of Bell's prescription, consider the analogy of flipping a loaded coin or die. After a single flip, one may ask Bell's question: suppose the loading had been different on that flip, what would the result have been? This question can only be answered in terms of probabilities, even though the underlying process in this case is thought to be causal. The experimental answer would be to change the loading and perform a large number of flips to determine the probability for heads and tails with each loading. In the case of the common random process of coin flips, multiple interactions and ranges of causal variable values control the final outcome. Further, in such processes, a change in parameter in any decimal place may switch outcomes at a boundary between different outcomes. Thus, causality does not necessarily imply predictability in a world of finite instrument precision.

In the Bell experiment case, quantum mechanics provides the predicted conditional probabilities for observations at different conditionally independent [11] instrument settings on the B-side of the apparatus using different particles, given one selected setting and outcome for particles on the A-side. The analogy with coin flipping indicates that the data for a second instrument setting requires data sets from two different experimental runs. However, to obtain $C(b,b')$ requires a different procedure than used to obtain the first two correlations occurring in the inequality. Since $C(b,b')$ must be computed from the previous data sets using conditional dependence, the result has a functional form different from the other correlations, a fact not recognized by Bell or experimentalists who have copied his assumptions and thus obtained the same result—violation of (2.2). An additional fact [4] not obvious from Bell's statistical derivation, will now be considered that further clarifies the situation.

3. A Bell Inequality Cannot Be Violated by Three Laboratory Data Sets

In laboratory experiments of the Bell type, correlations are not directly observed: finite data sets of ±1's are observed from which the correlations of (2.2) are estimated [5]. When the same algebraic steps are applied to the correlation estimates resulting from three finite data sets that Bell applied to correlations of three theoretically predicted infinite data sets, the same Bell inequality is obtained and is identically satisfied. This surprising result implies that for physical data that is intrinsically finite, the Bell inequality holds as a mathematical fact with or without Bell's assumptions of locality and representation of hidden variables λ. Bell's assumptions, widely believed to be intrinsic to the derivation of the Bell inequality because they were used in its derivation, are unnecessary, and separate from the simpler and more basic mathematical and physical facts that themselves imply the inequality. This is an unusual situation in which purely logical principles override the commonly accepted conclusion that Bell's assumptions determine the results of the Bell theorem. Given the important implications that follow from the more general result, it will now be re-derived for examination.

Assume that three data sets, random or deterministic, labeled *a*, *b*, and b' have been obtained so that they can be written on paper. (Only after these are obtained can the Bell inequality be applied in practice.) The data set items are denoted by a_i, b_i, and b_i', with N items in each set. Each datum equals ±1. One may begin by writing the equation

$$a_i b_i - a_i b_i' = a_i b_i \left(1 - a_i b_i a_i b_i'\right) \tag{3.1}$$

and sum it over the N data triplets of the data sets. After dividing by N, one obtains

$$\frac{1}{N}\sum_{i=1}^{N}\left(a_i b_i - a_i b_i'\right) = \frac{1}{N}\sum_{i=1}^{N} a_i b_i \left(1 - a_i b_i a_i b_i'\right). \tag{3.2}$$

Taking absolute values of both sides,

$$\left|\frac{1}{N}\sum_{i=1}^{N}\left(a_i b_i - a_i b_i'\right)\right| = \left|\frac{1}{N}\sum_{i=1}^{N} a_i b_i \left(1 - a_i b_i a_i b_i'\right)\right|$$
$$\leq \frac{1}{N}\sum_{i=1}^{N}\left|a_i b_i\right|\left|1 - a_i b_i a_i b_i'\right| \tag{3.3}$$
$$= \frac{1}{N}\sum_{i=1}^{N}\left|1 - a_i b_i a_i b_i'\right|,$$

or since $a_i^2 = 1$,

$$\left|\frac{\sum_i^N a_i b_i}{N} - \frac{\sum_i^N a_i b_i'}{N}\right| = \left|\frac{\sum_i^N a_i\left(b_i - b_i'\right)}{N}\right| = \left|\frac{\sum_i^N a_i b_i\left(1 - b_i b_i'\right)}{N}\right| \leq \frac{\sum_i^N \left|1 - b_i b_i'\right|}{N}. \tag{3.4}$$

The Bell inequality as applied to experimental data sets equal in number to Bell's variables, is thus a fact of algebra independent of the physical origin or properties of the data, and it holds for deterministic as well as random data. The sums on the two sides of (3.4) have the form of correlation estimates although the data may be random, deterministic, or a combination of the two. In the case where the data are all random, they may exhibit correlations due to a variety of circumstances, e.g., the correlations may result from correlation to other variables not indicated or known. The final correlation of (3.4) reuses the data used to compute correlations of variables (a,b) and (a,b'), since $(a_i b_i)(a_i b_i') = b_i b_i'$, a fact used explicitly in (3.3) as well as in Bell's derivation in (2.2). As a result, the (b,b') correlation is not expected to have the same form as the previous correlations, since it is computed from the product of their data pairs, and not the process that created their correlation.

To repeat: (3.4) holds independently of physical attributes of the data except for detector clicks recorded as ±1's. (The trigonometric identities of planar geometry do not change if the angles in them arise from AC circuits rather than surveyor data.) The data sets may represent nonsense or be severely corrupted due to nonlocal interference between detectors. However, once data are selected and subscripted to represent a Bell variable, it does not matter what physical effects influenced their behavior. A startling conclusion follows: nonlocal interaction (*i.e.*, pickup) between detectors cannot cause violation of the Bell inequality under the condition of cross-correlation of three data sets corresponding to the three cross-correlated variables assumed in Bell's derivation. Bell's locality assumption is irrelevant to satisfaction of the inequality. It will, however, affect the form of the correlations.

In the case where the data are random and estimates statistically converge to correlations in (3.4) as N becomes large, one has for the first correlation $C_1(a,b)$ of variables a and b,

$$C_1(a,b) = \lim_{N\to\infty}\frac{\sum_i^N a_i b_i}{N},$$

with similar results for the other correlations of (3.4). A more general form of the Bell inequality results:

$$\left|C_1(a,b) - C_2(a,b')\right| \leq 1 - C_3(b,b'). \tag{3.5}$$

The correlation arguments *a*, *b*, etc., now refer to instrument angular settings while when subscripts are added, as in (3.4), they indicate individual data outputs at those settings. Given that no data characteristics have been specified, the three correlation functions will in general have different functional forms, as indicated by their subscripts, but without violating (3.5). However, the final correlational forms are constrained by (3.5).

The physical situation to which (3.5) is applied now imposes restrictions on the data. In the physical experiment as suggested above, the two *B*-side data sets occur at mutually exclusive angular settings that, in Bell's prescription, can only be realized in independent trials. To apply the Bell inequality to this situation, outputs in different trails at two different settings on the *B*-side must be correlated, for each of the two possible outputs at a given setting on the *A*-side of the apparatus. The four data sets acquired in two experimental runs may then be contracted to three, and (3.5) holds with $C(b,b')$ data extracted from $C(a,b)$ and $C(a,b')$.

4. Bell Inequality Violation from Non-Cross-Correlated Data

The Bell inequality is widely believed to be violated by carefully recorded experimental data as in [5]. How can this be, given that the inequality is identically satisfied by any three cross-correlated, physically obtained data sets independently of whether they are deterministic or random? Note that from the derivation reproduced in Section 2 it is not immediately obvious that the general result of Section 3 is true. In the Bell derivation of Section 2, various physical attributes of the data were originally spelled out such as independence of data at *A* from settings at *B*, etc. As seen above, this has no bearing on the satisfaction of the inequality though it would certainly in general change the form of the correlations. It has not been recognized that the inequality results from the cross-correlation of three variables (or four resulting in the four variable case). It is commonly believed (following Bell) that the correlations all have the same form, and as a result, they may be computed using independent pairs of data as in [5]. This assumption suggests a belief that the basic process is second order stationary which is inconsistent with non-commutativity, and the resulting conditionality of probabilities.

The fact that three finite data sets, corresponding to Bell's three random variables, identically satisfy (3.5) proves that the correlations ordinarily inserted into the inequality that violate it are logically inconsistent with any three data sets that can exist. (Note, all experimental data sets are finite, and thus satisfy (3.4).) The correlations $C(a,b)$ and $C(a,b')$ follow from direct measurements of Bell pairs produced from the source. These forms have been confirmed in classic experiments such as [5]. The third correlation $C(b,b')$ must then be derived from the same data to be logically consistent with the Bell inequality in the form of (3.4) or (2.2). When that is accomplished using quantum data, the inequality is applied consistently with the mathematics of its derivation, and is satisfied.

The results of the above requirements for obtaining $C(b,b')$ will be described in detail in the next section.

5. How Two Quantum Mechanical Correlations Determine the Third

5.1. Derivation Using Bell's Notation

To convert Bell's interpretation of the inequality into the realm of experimental probability implies that variables $B(b,\lambda)$ and $B(b',\lambda)$ be obtained from two mutually exclusive settings and that each be correlated with the same value of $A(a,\lambda)$ in (2.2). The correlations must be obtained for each value of $A(a,\lambda)$ in turn. The contribution to the correlation of variables ($B(b,\lambda)$, $B(b',\lambda)$) is thus determined by two conditions that hold simultaneously. Since $A(a,\lambda_1)^2=1$, $A(a,\lambda_i)B(b,\lambda_i)A(a,\lambda_i)B(b',\lambda_i)=B(b,\lambda_1)B(b',\lambda_1)$ in (2.2). But for correlations computed by averaging over two different values of λ in statistically independent trials:

$$\begin{aligned}&C\left(A(a,\lambda_1)B(b,\lambda_1)A(a,\lambda_2)B(b',\lambda_2)\right)\\&=C\left(A(a,\lambda_1)B(b,\lambda_1)\right)C\left(A(a,\lambda_2)B(b',\lambda_2)\right),\end{aligned}\tag{5.1a}$$

where subscripts 1 and 2 on λ indicate the separate trials. The correlation then necessarily factors since probabilities in independent trials factor and must be multiplied. The condition $A(a,\lambda_1)=A(a,\lambda_2)$ must also be imposed, since the same value of $A(a,\lambda)$ multiplies both $B(b,\lambda)$ and $B(b',\lambda)$. It follows that

$$C\left(A(a,\lambda_1)B(b,\lambda_1)\right)C\left(A(a,\lambda_2)B(b',\lambda_2)\right)=C\left(B(b,\lambda_1)B(b',\lambda_2)\right).\tag{5.1b}$$

Since each of the correlations on the left is given by the well-known Bell correlation, the result is

$$C\left(B(b,\lambda_1)B(b',\lambda_2)\right)=\left[-\cos(b-a)\left(-\cos(b'-a)\right)\right]=\cos(b-a)\cos(b'-a)\tag{5.1c}$$

This important result will be re-derived below in more mathematically explicit detail using quantum probabilities. As stated, $B(b,\lambda)$ and $B(b',\lambda)$ are variables each correlated to fixed outcomes at *A*, and so are correlated to each other as conditionally independent.

5.2. Derivation Using Quantum Probabilities

The correlation of (5.1c) will now be re-derived using quantum mechanical probabilities to predict correlations at alternate variable setting pairs (a,b) and (a,b') on the two sides of the apparatus of Figure 1. These probabilities result from entanglement and are well known [9] (the subscripted pluses and minuses indicate ±1 outputs at instrument settings *a* and *b* respectively):

$$\begin{aligned}P_{++}(a,b)=P_{--}(a,b)=\frac{1}{2}\sin^2\frac{b-a}{2};\\P_{+-}(a,b)=P_{-+}(a,b)=\frac{1}{2}\cos^2\frac{b-a}{2}.\end{aligned}\tag{5.2a}$$

The angular setting difference divided by 2 holds for Bell's original case of entangled spins. (In the optical version that corresponds to most Bell experiments, the 2 does not occur, with the result that a factor of 2 occurs in the argument of the final correlation.) Note, the joint probabilities (5.2a) are expressed in terms of conditional probabilities. Again using ± subscripts on the probabilities to indicate ±1 outcomes, $P_+(a) = P_-(a) = 1/2$ and the conditional probabilities of outcomes on the *B*-side given those on the *A*-side from (5.2a) are

$$\begin{aligned} P_{++}(b \mid a) &= P_{--}(b \mid a) = \sin^2 \frac{b-a}{2}; \\ P_{+-}(b \mid a) &= P_{-+}(b \mid a) = \cos^2 \frac{b-a}{2}. \end{aligned} \tag{5.2b}$$

The probabilities at an alternative setting b' are obtained by inserting it in place of b in (5.2a,b). From the joint probabilities (5.2a), the correlation $C(a,b)$ is

$$\begin{aligned} C(a,b) &= \left[(+1)(+1)+(-1)(-1)\right]\frac{1}{2}\sin^2 \frac{b-a}{2} \\ &\quad +\left[(+1)(-1)+(-1)(+1)\right]\frac{1}{2}\cos^2 \frac{b-a}{2} \\ &= -\left(\cos^2 \frac{b-a}{2} - \sin^2 \frac{b-a}{2}\right) = -\cos(b-a), \end{aligned} \tag{5.3a}$$

and the correlation $C(a,b')$ is immediately

$$C(a,b') = -\cos(b'-a). \tag{5.3b}$$

To implement Bell's prescription of Section 2 for two variables on the *B*-side of **Figure 1**, two sets of observations must be performed just as in the case for observing probabilities in the analogous situation of two differently loaded coins. To be relatable to observations in Bell experiments, the probability densities used in the Bell notation must be replaced by quantum probabilities as appropriate to an ensemble of observations. The conditional probabilities of (5.2b) for setting coordinates (b, b') on the same side of a Bell apparatus are then required.

From (3.3-3.4) and corresponding steps in (2.2), the correlation of outcomes at (b,b') is the sum of conditional averages for $A_+(a)=1$ and $A_-(a)=-1$ each occurring with probability ½. In each case, the value observed at setting a is a parameter for the conditional probabilities now used to provide the products of probabilities for outcomes in the two independent trials. The normalization of these probabilities equals 1 for $A_+(a)=1$:

$$\begin{aligned} & P_{++}(b \mid a)P_{++}(b' \mid a) + P_{-+}(b \mid a)P_{-+}(b' \mid a) \\ & + P_{++}(b \mid a)P_{-+}(b' \mid a) + P_{-+}(b \mid a)P_{++}(b' \mid a) \\ & = \sin^2\left[(b-a)/2\right]\sin^2\left[(b'-a)/2\right] + \cos^2\left[(b-a)/2\right]\cos^2\left[(b'-a)/2\right] \\ & + \sin^2\left[(b-a)/2\right]\cos^2\left[(b'-a)/2\right] + \cos^2\left[(b-a)/2\right]\sin^2\left[(b'-a)/2\right] \\ & = 1, \end{aligned} \tag{5.4a}$$

A similar normalization may be computed for the opposite outcome at $A_-(a)=-1$. Using the conditional probabilities from (5.2b), the correlation of the conditionally independent [11] variables (b_i, b_i') is $C(bb'|a,1)$ where the 1 after setting *a* denotes the numerical output selected at that setting:

$$\begin{aligned} C(bb'|a,1) = {} & (1)(1)\sin^2[(b-a)/2]\sin^2[(b'-a)/2] \\ & +(-1)(-1)\cos^2[(b-a)/2]\cos^2[(b'-a)/2] \\ & +(1)(-1)\sin^2[(b-a)/2]\cos^2[(b'-a)/2] \\ & +(-1)(1)\cos^2[(b-a)/2]\sin^2[(b'-a)/2] \\ = {} & \cos(b'-a)\cos(b-a). \end{aligned} \tag{5.4b}$$

Similarly:

$$C(bb'|a,-1) = \cos(b-a)\cos(b'-a). \tag{5.4c}$$

Since the two values at a occur with probability ½ the overall correlation is [12]

$$\begin{aligned} C(b,b'|a) &= (1/2)C(bb'|a,1) + (1/2)C(bb'|a,-1) \\ &= \cos(b-a)\cos(b'-a). \end{aligned} \tag{5.5}$$

The steps of the calculation may be grouped differently to simplify the interpretation of the result. Correlation $C(bb'|a,1)$ may be written

$$C(bb'|a,1) = \sum_{b_i b_i'} b_i b_i' P(b_i, b_i'|a,1) = \sum_{b_i b_i'} b_i b_i' P(b_i|a,1) P(b_i'|a,1), \tag{5.6a}$$

where $P(b_i, b_i'|a,1)$ factors since b_i and b_i' are outputs at different settings in two independent experiments with data collected only for outcome +1 at *a*. The result is

$$C(bb'|a,1) = \sum_{b_i} b_i P(b_i|a,1) \sum_{b_i'} b_i' P(b_i'|a,1) = \bar{b}(a,1)\bar{b}'(a,1), \tag{5.6b}$$

where

$$\begin{aligned} \bar{b}(a,1) &= 1P(1|a,1) - 1P(-1|a,1) = \sin^2(b-a)/2 - \cos^2(b-a)/2 = -\cos(b-a), \\ \bar{b}'(a,1) &= 1P(1|a,1) - 1P(-1|a,1) = \sin^2(b'-a)/2 - \cos^2(b'-a)/2 = -\cos(b'-a), \end{aligned} \tag{5.6c}$$

and

$$C(bb'|a,1) = \cos(b-a)\cos(b'-a). \tag{5.6d}$$

The same result is obtained for $C(bb'|a,-1)$ so that (5.5) again follows.

The point of this exercise is to show that the final result of (5.5) is analogous to that obtained in coin flipping experiments to compare probabilities for two different coin loadings. The more elaborate data matching procedure mentioned previously may thus be bypassed. After using appropriate trig identities, (5.5) has been shown to satisfy the Bell inequality given by (2.2) [13].

6. Discussion

A summary of logical steps associated with the derivation and use of the Bell inequality may prove useful to the reader. Reviewing the logic of the Bell inequality begins with Bell's derivation using three cross-correlated random variables, *A*, *B*, *B'* each having values of ±1 only. Three correlations $C(ab)$, $C(ab')$, and $C(bb')$ result with each variable correlated with two others in a rather symmetrical form.

However, it has not been noticed from the mathematical steps with which the computation proceeds, that the difference of the first two correlations alone determines the form of the inequality, and that their two functional forms determine the different functional form of the third correlation. This follows because the values of $B(b)$) and $B(b')$ that occur for each fixed value of $A(a)$ determine $C(bb')$ also since $C(abab') = C(bb')$, given that $A(a)^2 = 1$. *Thus, correlated particle-pair data initially produce correlations* $C(ab)$ *and* $C(ab')$, *but the third correlation* $C(bb')$ *then results through reuse of the previous data values*. Bell did not directly compute the final correlation but seemingly assumed that it had the same form as $C(ab)$ and $C(ab')$, as have generations of experimentalists. This is in contradiction of the Bell inequality derivation.

An additional significant fact emerges when Bell's sequence of mathematical steps is applied to three finite (laboratory) data sets, random or deterministic, each consisting of ± 1's. The same Bell inequality results for the finite correlation estimates, and the inequality must be identically satisfied by any three data sets, random or deterministic. Further, the correlations may have different functional forms. These results are independent of nonlocal interactions among the detectors, *i.e.*, pickup, which would affect the form of correlations, but not their satisfaction of the inequality. Thus, when applied to actual physical data rather than assumed correlations, the inequality in three variables cannot be violated. (The same result is easily proven to hold in the four variable case.)

Given the physical experiment to which the inequality pertains, a further complication arises: a mathematical inequality in three variables is applied to an experiment based on particle pairs in which only one measurement per particle may be obtained. It is intrinsically impossible to observe variables b and b' at mutually exclusive settings in one random realization as required by Bell's derivation. To obtain data at such alternative instrument settings, two experimental runs are required, one experimental run for each pair of settings (a,b) and (a,b'). Since outputs at both b and b' are obtained and correlated for a given output value at a in the derivation of the Bell inequality, they are then correlated to each other, *i.e.*, are "conditionally independent" in the mathematical usage. This is consistent with the Bell inequality as expressed in a form corresponding to data observations. The Bell version does not correspond to actual physical observation. That is due to the fact that one cannot undo a random experimental event to obtain a result at an alternative mutually exclusive instrument setting.

Given that particles are produced in pairs in the experiments to which the inequality has been applied, it has been assumed that it is correct to insert three independently realized correlations of data pairs into the inequality. This is mathematically inconsistent with the Bell derivation. Since its use is at variance with the derivation, the inequality may be violated.

When quantum probability predictions are used to describe the merging of data from two experiments as necessary to apply the inequality, a different form results for the third correlation than that assumed by Bell, and later by experimentalists. The form in question is the product of the contributions that produced the correlations $C(ab)$ and $C(ab')$ for fixed outputs at a, as derived above. Thus, correction of Bell experimental results so that they are logically consistent with quantum mechanics requires logical application of experimental observations already known. The inequality, if used consistently with its derivation, is satisfied. This is accomplished without assumptions of non-locality or non-reality.

7. Conclusions

1) The Bell three-random-variable inequality, if inextricably tied to bell's hidden variable notation cannot be applied to any experiment, since one cannot undo the results at one instrument setting to obtain results at an alternative setting. 2) However, the same steps that Bell used in the derivation assuming three variables may be applied to three arbitrary data sets, random or deterministic, with the result that the Bell inequality when applied to actual data must be identically satisfied independently of Bell's various assumptions. The estimates now taking the place of Bell's assumed correlations may clearly have different functional forms. 3) When applied to an actual experiment in which only one pair of data is obtained per random realization, application of the inequality to answer Bell's question regarding results at alternative settings requires two experimental runs. A fixed setting and fixed random output are chosen on one side of an apparatus, and two alternate mutually exclusive settings and outcomes are observed for a second variable. The alternate variable outcomes are now "conditionally independent", or conditionally dependent on the fixed variable, and the Bell inequality for correlations is satisfied by quantum mechanical results logically employed.

Anyone who, in spite of the above, believes that the Bell inequalities may in principle be violated by physical data needs to perform the following task: write three very small data sets (or four in the four variable case) on the back of an envelope that violate the inequality and show them to the world.

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Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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